Telecom ParisTech SOCOM205
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ASSIGNMENT 5: SOLUTION

Exercise 1 (Capacity of two channels)**.** Consider two DMCs $(\mathcal{X}_1, p(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2, respectively. A new channel $(\mathcal{X}_1 \times \mathcal{X}_2, p(y_1|x_1) \times p(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ are sent simultaneously, resulting in y_1, y_2. Find the capacity of this channel.

Solution. Note that due to the channel we have

$$p(x_1, x_2, y_1, y_2) = p(x_1, x_2)p(y_1|x_1)p(y_2|x_2)$$

which implies the Markov chain

$$Y_1 - X_1 - X_2 - Y_2.$$

Now,

$$\begin{aligned}
C &= \max_{p(x_1,x_2)} I(X_1, X_2; Y_1, Y_2) \\
&= \max_{p(x_1,x_2)} I(X_1, X_2; Y_1) + I(X_1, X_2; Y_2|Y_1) \\
&\overset{(a)}{=} \max_{p(x_1,x_2)} I(X_1; Y_1) + I(X_1, X_2; Y_2|Y_1) \\
&= \max_{p(x_1,x_2)} I(X_1; Y_1) + H(Y_2|Y_1) - H(Y_2|Y_1, X_1, X_2) \\
&\overset{(b)}{=} \max_{p(x_1,x_2)} I(X_1; Y_1) + H(Y_2|Y_1) - H(Y_2|X_2) \\
&\overset{(c)}{\leq} \max_{p(x_1,x_2)} I(X_1; Y_1) + H(Y_2) - H(Y_2|X_2) \\
&= \max_{p(x_1,x_2)} I(X_1; Y_1) + I(X_2; Y_2) \\
&\leq \max_{p(x_1,x_2)} I(X_1; Y_1) + \max_{p(x_1,x_2)} I(X_2; Y_2) \\
&= \max_{p(x_1)} I(X_1; Y_1) + \max_{p(x_2)} I(X_2; Y_2) \\
&= C_1 + C_2
\end{aligned}$$

where (a) is due to the Markov chain $X_2 - X_1 - Y_1$, (b) is due to the Markov chain $Y_2 - X_2 - (X_1, Y_1)$ and (c) is derived using conditioning inequality.

The equality can be achieved in all steps by choosing X_1 and X_2 independent, *i.e.*, $p(x_1, x_2) = p(x_1)p(x_2)$. So, $C = C_1 + C_2$. □

Exercise 2 (Choice of channels)**.** Find the capacity C of the union of two channels $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$, where at each time, one can send a symbol over channel 1 or channel 2 but not both. Assume that the output alphabets are distinct and do not intersect. Show that $2^C = 2^{C_1} + 2^{C_2}$. Thus, 2^C is the effective alphabet size of a channel with capacity C.

Solution. Let

$$\theta = \begin{cases} 1 & \text{with probability } p \\ 2 & \text{with probability } 1-p \end{cases}$$

be the indicator that shows we are using which channel. Also, define $X = X_\theta$ and $Y = Y_\theta$. The capacity of the channel is thus

$$C = \max_{p(x)} I(X;Y).$$

Now notice that

$$I(X,\theta;Y) = I(\theta;Y) + I(X;Y|\theta) = I(X;Y) + I(\theta;Y|X).$$

So,

$$\begin{aligned}
I(X;Y) &= I(\theta;Y) + I(X;Y|\theta) - I(\theta;Y|X) \\
&\overset{(a)}{=} H(\theta) + I(X;Y|\theta) \\
&= H(\theta) + I(X_1;Y_1|\theta=1)Pr(\theta=1) + I(X_2;Y_2|\theta=2)Pr(\theta=2) \\
&= H(\theta) + I(X_1;Y_1)Pr(\theta=1) + I(X_2;Y_2)Pr(\theta=2) \\
&= H(\theta) + p \cdot I(X_1;Y_1) + (1-p) \cdot I(X_2;Y_2)
\end{aligned}$$

where (a) is due to the facts that $H(\theta|Y) = H(\theta|X) = 0$ since the outputs (and also inputs) are different for two channels.

Now, we have

$$\begin{aligned}
C &= \max_{p(x)} I(X;Y) \\
&= \max_{p}[\max_{p(x_1,x_2)} [H(\theta) + p \cdot I(X_1;Y_1) + (1-p) \cdot I(X_2;Y_2)]] \\
&= \max_{p}[H(\theta) + p \cdot \max_{p(x_1)} I(X_1;Y_1) + (1-p) \cdot \max_{p(x_2)} I(X_2;Y_2)] \\
&= \max_{p}[H(\theta) + p \cdot C_1 + (1-p) \cdot C_2] \\
&= \max_{p}[-p\log p - (1-p)\log(1-p) + p \cdot (C_1 - C_2) + C_2]
\end{aligned}$$

Taking derivative with respect to p and let it to zero, we have

$$\log(\frac{p^*}{1-p^*}) = C_1 - C_2,$$

$$p^* = \frac{2^{C_1-C_2}}{2^{C_1-C_2}+1},$$

So,

$$\begin{aligned}
C &= -p^*\log(\frac{p^*}{1-p^*}) - \log(1-p^*) + p^* \cdot (C_1 - C_2) + C_2 \\
&= -\log(1-p^*) + C_2
\end{aligned}$$

Hence,

$$2^C = 2^{C_2 - \log(1-p^*)} = \frac{2^{C_2}}{1-p^*} = 2^{C_2} \cdot (2^{C_1 - C_2} + 1) = 2^{C_1} + 2^{C_2}.$$

□

Exercise 3 (Z-channel)**.** The Z-channel has binary input and output alphabets and transition probabilities $p(y|x)$ given by the following matrix:

$$Q = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, x, y \in \{0, 1\}$$

a. Find the capacity of the Z-channel and the maximizing input probability distribution.

b. Assume that we choose a $(2^{nR}, n)$ code at random, where each codeword is a sequence of fair coin tosses. This will not achieve capacity. Find the maximum rate R such that the probability of error $P_e^{(n)}$, averaged over the randomly generated codes, tends to zero as the block length n tends to infinity.

Solution. a. Let $p = Pr(X = 1)$. So, $Pr(Y = 1) = 1 - Pr(Y = 0) = \frac{p}{2}$

$$H(Y|X) = H(Y|X = 0)Pr(X = 0) + H(Y|X = 1)Pr(X = 1) = 0 + 1 \cdot p = p$$

$$H(Y) = h_b(\frac{p}{2})$$

$$I(X;Y) = H(Y) - H(Y|X) = h_b(\frac{p}{2}) - p$$

Taking derivative with respect to p, it can be seen that the mutual information gets maximized for $p = \frac{2}{5}$ and the capacity is thus $C \simeq 0.32$.

b. From the proof of channel coding theorem, it can be seen that if we choose the codewords with probability $p(x)$, the rate $I(X;Y)$ can be achieved. Here, we choose the codewords with $Pr(X = 0) = \frac{1}{2}$, so

$$I(X;Y) = H(Y) - H(Y|X) = h_b(\frac{1}{4}) - \frac{1}{2} = \frac{3}{2} - \frac{3}{4}\log 3$$

is achievable which is less than the capacity.

□

Exercise 4 (Erasures and errors in a binary channel)**.** Consider a channel with binary inputs that has both erasures and errors. Let the probability of error be α and the probability of erasure be β, which means that when we send symbol 0, with probability $1 - \alpha - \beta$ we receive symbol 0, with probability α we receive symbol 1 and with probability β we receive an erasure symbol. Find the capacity of this channel.

Solution. The alphabet of $\mathcal{Y} = \{0, e, 1\}$ and the transition probability matrix is

$$Q(y|x) = \begin{pmatrix} 1-\alpha-\epsilon & \alpha & \epsilon \\ \epsilon & \alpha & 1-\alpha-\epsilon \end{pmatrix}$$

Due to the symmetry of transition probability matrix for inputs 0 and 1, $I(X;Y)$ is the same for $Pr(X=p)$ and $Pr(X=1)=1-p$, so is symmetric with respect to $\frac{1}{2}$, moreover $I(X;Y)$ is concave with respect to p. These, yields that $I(X;Y)$ is maximized for $p=\frac{1}{2}$. With $p=\frac{1}{2}$, we have

$$Pr(Y=0)=\frac{1}{2}(1-\alpha)$$
$$Pr(Y=1)=\frac{1}{2}(1-\alpha)$$
$$Pr(Y=e)=\alpha$$
$$H(Y)=-(1-\alpha)\log(\frac{1-\alpha}{2})-\alpha\log(\alpha)$$
$$H(Y|X)=H(1-\alpha-\epsilon,\epsilon,\alpha)$$
$$C=H(Y)-H(Y|X)$$

□

Exercise 5 (Binary multiplier channel)**.** Consider the channel $Y=X\cdot Z$, where X and Z are independent binary random variables that take on values 0 and 1. Z is Bernoulli(α) [i.e., $P(Z=1)=\alpha$].

a. Find the capacity of this channel and the maximizing distribution on X.

b. Now suppose that the receiver can observe Z as well as Y. What is the capacity?

Solution. a. The transition probability matrix is

$$Q=\begin{pmatrix}1 & 0\\ 1-\alpha & \alpha\end{pmatrix}$$

So, if $Pr(X=1)=p$, then

$$H(Y|X)=H(Y|X=0)Pr(X=0)+H(Y|X=1)Pr(X=1)=0\cdot(1-p)+h_b(\alpha)\cdot p=ph_b(\alpha).$$
$$H(Y)=h_b(\alpha\cdot p)$$
$$I(X;Y)=H(Y)-H(Y|X)=h_b(\alpha\cdot p)-ph_b(\alpha)$$

Taking derivative with respect to p and let it equal to zero, we have

$$\alpha\log(\frac{1-\alpha p^*}{\alpha p^*})=h_b(\alpha),$$
$$p^*=\frac{1}{\alpha(2^{\frac{h_b(\alpha)}{\alpha}}+1)}.$$

and

$$\begin{aligned}C=I(X;Y)|_{p=p^*}&=h_b(\alpha\cdot p^*)-p^*h_b(\alpha)\\&=p^*\alpha\log(\frac{1-\alpha p^*}{\alpha p^*})-\log(1-\alpha p^*)-p^*h_b(\alpha)\\&=-\log(1-\alpha p^*)\\&=\log(\frac{2^{\frac{h_b(\alpha)}{\alpha}}+1}{2^{\frac{h_b(\alpha)}{\alpha}}})\end{aligned}$$

Note that for $\alpha=\frac{1}{2}$, this channel is the same as Z-channel.

b. In this case,

$$C = \max_{p(x)} I(X;Y,Z) = \max_{p(x)}[I(X;Z) + I(X;Y|Z)] = \max_{p(x)} I(X;Y|Z).$$

Assuming $Pr(X = 1) = p$,

$$H(Y|Z) = H(Y|Z = 0)Pr(Z = 0) + H(Y|Z = 1)Pr(Z = 1) = 0 \cdot (1-\alpha) + H(X|Z = 1) \cdot \alpha = \alpha H(X) = \alpha h_b(p)$$

$$H(Y|X,Z) = 0$$

$$I(X;Y|Z) = H(Y|Z) - H(Y|X,Z) = \alpha h_b(p)$$

which is maximized for $p = \frac{1}{2}$ and hence, $C = \alpha$.

$\square$