

ASSIGNMENT 4

Exercise 1 (Additive channel). Consider a channel $Y = Z + X$, where X is a binary input, Y a binary output, and Z , the noise in the channel, a random variable that is independent of X and that takes values $\{0, a\}$ with probabilities $Pr(Z = 0) = 1 - Pr(z = a) = \frac{1}{3}$. Find the capacity of this channel with respect to a .

Exercise 2 (Mod 11 channel). Let $U \in \{0, 1, \dots, 10\}$ be the input of a DMC. The output is determined by the formula $V = U + Z \pmod{11}$, where Z is independent of U , and

$$P\{Z = 1\} = P\{Z = 2\} = P\{Z = 3\} = \frac{1}{3}.$$

Find the channel capacity.

Exercise 3. Find the capacity and optimal input distribution for the three-input, three-output channel whose transition probabilities are given by

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

Exercise 4 (Cascade of BSCs). Consider the cascade of k binary symmetric channels, each having the same crossover probability p :

1. Show that the binary channel obtained is equivalent to a BSC with crossover probability $\frac{1}{2}(1 - (1 - 2p)^k)$. This shows that the capacity of the cascade channel goes to zero as $k \rightarrow \infty$.
2. Suppose now that at the output of each channel arbitrary information processing is allowed. What is the capacity of the cascade channel?

Exercise 5 (Repetition Code). The symbols 0 or 1 are to be transmitted through a binary symmetric channel with crossover probability $p < 1/2$. The channel code has two codewords (000) and (111), the code (000) is transmitted if the message is 0, and (111) is transmitted if the message is 1. The decoder is using a majority decision; if the received word has more than one zero, then the decoder output is 0, otherwise it is 1. What are the probabilities of incorrect decoding $P_{e,1}$ and $P_{e,2}$?

Exercise 6 (Byte channel). The input to a channel Q is a word of 8 bits. The output is also a word of 8 bits. Each time it is used, the channel flips exactly one of the transmitted bits, but the receiver does not know which one. The other seven bits are received without error. All 8 bits are equally likely to be flipped. Derive the capacity of this channel.

Exercise 7 (Zero-error, pentagon channel). Consider the “pentagon” channel with input and output alphabets $\{0, 1, 2, 3, 4\}$ and conditional probabilities given by

$$Q(y|x) = \begin{cases} 1/2 & \text{if } y = x \pm 1 \pmod{5} \\ 0 & \text{otherwise.} \end{cases}$$

For instance, symbol “4” becomes “3” or “0” with probability $1/2$.

1. Compute the capacity of this channel.
2. The “zero-error” capacity is the number of information bits per channel use that can be transmitted with exactly zero error probability.
 - (a) By considering coding with block length equal to 1, show that the zero-error capacity of this channel is at least 1 bit per channel use.
 - (b) Exhibit a code with 5 codewords of length 2 which shows that the zero-error capacity is strictly greater than 1 bit per channel use.

Shannon showed in 1956 that $0.5 \log 5$ bits per channel use is achievable without error over the pentagon channel. But it was more than twenty years after (1979(!)) that Lovász showed that $0.5 \log 5$ is indeed the highest rate one can get (*i.e.*, it is the zero-error capacity), even when considering codewords of length ≥ 2 .

Exercise 8 (Processing the output). Consider a channel with a conditional probability distribution $\{Q(y|x)\}$. Instead of using directly the output Y , we first process the output as $\tilde{Y} = g(Y)$, and then use \tilde{Y} for decoding. Is this processing useful?