Telecom ParisTech SOCOM205
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Assignment 3: Solution

Exercise 1 (Conditional entropy)**.** Show that if $H(Y|X) = 0$, then Y is a function of X.

Proof.

$$0 = H(Y|X) = \sum_{x \in \mathcal{X}} H(Y|X = x) \cdot p(x).$$

Hence, for each $p(x) > 0$ we should have $H(Y|X = x) = 0$, which means $\exists y_x \in Y$ such that $p(y_x|x) = 1$. So, for each $x \in \mathcal{X}$ that $p(x) > 0$ and its corresponding y_x, define $f(x) = y_x$, which shows that $Y = f(X)$ is a function of X. □

Exercise 2 (Mutual information)**.**

a. Let X be a uniform random variable over $\{1, 2, 3, 4\}$. Let

$$Y = \begin{cases} 0 \text{ if } X \text{ is odd} \\ 1 \text{ otherwise.} \end{cases} \quad Z = \begin{cases} 0 \text{ if } X \text{ is even} \\ 1 \text{ otherwise.} \end{cases}$$

Find $I(Y; Z)$.

b. We roll a fair die which has six sides. What is the mutual information between the top side and the one facing you?

Proof.

a. Note that always $Y \neq Z$, which means knowing Z let us know Y, *i.e.* $H(Y|Z) = 0$.

$$I(Y; Z) = H(Y) - H(Y|Z) = 1 - 0 = 1.$$

b. Top side X_T can take any of $\{1, 2, 3, 4, 5, 6\}$ with same probability. Moreover, knowing the one facing us, X_F, X_T can take four values with same probability, so

$$I(X_T; X_F) = H(X_T) - H(X_T|X_F) = \log(6) - \log(4)$$

□

Exercise 3 (Conditional mutual information)**.** Consider a sequence of n binary random variables $X_1, X_2, \cdots, X_n$. Each sequence with an even number of 1's has probability $2^{-(n-1)}$, and each sequence with an odd number of 1's has probability 0. Find the mutual informations $I(X_1; X_2)$, $I(X_2; X_3|X_1), \cdots, I(X_{n-1}; X_n|X_1, ..., X_{n-2})$.

Proof. We have always $X_n = X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1}$,[1] since the sequences with odd number of ones have zero probability, and since each sequence with even number of 1s is equiprobable, X_1, X_2, $\cdots$, X_n are independent Bernoulli($\frac{1}{2}$) random variables. So, for $2 \leq i \leq n-2$,

$$\begin{aligned} I(X_i; X_{i+1}|X_1, \cdots, X_{i-1}) &= H(X_{i+1}|X_1, \cdots, X_{i-1}) - H(X_{i+1}|X_1, \cdots, X_{i-1}, X_i) \\ &= H(X_{i+1}) - H(X_{i+1}) = 0 \end{aligned}$$

[1] $\oplus$ is sum modulo 2.

and for $i = n-1$,

$$\begin{aligned} I(X_{n-1};X_n|X_1,\cdots,X_{n-2}) &= H(X_n|X_1,\cdots,X_{n-2}) - H(X_n|X_1,\cdots,X_{n-2},X_{n-1}) \\ &= H(X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1}|X_1,\cdots,X_{n-2}) - H(X_1 \oplus X_2 \oplus \cdots \oplus X_{n-1}|X_1,\cdots,X_{n-2},X_{n-1}) \\ &= H(X_{n-1}|X_1,\cdots,X_{n-2}) - 0 \\ &= H(X_{n-1}) = 1 \end{aligned}$$

□

Exercise 4 (Entropy and pairwise independence)**.** Let X, Y, Z be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X;Y) = I(X;Z) = I(Y;Z) = 0$.

a. Under this constraint, what is the minimum value for $H(X,Y,Z)$?

b. Give an example achieving this minimum.

Proof. a.

$$\begin{aligned} H(X,Y,Z) &= H(X) + H(Y|X) + H(Z|Y,X) \\ &= H(X) + H(Y) + H(Z|Y,X) \\ &\geq H(X) + H(Y) \\ &= 2 \end{aligned}$$

b. Let $Z = X \oplus Y$.

□

Exercise 5 (Typicality)**.** To clarify the notion of typical set $A_\epsilon^{(n)}$, we will calculate the set for a simple example. Consider a sequence of i.i.d. binary random variables, $X_1, X_2, \cdots, X_n$, where the probability that $X_i = 1$ is 0.7.

a. Compute $H(X)$.

b. With $n = 8$ and $\epsilon = 0.1$, which sequences fall in the typical set $A_\epsilon^{(n)}$? What is the probability of the typical set? How many elements are there in the typical set?

Solution. a. $H(X) = h_b(0.3) \simeq 0.88$

b. Suppose there are r ones in the sequence. The probability of this sequence is

$$p(x^n) = 0.7^r \cdot 0.3^{8-r}.$$

So, for being typical we should have

$$|-\frac{1}{8}\log(p(x^n)) - H(X)| \leq \epsilon,$$

which gives that

$$|-\frac{r \cdot \log(0.7) + (8-r) \cdot \log(0.3)}{8} - 0.88| = |0.857 - 0.152r| \leq 0.1$$

It can be verified that $r = 5$ and $r = 6$ satisfy this condition, *i.e.* sequences with 5 or 6 ones. The probability of typical set is

$$Pr(A_{0.1}^{(8)}) = \binom{8}{5}(0.7)^5(0.3)^3 + \binom{8}{6}(0.7)^6(0.3)^2,$$

and the number of elements is

$$|A_{0.1}^{(8)}| = \binom{8}{5} + \binom{8}{6}.$$

□

Exercise 6 (AEP)**.** Let(X_i, Y_i) be i.i.d. $\sim p(x, y)$. Compute

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{p(X^n)p(Y^n)}{p(X^n, Y^n)}.$$

Solution.

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{p(X^n)p(Y^n)}{p(X^n, Y^n)} = \mathbb{E} \log \frac{p(X)p(Y)}{p(X, Y)} = -I(X; Y)$$

□

Exercise 7 (Desintegration)**.** An entity of size 1 splits each second into two parts, with proportion of sizes having the following distribution:

$$\text{Proportion} = \begin{cases} (\frac{3}{4}, \frac{1}{4}) & \text{with probability } \frac{2}{5} \\ (\frac{2}{3}, \frac{1}{3}) & \text{with probability } \frac{3}{5} \end{cases}$$

At each time, the bigger part remains, and the smaller part will disapear. Thus, for example, a splitting in the first second may result in a part of size $\frac{3}{4}$. In the 2nd second, the size might reduce to $(\frac{3}{4}) \cdot (\frac{2}{3})$, and so on. How large, to first order in the exponent, is the remained part after n splitting?

Solution. Consider $C_n = X_1 \cdot X_2 \cdots X_n$ where

$$X_i = \begin{cases} \frac{3}{4} & \text{with probability } \frac{2}{5} \\ \frac{2}{3} & \text{with probability } \frac{3}{5} \end{cases}.$$

We know that $\frac{1}{n}\log(C_n) = \frac{1}{n}\sum_i \log(X_i) = \mathbb{E}\log X \pm \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. So

$$C_n = 2^{n(\mathbb{E}\log X \pm \varepsilon_n)}$$

where $\mathbb{E}\log X = \frac{2}{5}\log\frac{3}{4} + \frac{3}{5}\log\frac{2}{3}$.

□