Telecom ParisTech
Prof. Aslan Tchamkerten

SOCOM205, Information Theory
February 2017

# ASSIGNMENT 1

**Exercise 1** (Alternative definition of unique decodability)**.** An $f : \mathcal{X} \to \mathcal{Y}$ code is called uniquely decodable if for any messages $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_k$ (where $u_1, v_1, \cdots, u_k, v_k \in \mathcal{X}$) with

$$f(u_1)f(u_2)\cdots f(u_k) = f(v_1)f(v_2)\cdots f(v_k),$$

we have $u_i = v_i$ for all $i$. That is, as opposed to the definition given in class, we require that the codes of any pair of messages with the same length are equal. Prove that the two definitions are equivalent.

**Exercise 2** (Bad codes)**.** Which of the following binary codes cannot be a Huffman code for any distribution? Verify your answer.

a. 0, 10, 111, 101

b. 00, 010, 011, 10, 110

c. 1, 000, 001, 010, 011

**Exercise 3** (Shannon code, divergence)**.** Suppose we wrongly estimate the probability of a source of information, and that we use a Shannon code for a distribution $Q$ whereas the true distribution is $P$. Show that

$$H(P) + D(P||Q) \leq L(C) \leq H(P) + D(P||Q) + 1\,.$$

So $D(P||Q)$ can be interpreted as the increase in descriptive complexity due to incorrect information.

**Exercise 4** (Huffman code for a wrong source)**.** The purpose of this problem is to see what happens when you design a code for the wrong set of probabilities. Consider a Huffman code that is designed for a three symbol source whose probability is given by $(0.5, 0.3, 0.2)$. Suppose that we use this code for the source $(0.15, 0.2, 0.65)$. Find the average number of binary code symbols per source symbol and compare it with the entropy of the source.

**Exercise 5** (Entropy)**.** Let $X$ and $Y$ be the outcomes of a pair of dice thrown independently (hence each independently takes on values in $\{1, 2, 3, 4, 5, 6\}$ with equal probabilities). Let $Z = X + Y$ and let $Q = Z \mod 2$. Compute the following entropies: $H(X)$, $H(Y)$, $H(Z)$, $H(Q)$.

**Exercise 6** (Entropy)**.** Let $X$ be a random variable taking values in $M$ points $a_1, \ldots, a_M$ and let $p_X(a_M) = \alpha$. Show that

$$H(X) = -\alpha \log \alpha - (1-\alpha)\log(1-\alpha) + (1-\alpha)H(Y)$$

where $Y$ is a random variable taking values in $M - 1$ points $a_1, \ldots, a_{M-1}$ with probabilities $P_Y(a_j) = P_X(a_j)/(1-\alpha)$ for $1 \leq j \leq M-1$. Show that

$$H(X) \leq -\alpha \log \alpha - (1-\alpha)\log(1-\alpha) + (1-\alpha)\log(M-1)$$

and determine the condition for equality.

**Exercise 7** (Huffman Codes)**.** The sequence of six independent realizations of source $X$ is encoded symbol-by-symbol using a binary Huffman code. The resulted string is $10110000101$. We know that the alphabet of $X$ has five elements and that its distribution is either $\{0.4, 0.3, 0.2, 0.05, 0.05\}$ or $\{0.3, 0.25, 0.2, 0.2, 0.05\}$. Which of them is the distribution of $X$?