Rationality & Recognisability

An introduction to weighted automata theory

Tutorial given at post-WATA 2014 Workshop

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Part I

The model of weighted automata

Part II

Rationality

Part III

Recognisability

Outline of Part III

- Representation and recognisable series.
  - KS Theorem
- The reachability space and the control morphism
  - The notion of action
- The observation morphism
  - The notion of quotient and the minimal automaton
  - The representation theorem
- The reduced representation
  - The exploration procedure
  - Decidability of equivalence for weighted automata

Recognisable series

$\mathbb{K}$ semiring A^* free monoid

Recognisable series

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

$\mathbb{K}$-representation

Q finite $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ $\qquad$ morphism

(I, μ, T) $\qquad$ $I \in \mathbb{K}^{1\times Q}$ $\qquad$ $\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ $\qquad$ $T \in \mathbb{K}^{Q\times 1}$

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(I, μ, T) realises (recognises) $s \in \mathbb{K}\langle\langle A^*\rangle\rangle$

$$\forall w \in A^* \qquad \langle s, w\rangle = I \cdot \mu(w) \cdot T$$

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$\mathbb{K}\mathrm{Rec}\, A^* \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ $\qquad$ submodule of recognisable series

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Example

$$I = \begin{pmatrix} 1 & 0 \end{pmatrix} , \quad \mu(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \mu(b) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(I, μ, T) realises $\displaystyle\sum_{w \in A^*} |w|_b \, w \quad \in \mathbb{K}\mathrm{Rec}\, A^*$

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$\mathbb{K}\mathrm{Rec}\, M \subseteq \mathbb{K}\langle\langle M\rangle\rangle$ submodule of recognisable series

The key lemma

$\mathbb{K}$ semiring $\qquad$ A^* free monoid

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$\mu\colon A^* \to \mathbb{K}^{Q\times Q}$ defined by $\{\mu(a)\}_{a\in A}$ arbitrary

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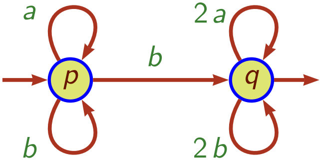
Lemma

$$\mu\colon A^* \to \mathbb{K}^{Q\times Q} \qquad X = \sum_{a\in A} \mu(a)\, a$$

$$\forall w \in A^* \qquad \langle X^*, w\rangle = \mu(w)$$

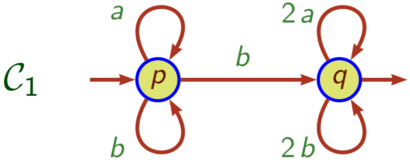
Automata are matrices

$\mathcal{C}_1$



$$\mathcal{C}_1 = \langle\, I_1, E_1, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} a+b & b \\ 0 & 2\,a + 2\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

Automata over free monoids are representations



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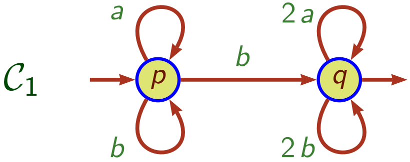

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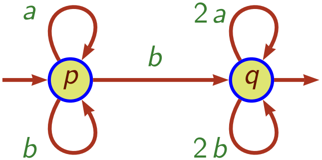

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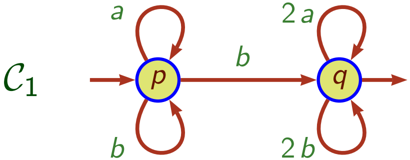
$$|\mathcal{C}_1| = I_1 \cdot {E_1}^* \cdot T_1 = \sum_{w \in A^*} (I_1 \cdot \mu_1(w) \cdot T_1)\, w$$

Automata over free monoids are representations



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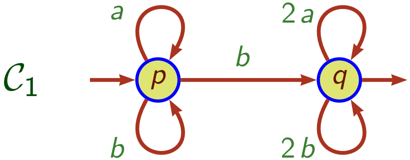
$$|\mathcal{C}_1| = I_1 \cdot {E_1}^* \cdot T_1 = \sum_{w \in A^*} (I_1 \cdot \mu_1(w) \cdot T_1)\, w \qquad\qquad |\mathcal{C}_1| \in \mathbb{K}\mathrm{Rec}\, A^*$$

Automata over free monoids are representations



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Conversely, representations are automata

The Kleene-Schützenberger Theorem

Fundamental Theorem of Finite Automata and *Key Lemma*

yield

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Theorem

A *finite* $\Rightarrow$ $\mathbb{K}\mathrm{Rec}\, A^* = \mathbb{K}\mathrm{Rat}\, A^*$

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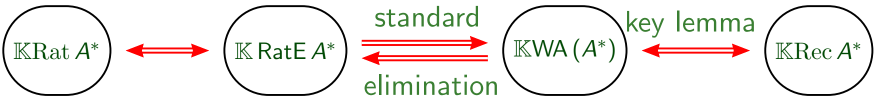

Action of a monoid on a set

The reachability set

$$\mathcal{A} = (\ I, \mu, T\ )$$

The reachability set

$$\mathcal{A} = (\, I, \mu, T \,)$$

Reachability set

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$$

$$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^{Q}$$

Reachability space

$$\langle \mathbf{R}_{\mathcal{A}} \rangle$$

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Reachability space

$$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^{Q} \qquad \langle \mathbf{R}_{\mathcal{A}} \rangle$$

A^* acts on $\mathbf{R}_{\mathcal{A}}$: $(I \cdot \mu(w)) \bullet a = (I \cdot \mu(w)) \cdot \mu(a) = I \cdot \mu(w\, a)$

The reachability set

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Reachability set | Reachability space

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A^* acts on $\mathbf{R}_{\mathcal{A}}$: $\quad (I \cdot \mu(w)) \bullet a = (I \cdot \mu(w)) \cdot \mu(a) = I \cdot \mu(w\,a)$

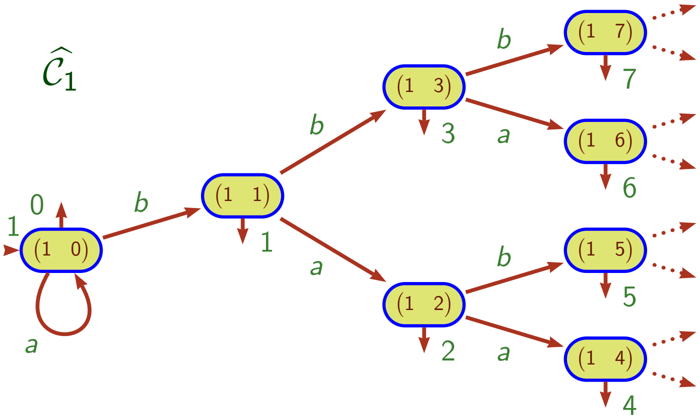
This action turns

$\mathbf{R}_{\mathcal{A}}$ into a deterministic automaton $\widehat{\mathcal{A}}$

(possibly infinite)

The reachability set

$$\mathcal{C}_1 = (\, I_1, \mu_1, T_1 \,)$$



The reachability set

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Reachability set

Reachability space

$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \}$ $\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q$ $\langle \mathbf{R}_{\mathcal{A}} \rangle$

$\mathbf{R}_{\mathcal{A}}$ is turned into a deterministic automaton $\widehat{\mathcal{A}}$

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$\mathbf{R}_{\mathcal{A}}$ is turned into a deterministic automaton $\widehat{\mathcal{A}}$

If $\mathbb{K} = \mathbb{B}$, $\widehat{\mathcal{A}}$ is the (classical) determinisation of $\mathcal{A}$

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If $\mathbb{K}$ is *locally finite*, $\mathbf{R}_{\mathcal{A}}$ and $\widehat{\mathcal{A}}$ are finite.

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$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \,|\, w \in A^* \}$

Reachability space

$\mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q$ $\langle \mathbf{R}_{\mathcal{A}} \rangle$

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Counting in a locally finite semiring is not really counting

The control morphism

$$\mathcal{A} = (\, I, \mu, T \,)$$

Reachability set

Reachability space

$$\mathbf{R}_{\mathcal{A}} = \{ I \cdot \mu(w) \mid w \in A^* \} \qquad \mathbf{R}_{\mathcal{A}} \subseteq \mathbb{K}^Q \qquad \langle \mathbf{R}_{\mathcal{A}} \rangle$$

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$$\Psi_{\mathcal{A}} \colon \mathbb{K}\langle A^* \rangle \longrightarrow \mathbb{K}^Q \qquad \forall w \in A^* \quad \Psi_{\mathcal{A}}(w) = I \cdot \mu(w)$$

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$$\begin{array}{c} \mathbb{K}\langle A^* \rangle \\ \Psi_{\mathcal{A}} \big\downarrow \\ \mathbb{K}^{Q} \end{array}$$

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$$\operatorname{Im} \Psi_\mathcal{A} = \Psi_\mathcal{A}(\mathbb{K}\langle A^* \rangle) = \langle \mathbf{R}_\mathcal{A} \rangle$$

$$\mathbb{K}\langle A^* \rangle \xrightarrow{\Psi_\mathcal{A}} \mathbb{K}^Q \qquad w \xrightarrow{\Psi_\mathcal{A}} x$$

The control morphism

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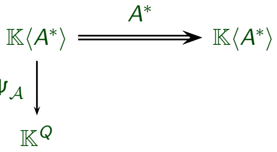

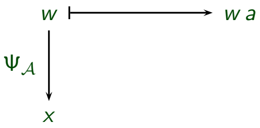

The control morphism

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$$\begin{array}{ccc} \mathbb{K}\langle A^* \rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle A^* \rangle \\ \Psi_{\mathcal{A}} \downarrow & & \downarrow \Psi_{\mathcal{A}} \\ \mathbb{K}^Q & \overset{A^*}{\Longrightarrow} & \mathbb{K}^Q \end{array} \qquad \begin{array}{ccc} w & \longmapsto & w\,a \\ \Psi_{\mathcal{A}} \downarrow & & \downarrow \Psi_{\mathcal{A}} \\ x & \longmapsto & x \cdot \mu(a) \end{array}$$

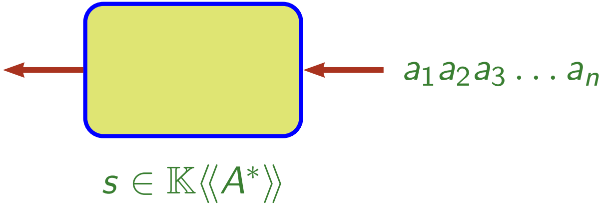
The control morphism is a morphism of actions

Quotient of series

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

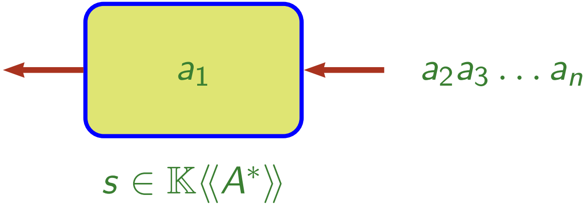
$$u \in A^* \qquad u^{-1} s = \sum_{w \in A^*} \langle s, u\, w \rangle\, w$$

Quotient of series



The input belongs to a free monoid A^*

Quotient of series



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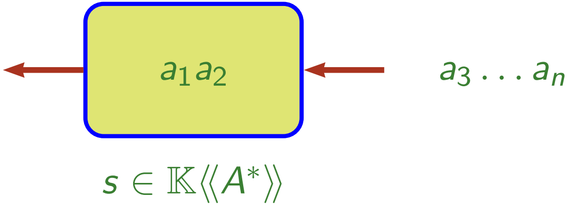

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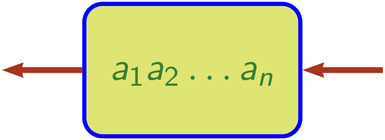

$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

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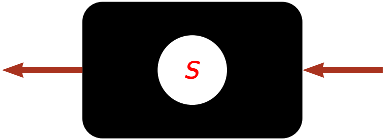
Quotient of series



$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$

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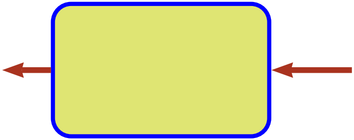
Quotient of series

$\langle s, a_1 \dots a_n \rangle = k$



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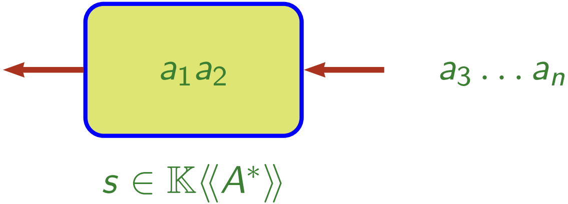
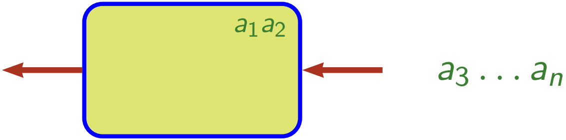


Quotient of series



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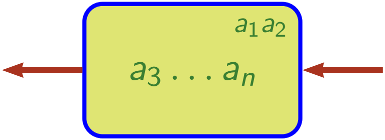

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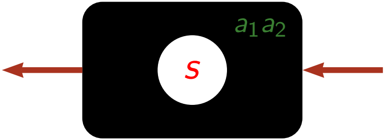

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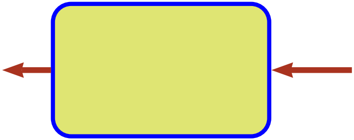

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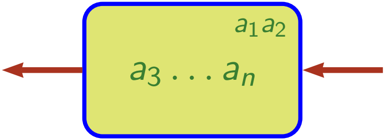
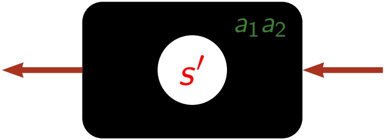


$$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

Quotient of series



$$s' \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

Quotient of series

$$k = \langle s', a_3 \dots a_n \rangle = \langle s, a_1 a_2 a_3 \dots a_n \rangle$$

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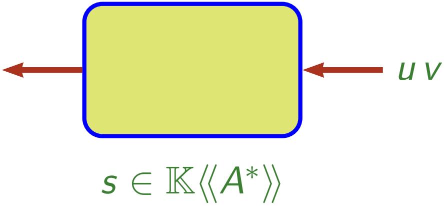
$$s' = [a_1 a_2]^{-1} s$$

The series s' is *the quotient* of s by $a_1 a_2$

Quotient of series



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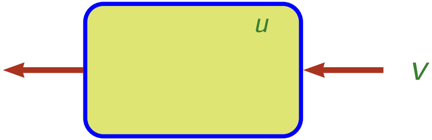

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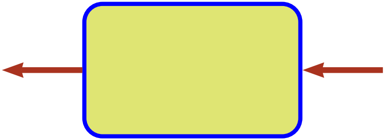
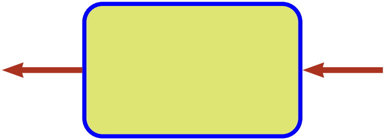


$$k = \langle s', v \rangle = \langle s, u\, v \rangle$$

Quotient of series



$$k = \langle s', v \rangle = \langle s, u\, v \rangle$$

$$s' = u^{-1} s$$

The series s' is *the quotient* of s by u

Quotient of series

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

$$u \in A^* \qquad u^{-1} s = \sum_{w \in A^*} \langle s, u\,w \rangle\, w$$

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$$u^{-1}(s + t) = u^{-1} s + u^{-1} t \qquad u^{-1}(k\, s) = k\,(u^{-1} s)$$

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Quotient is a (right) action of A^* on $\mathbb{K}\langle\langle A^* \rangle\rangle$

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Quotient is a (right) action of A^* on $\mathbb{K}\langle\langle A^* \rangle\rangle$

$$(u\,v)^{-1} s = v^{-1} (u^{-1} s)$$

The minimal automaton

$$s \in \mathbb{K}\langle\langle A^* \rangle\rangle$$

$$\mathbf{R}_s = \left\{ u^{-1} s \,\middle|\, u \in A^* \right\}$$

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$$\mathbf{R}_s = \left\{ u^{-1} s \,\middle|\, u \in A^* \right\}$$

Quotient turns

$\mathbf{R}_s$ into the minimal automaton $\mathcal{A}_s$ of s

(possibly infinite)

The observation morphism

$$\mathcal{A} = (\, I, \mu, T \,)$$

$$\Phi_{\mathcal{A}} \colon \mathbb{K}^{Q} \longrightarrow \mathbb{K}\langle\!\langle A^* \rangle\!\rangle \qquad \Phi_{\mathcal{A}}(x) = \left|(\, x, \mu, T \,)\right| = \sum_{w \in A^*} (x \cdot \mu(w) \cdot T)\, w$$

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$$\begin{array}{c} \mathbb{K}^{Q} \\ \Phi_{\mathcal{A}} \big\downarrow \\ \mathbb{K}\langle\langle A^* \rangle\rangle \end{array} \qquad \begin{array}{c} x \\ \Phi_{\mathcal{A}} \big\downarrow \\ t \end{array}$$

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$$\begin{array}{ccc} \mathbb{K}^{Q} & \xRightarrow{\;A^*\;} & \mathbb{K}^{Q} \\ \downarrow{\scriptstyle \Phi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{A}}} \\ \mathbb{K}\langle\langle A^* \rangle\rangle & \xRightarrow{\;A^*\;} & \mathbb{K}\langle\langle A^* \rangle\rangle \end{array} \qquad\qquad \begin{array}{ccc} x & \longmapsto & x \cdot \mu(a) \\ \downarrow{\scriptstyle \Phi_{\mathcal{A}}} & & \downarrow{\scriptstyle \Phi_{\mathcal{A}}} \\ t & \longmapsto & a^{-1}t \end{array}$$

The observation morphism is a morphism of actions

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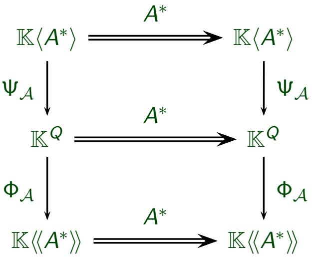

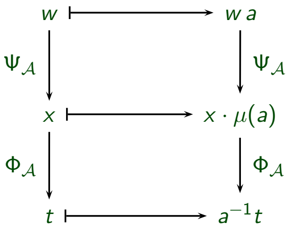

The observation morphism is a morphism of actions

The representation theorem

$U \subseteq \mathbb{K}\langle\langle A^* \rangle\rangle$ submodule U stable (by quotient)

Theorem (Fliess 71, Jacob 74)

$s \in \mathbb{K}\mathrm{Rec}\, A^* \iff \exists U$ *stable* *finitely generated* $s \in U$

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$$\begin{array}{ccc}
\mathbb{K}\langle A^* \rangle & \xrightarrow{\;A^*\;} & \mathbb{K}\langle A^* \rangle \\
\downarrow \Psi_{\mathcal{A}} & & \downarrow \Psi_{\mathcal{A}} \\
\mathbb{K}^Q & \xrightarrow{\;A^*\;} & \mathbb{K}^Q \\
\downarrow \Phi_{\mathcal{A}} & & \downarrow \Phi_{\mathcal{A}} \\
\mathbb{K}\langle\langle A^* \rangle\rangle & \xrightarrow{\;A^*\;} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}$$

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$$
\begin{array}{lccc}
1_{A^*} \in & \mathbb{K}\langle A^* \rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle A^* \rangle \\
 & \downarrow \Psi_{\mathcal{A}} & & \downarrow \Psi_{\mathcal{A}} \\
I \in \mathrm{Im}\, \Psi_{\mathcal{A}} & \mathbb{K}^Q & \overset{A^*}{\Longrightarrow} & \mathbb{K}^Q \\
 & \downarrow \Phi_{\mathcal{A}} & & \downarrow \Phi_{\mathcal{A}} \\
s \in \Phi_{\mathcal{A}}(\mathrm{Im}\, \Psi_{\mathcal{A}}) & \mathbb{K}\langle\langle A^* \rangle\rangle & \overset{A^*}{\Longrightarrow} & \mathbb{K}\langle\langle A^* \rangle\rangle
\end{array}
$$

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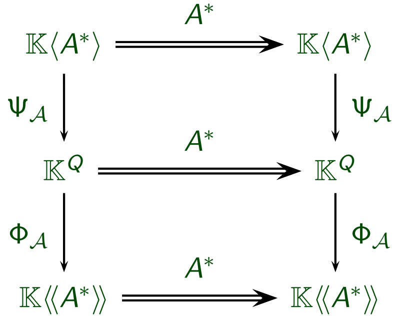

The representability theorem for recognisable series

Proposition

$\mathcal{A} = \langle\, I, \mu, T \,\rangle$ *dimension* Q $\qquad s = |\mathcal{A}|$

$\langle \mathbf{R}_{\mathcal{A}} \rangle$ *generated by* $G \subset \mathbb{K}^Q$

$\exists\ \mathcal{A}_G$ *of dimension* G $\qquad s = |\mathcal{A}_G|$ $\qquad \mathcal{A} \stackrel{M_G}{\Longleftarrow} \mathcal{A}_G$

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$\mathbb{K}\langle A^* \rangle$ $\xrightarrow{\Psi_{\mathcal{A}}}$ $\mathbb{K}^Q$

P $\xrightarrow{\Psi_{\mathcal{A}}}$ $\Psi_{\mathcal{A}}(P)$

$\operatorname{Im} \Psi_{\mathcal{A}}$ $\mathbb{K}^Q$ $\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im} \Psi_{\mathcal{A}}$

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$\begin{array}{cccc} & \mathbb{K}\langle A^*\rangle & & P \\ & \Big\downarrow \Psi_{\mathcal{A}} & & \Big\downarrow \Psi_{\mathcal{A}} \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^{Q} & & \Psi_{\mathcal{A}}(P) \end{array}$$

generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Halting criterium

The exploration procedure

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$\operatorname{Im}\Psi_{\mathcal{A}}$

$\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite finite $\operatorname{Im}\Psi_{\mathcal{A}}$

The exploration procedure

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$$\mathbb{K}\langle A^* \rangle \xrightarrow{\Psi_{\mathcal{A}}} \mathbb{K}^Q \qquad P \xrightarrow{\Psi_{\mathcal{A}}} \Psi_{\mathcal{A}}(P)$$

$\operatorname{Im}\Psi_{\mathcal{A}}$ $\mathbb{K}^Q$ $\Psi_{\mathcal{A}}(P)$ generating set of $\operatorname{Im}\Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite — finite $\operatorname{Im}\Psi_{\mathcal{A}}$
- $\mathbb{F}$ field — finite dimension

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

$$\begin{array}{cccc} & \mathbb{K}\langle A^* \rangle & P & \\ & \Psi_{\mathcal{A}} \big\downarrow & \Psi_{\mathcal{A}} \big\downarrow & \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^Q & \langle \Psi_{\mathcal{A}}(P) \rangle & \text{generating set of } \operatorname{Im}\Psi_{\mathcal{A}} \end{array}$$

Halting criterium

- $\mathbb{B}$ finite — finite $\operatorname{Im}\Psi_{\mathcal{A}}$
- $\mathbb{F}$ field — finite dimension
- $\mathbb{Z}$ ED — Noetherian

The exploration procedure

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$$\mathbb{K}\langle A^* \rangle \xrightarrow{\Psi_{\mathcal{A}}} \mathbb{K}^Q \qquad P \xrightarrow{\Psi_{\mathcal{A}}} \langle \Psi_{\mathcal{A}}(P) \rangle$$

$\operatorname{Im} \Psi_{\mathcal{A}}$ $\mathbb{K}^Q$ $\langle \Psi_{\mathcal{A}}(P) \rangle$ generating set of $\operatorname{Im} \Psi_{\mathcal{A}}$

Halting criterium

- $\mathbb{B}$ finite — finite $\operatorname{Im} \Psi_{\mathcal{A}}$
- $\mathbb{F}$ field — finite dimension
- $\mathbb{Z}$ ED — Noetherian
- $\mathbb{N}$ — well partial ordered set

The exploration procedure

$\mathbb{K}$-automaton $\mathcal{A} = \langle\, I, \mu, T \,\rangle$ Search for $P \subseteq A^*$

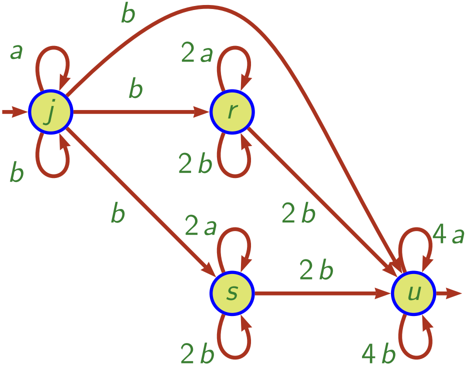
$$\begin{array}{ccccc} & \mathbb{K}\langle A^* \rangle & & P & \\ & \Big\downarrow {\scriptstyle \Psi_{\mathcal{A}}} & & \Big\downarrow {\scriptstyle \Psi_{\mathcal{A}}} & \\ \operatorname{Im}\Psi_{\mathcal{A}} & \mathbb{K}^Q & & \langle \Psi_{\mathcal{A}}(P) \rangle & \text{generating set of } \operatorname{Im}\Psi_{\mathcal{A}} \end{array}$$

Result

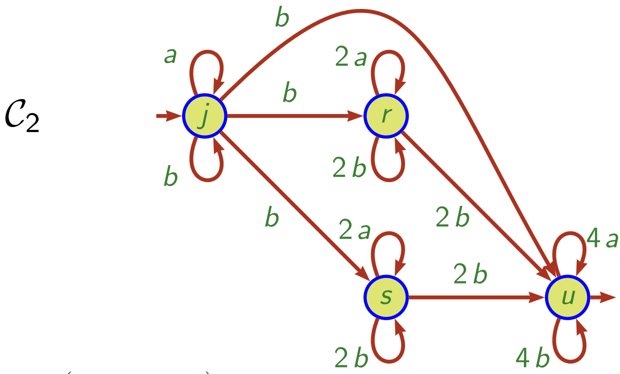
$$\mathcal{A} \overset{M_P}{\Longleftarrow} \mathcal{C}$$

Computation of an example

$\mathcal{C}_2$



Computation of an example



$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\mu(a) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \mu(b) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Reduced representation

$$\mathcal{A} = (\, I, \mu, T\,)$$

$\mathcal{A}$ is *reduced* if its *dimension* is minimal

(among all equivalent representations)

We suppose now that $\mathbb{K}$ is a (skew) field

Proposition

$\mathcal{A}$ is reduced iff $\Psi_{\mathcal{A}}$ is surjective and $\Phi_{\mathcal{A}}$ injective

Theorem

A reduced representation of $|\mathcal{A}|$ is effectively computable
(with cubic complexity)

Corollary

Equivalence of $\mathbb{K}$-recognisable series is decidable

Equivalence of weighted automata

Equivalence of weighted automata with weights in

the Boolean semiring $\mathbb{B}$	decidable
a subsemiring of a field	decidable
$(\mathbb{Z}, \min, +)$	undecidable
$\mathrm{Rat}\, B^*$	undecidable
$\mathbb{N}\mathrm{Rat}\, B^*$	decidable

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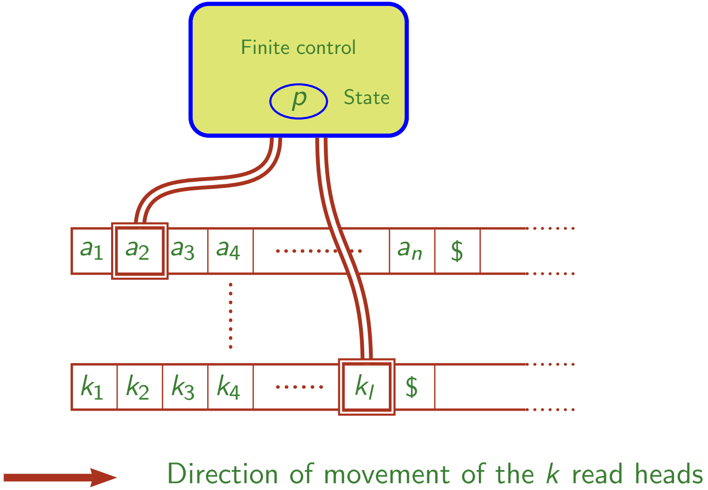
Equivalence of

transducers	undecidable
transducers with multiplicity in $\mathbb{N}$	decidable
functional transducers	decidable
finitely ambiguous $(\mathbb{Z}, \min, +)$	decidable

The 1W kT Turing machine



The 1-way k-tape Turing Machine (1W kT TM)