The validity of weighted automata

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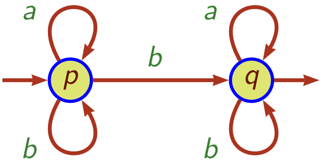
The automaton model

$\mathcal{B}_1$



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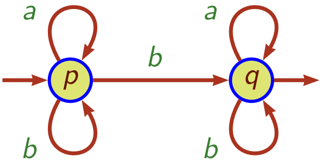
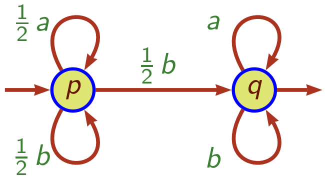


$L(\mathcal{B}_1) = A^* b A^*$

$$\longrightarrow p \xrightarrow{b} p \xrightarrow{a} p \xrightarrow{b} q \longrightarrow$$

The weighted automaton model

$\mathcal{C}_1$



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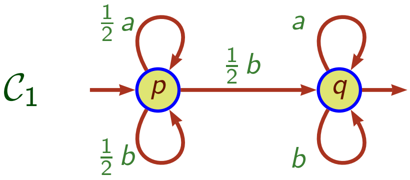

$$\xrightarrow{1} p \xrightarrow{\frac{1}{2}\,b} p \xrightarrow{\frac{1}{2}\,a} p \xrightarrow{\frac{1}{2}\,b} q \xrightarrow{1}$$

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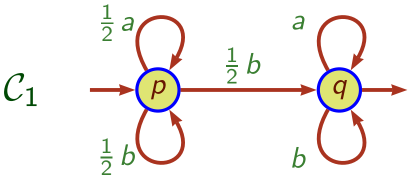

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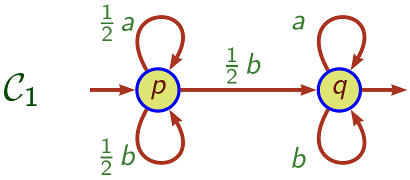
- Weight of a path c: *product* of the weights of transitions in c
- Weight of a word w: *sum* of the weights of paths with label w

$$b\,a\,b \quad \longmapsto \quad \frac{1}{2} + \frac{1}{8} = \frac{5}{8}$$

The weighted automaton model



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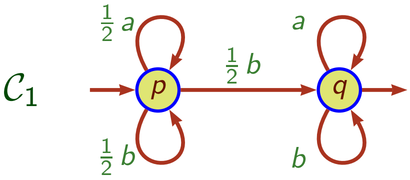

$|\mathcal{C}_1| \in \mathbb{Q}\langle\langle A^* \rangle\rangle$

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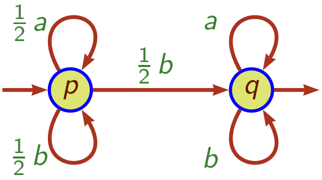
$$|\mathcal{C}_1| \colon A^* \longrightarrow \mathbb{Q}$$

The weighted automaton model

$\mathcal{C}_1$



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- Weight of a path c: *product* of the weights of transitions in c
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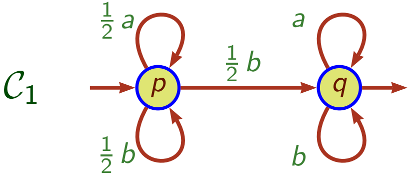
$$|\mathcal{C}_1| = \frac{1}{2}\, b + \frac{1}{4}\, a\, b + \frac{1}{2}\, b\, a + \frac{3}{4}\, b\, b + \frac{1}{8}\, a\, a\, b + \frac{1}{4}\, a\, b\, a + \frac{3}{8}\, a\, b\, b + \frac{1}{2}\, b\, a\, a + \ldots$$

The weighted automaton model



$$\mathcal{C}_1 = \langle\, I_1, \underline{E_1}, T_1 \,\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\,a + \frac{1}{2}\,b & \frac{1}{2}\,b \\ 0 & a+b \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle$$

The weighted automaton model

$$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle \qquad \underline{E} = \text{incidence matrix}$$

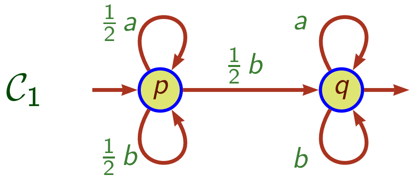
$$\underline{E}_{p,q} = \sum \{\mathbf{wl}(e) \,|\, e \quad \text{transition from } p \text{ to } q\}$$

$$\underline{E}^{n}_{p,q} = \sum \{\mathbf{wl}(c) \,|\, c \quad \text{computation from } p \text{ to } q \text{ of length } n\}$$

$$\underline{E}^{*} = \sum_{n \in \mathbb{N}} \underline{E}^{n}$$

$$\underline{E}^{*}_{p,q} = \sum \{\mathbf{wl}(c) \,|\, c \quad \text{computation from } p \text{ to } q\}$$

The weighted automaton model



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$$\left|\mathcal{C}_1\right| = I_1 \cdot \underline{E_1}^{\,*} \cdot T_1$$

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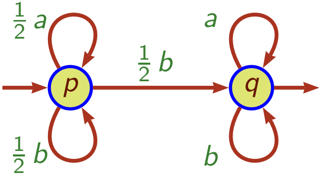

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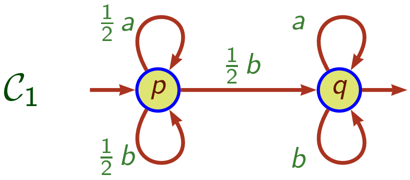
Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$

whose coefficients are effectively computable

The weighted automaton model



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Every $\mathbb{K}$-automaton defines a series in $\mathbb{K}\langle\langle A^* \rangle\rangle$ whose coefficients are effectively computable

Where is the problem ?

The weighted automaton model

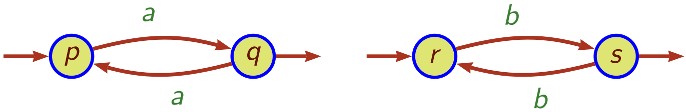
The problem comes from the fact that

human beings cannot stay

with a model that works

Pascal

The need for a richer model: eg, the concatenation product



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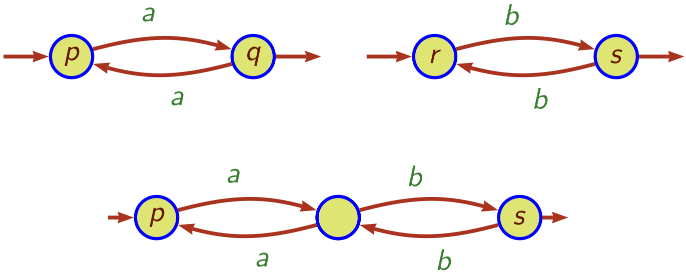

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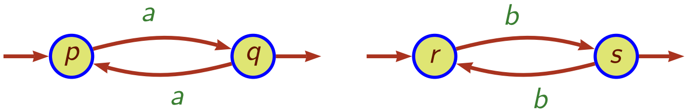

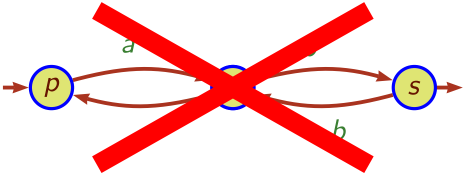

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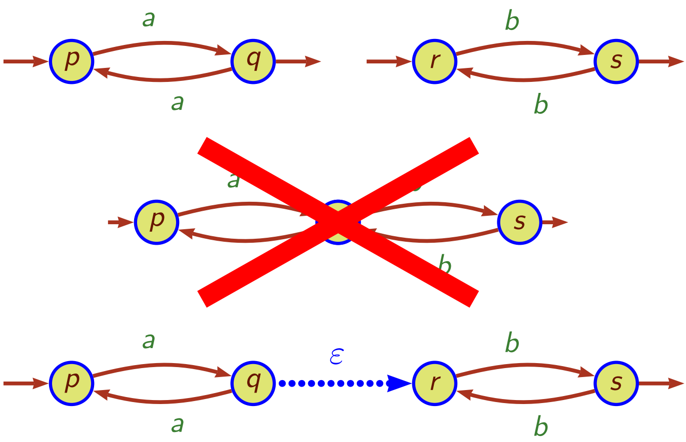

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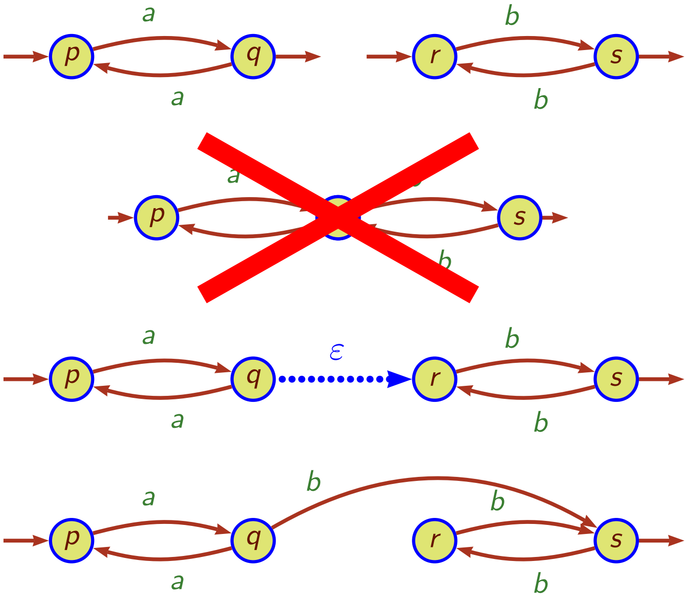

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

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Usefulness of ε-transitions:

Preliminary step for many constructions on NFA's:

- Product and star of position automata
- Thompson construction
- Construction of the universal automaton
- Computation of the image of a transducer
- ...

May correspond to the *structure* of the computations

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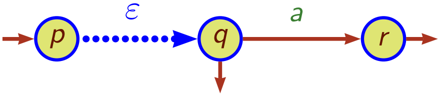
May correspond to the *structure* of the computations

Removal of ε-transitions is implemented in all automata software

A basic result in (classical) automata theory



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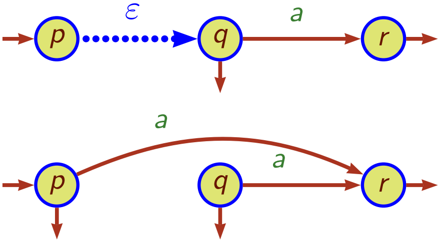

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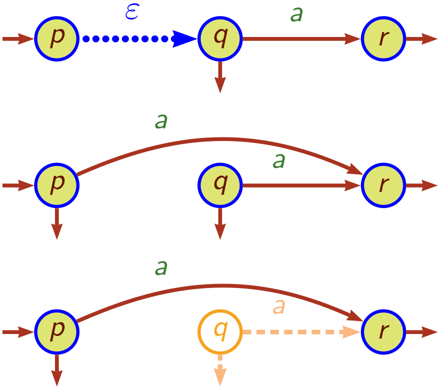

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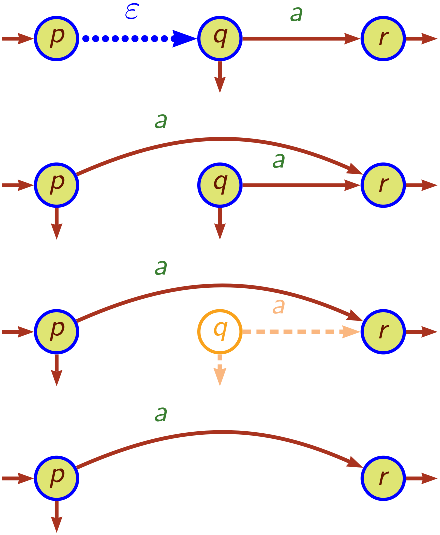

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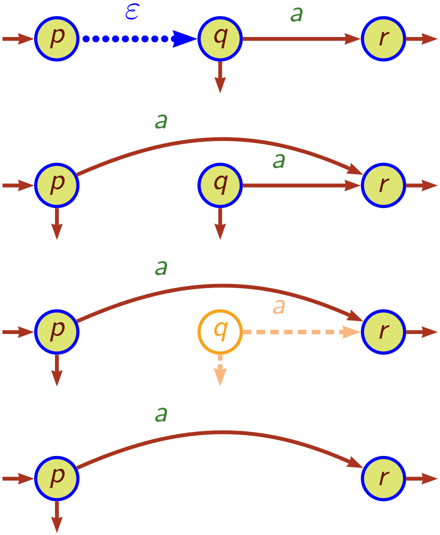

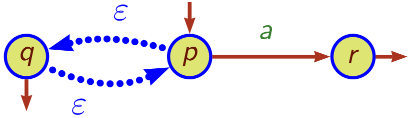

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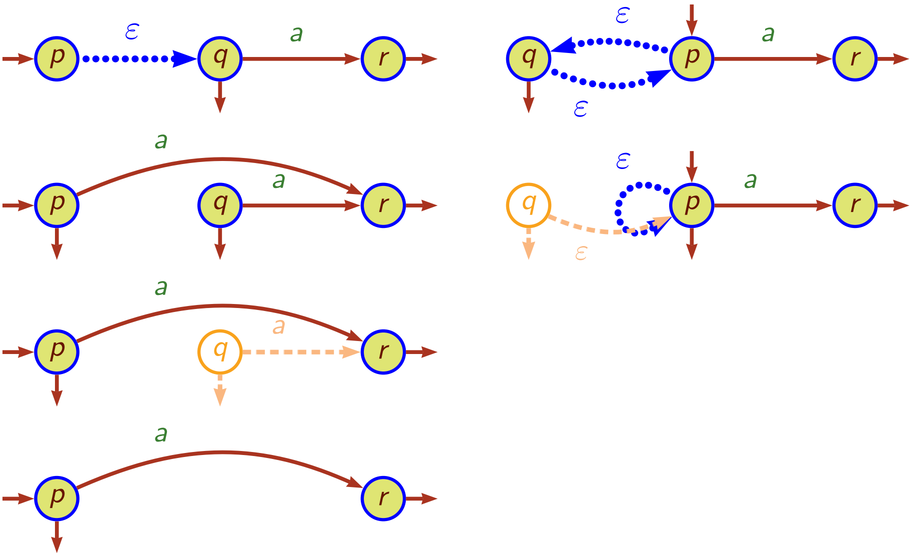

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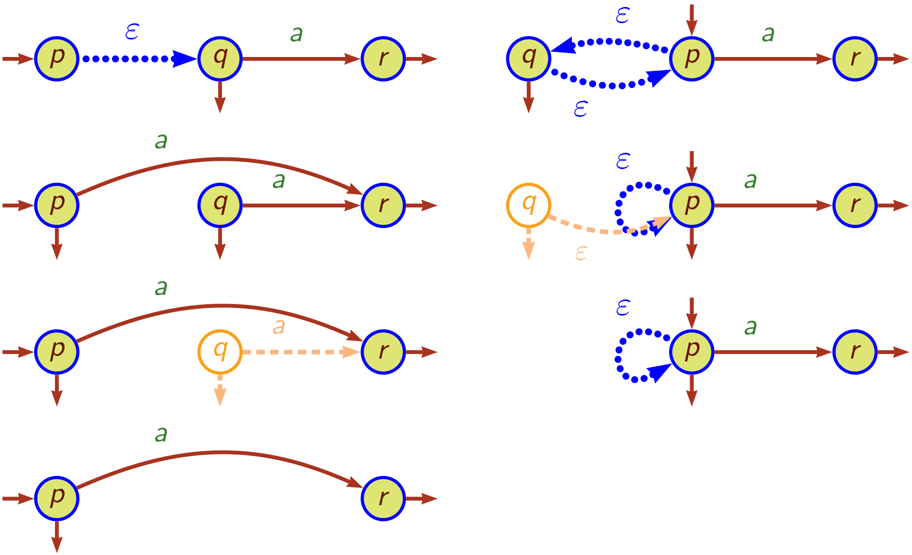

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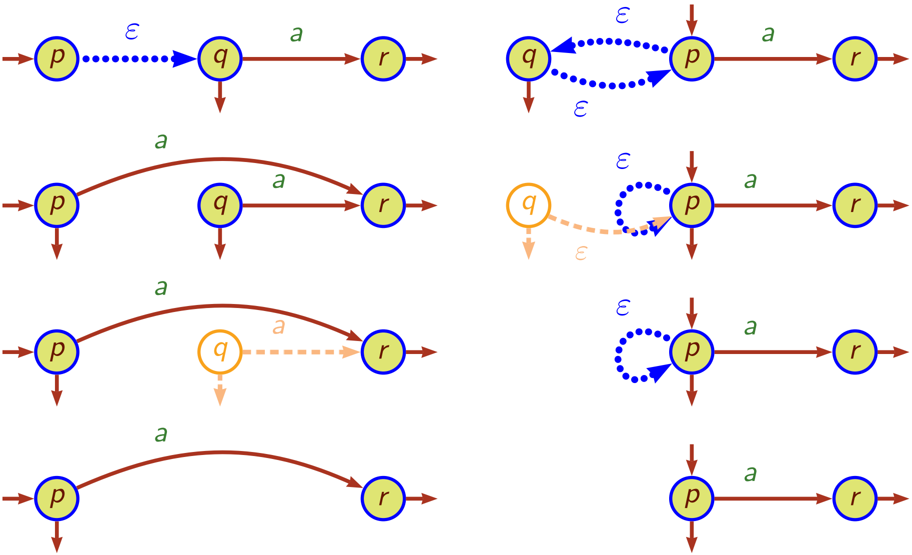

A basic result in (classical) automata theory

Theorem

Every ε-NFA is equivalent to an NFA

A proof

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ $\qquad$ $\underline{E}$ transition matrix of $\mathcal{A}$

Entries of $\underline{E}$ = subsets of $A \cup \{\varepsilon\}$

$$L(\mathcal{A}) = I \cdot \underline{E}^* \cdot T$$

$$\underline{E} = \underline{E}_0 + \underline{E}_{\mathsf{p}}$$

$$L(\mathcal{A}) = I \cdot (\underline{E}_0 + \underline{E}_{\mathsf{p}})^* \cdot T = I \cdot (\underline{E}_0^* \cdot \underline{E}_{\mathsf{p}})^* \cdot \underline{E}_0^* \cdot T$$

$\mathcal{A} = \langle\, I, \underline{E}, T \,\rangle$ equivalent to $\mathcal{B} = \left\langle\, I, \underline{E}_0^* \cdot \underline{E}_{\mathsf{p}}, \underline{E}_0^* \cdot T \,\right\rangle$

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One *proof* = several *algorithms* for *computing* $\underline{E}_0^*$ or $\underline{E}_0^* \cdot \underline{E}_\mathrm{p}$

A basic question in weighted automata theory

Question

Is every ε-WFA is equivalent to a WFA?

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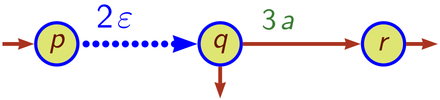

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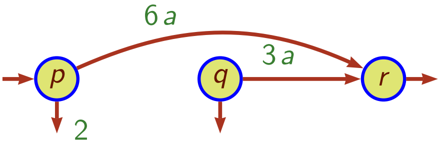

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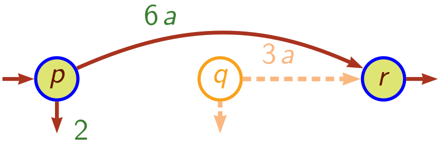

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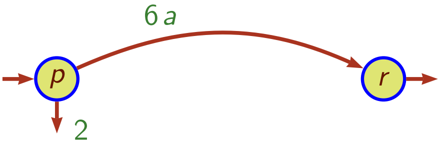

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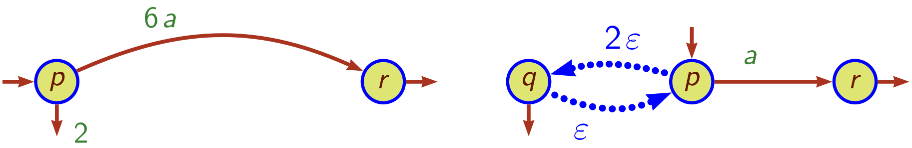

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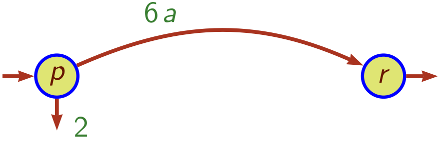
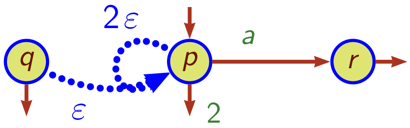
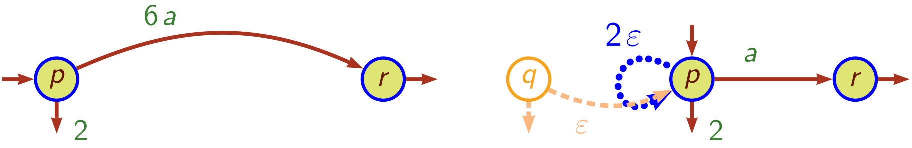




A basic question in weighted automata theory

Question

Is every ε*-WFA is equivalent to a WFA?*



A basic question in weighted automata theory

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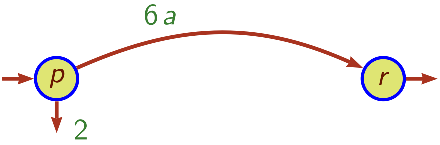

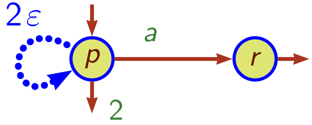

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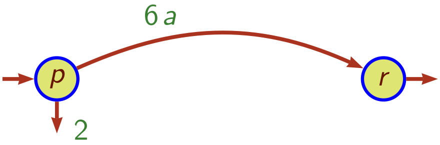

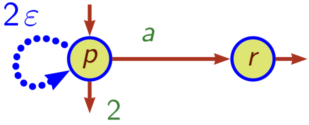

$\xrightarrow{1} q \xrightarrow{a} r \xrightarrow{1}$,

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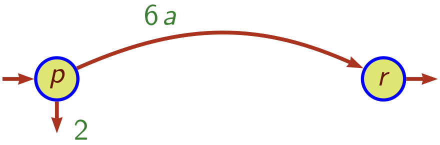

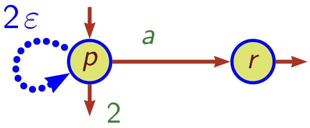

$$\xrightarrow{1} q \xrightarrow{a} r \xrightarrow{1} \quad , \qquad \xrightarrow{1} q \xrightarrow{2\,\varepsilon} q \xrightarrow{a} r \xrightarrow{1} \quad ,$$

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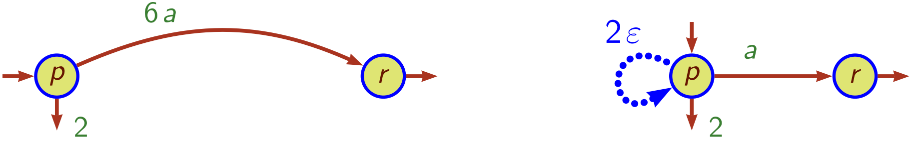

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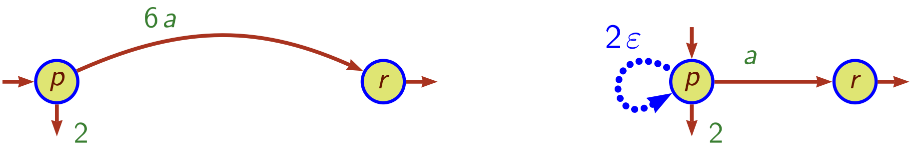

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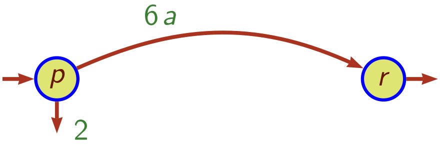
$$\xrightarrow{1} q \xrightarrow{2\varepsilon} q \xrightarrow{2\varepsilon} q \xrightarrow{a} r \xrightarrow{1} \quad , \quad \ldots$$

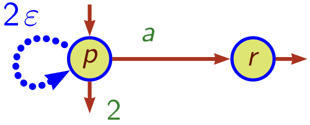
$$a \quad \longmapsto \quad 1 + 2 + 4 + \cdots$$

A basic question in weighted automata theory

Question

Is every ε-WFA is equivalent to a WFA?





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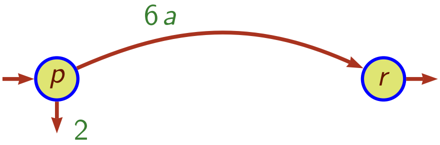

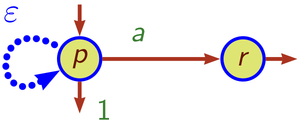

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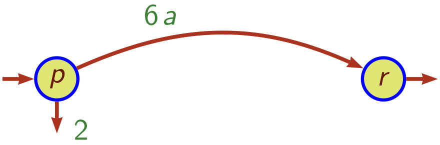

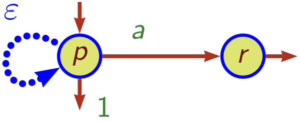

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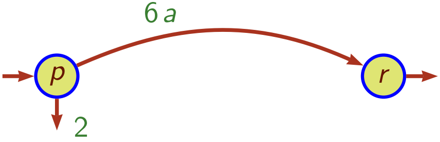

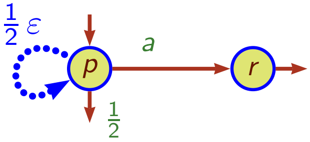

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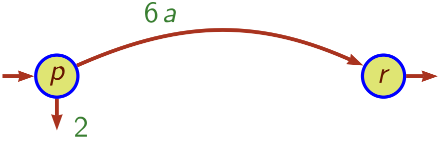

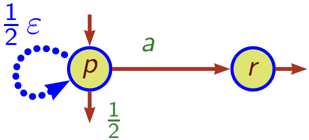

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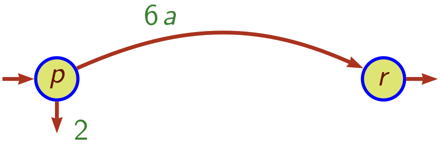

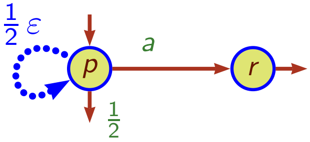

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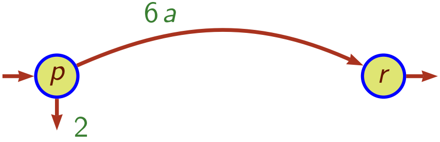

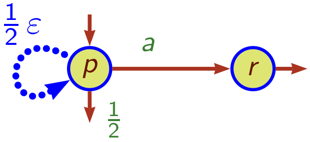

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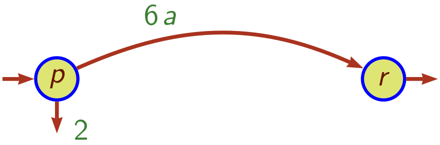

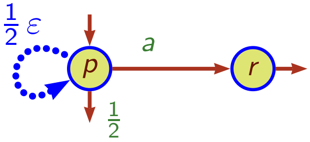

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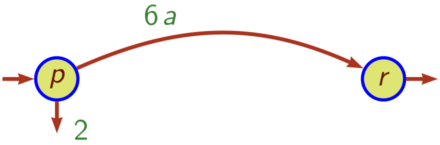

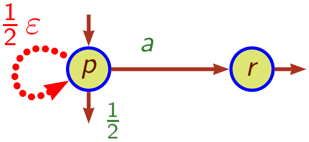

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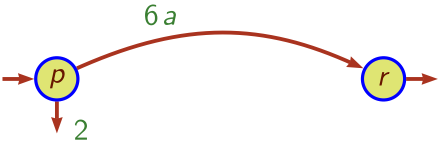

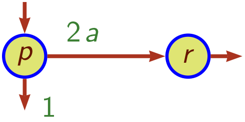

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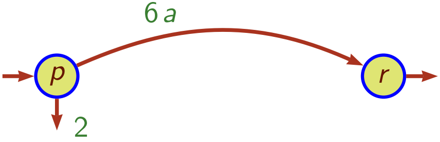

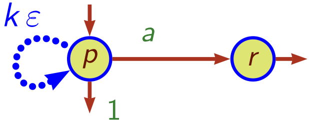

A basic question in weighted automata theory

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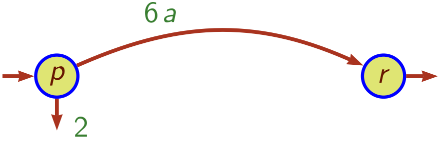

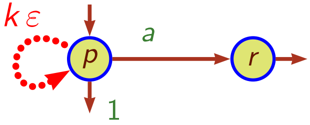

A basic question in weighted automata theory

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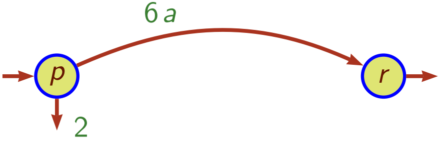

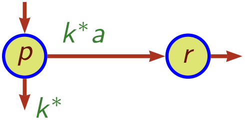

A basic question in weighted automata theory

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certainly not !

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Which ε-WFAs have a *well-defined* behaviour?

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How to compute the behaviour of an ε-WFA (when it is *well-defined*)?

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How to decide if the behaviour of an ε-WFA is *well-defined*?

Behaviour of weighted automata

$\mathcal{A} = \langle\, \mathbb{K}, Q, A, E, I, T \,\rangle$ possibly with ε-transitions

$u \in A^*$ paths labelled by u in $\mathcal{A}$ possibly infinitely many

$\langle |\mathcal{A}|, u \rangle$ sum of weights of computations labelled by u in $\mathcal{A}$

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Legitimate, as far as the behaviours of the automata are concerned
(Kuich–Salomaa 86, Berstel–Reutenauer 84-88; 11)

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$\mathcal{A}$ not acyclic $\Rightarrow$ weight of u in $\mathcal{A}$ may be an infinite sum.

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First point of view (algebraico-logic)

- Definition of a *new operator for infinite sums* $\sum_I$
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Works of Bloom, Ésik, Kuich (90's –)

based on the axiomatisation described by Conway (72)

Behaviour of weighted automata

Second point of view (more analytical)

Infinite sums are given a meaning via a topology on $\mathbb{K}$

Topology allows to define summable families in $\mathbb{K}$

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$\{t_i\}_{i \in I}$ *summable* of sum t :

$$\forall V \in \mathrm{N}(t)\,,\ \exists J_V \text{ finite}\,,\ J_V \subset I\,,\ \forall L \text{ finite}\,,\ J_V \subseteq L \subset I \quad \sum_{i \in L} t_i \in V\,.$$

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Lemma (Associativity)

$\{t_i\}_{i \in I}$ summable of sum t ,
$I = \bigcup_{j \in J} K_j \quad \forall j \in J \quad \{t_i\}_{i \in K_j}$ summable of sum s_j ,
then $\{s_j\}_{j \in J}$ summable of sum t

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This solution (Lombardy, S. 03 –)

- Yields a consistent theory

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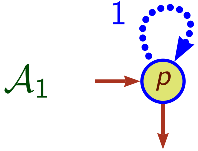
This solution (Lombardy, S. 03 –)

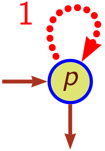
- Yields a consistent theory
- Two pitfalls for effectivity
  - *effective computation* of a summable family may not be possible
  - *effective computation* may give values to non summable families

Problems in computing the behaviour of a weighted automaton



Problems in computing the behaviour of a weighted automaton

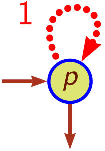
$\mathcal{A}_1$



$(1)^* =$ undefined

$\mathbb{N}$ natural integers $|\mathcal{A}_1|$ not defined

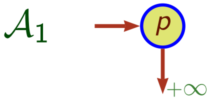
Problems in computing the behaviour of a weighted automaton



$\mathcal{A}_1$ $\qquad (1)^* = +\infty$

$\mathbb{N}$		natural integers	$\lvert\mathcal{A}_1\rvert$	not defined
$\mathcal{N}$	$\mathbb{N} \cup +\infty$	compact topology	$\lvert\mathcal{A}_1\rvert$	defined

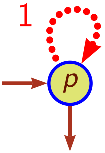
Problems in computing the behaviour of a weighted automaton



$\mathbb{N}$		natural integers	$\lvert\mathcal{A}_1\rvert$	not defined
$\mathcal{N}$	$\mathbb{N} \cup +\infty$	compact topology	$\lvert\mathcal{A}_1\rvert$	defined

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_1$



$(1)^* = $ undefined

$\mathbb{N}$		natural integers	$\lvert\mathcal{A}_1\rvert$	not defined
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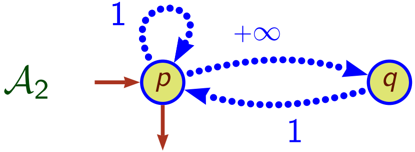

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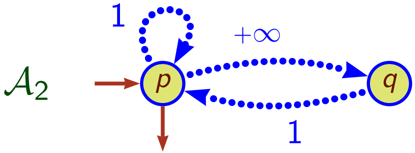

$\mathcal{N}$ $\mathbb{N} \cup +\infty$ compact topology $|\mathcal{A}_1|$ defined

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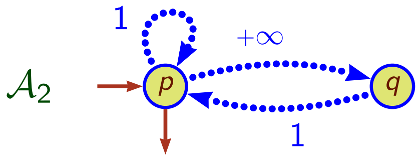

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Problems in computing the behaviour of a weighted automaton

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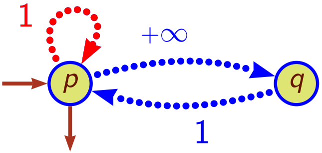

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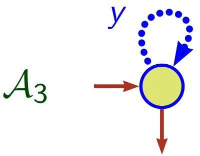


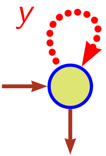
$$\mathbb{S} \subset \mathbb{N}^{2\times 2}, \quad x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1_{\mathbb{S}}, \quad y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad x+y = \infty_{\mathbb{S}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$\mathbb{S}$ equipped with the discrete topology

$0_{\mathbb{S}}$, y, and $\infty_{\mathbb{S}}$ starable $\qquad x = y^2 \qquad x$ not starable

Problems in computing the behaviour of a weighted automaton

$\mathcal{A}_3$



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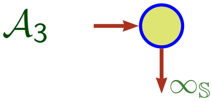

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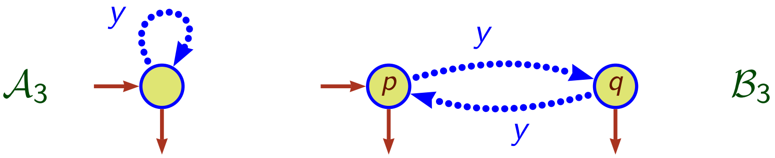

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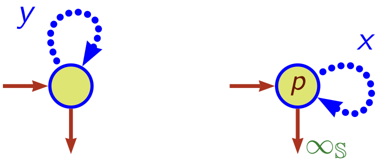

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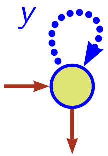

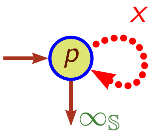

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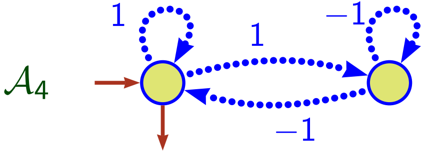

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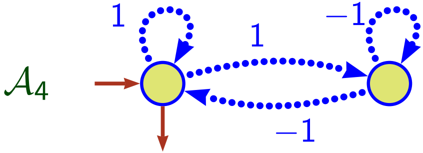

$$\mathcal{A}_4 = \left\langle\, I_4, \underline{E_4}, T_4 \,\right\rangle = \left\langle \begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

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$$\underline{E_4}^2 = 0 \quad \Longrightarrow \quad \underline{E_4}^* = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow \quad |\mathcal{A}_4| = 2$$

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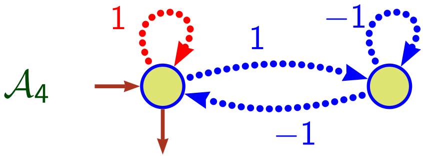

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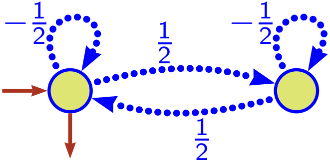
Problems in computing the behaviour

$\mathcal{A}_5$



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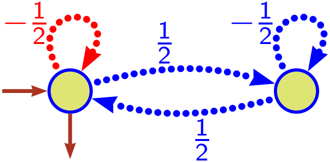

$$\left(-\frac{1}{2}\right)^* = \frac{2}{3}$$

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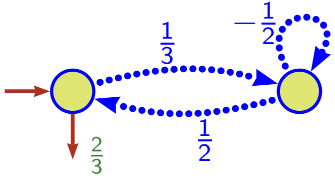

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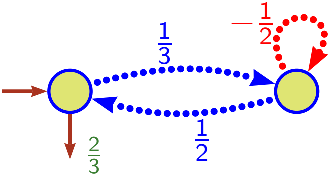

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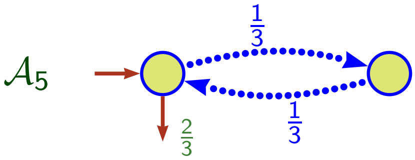

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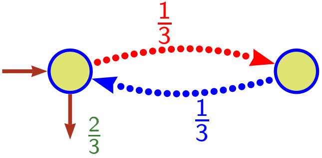

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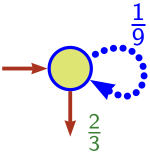

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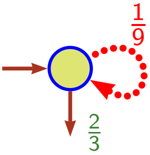

$$\left(\frac{1}{9}\right)^* = \frac{9}{8}$$

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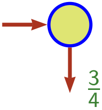

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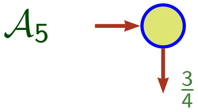

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A chicken and egg problem

automaton	**algorithm**
$\mathcal{A}$	A

A chicken and egg problem

automaton $\mathcal{A}$

algorithm A

valid ?

success ?

A chicken and egg problem

automaton	algorithm
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valid	

A chicken and egg problem

automaton		algorithm
$\mathcal{A}$		A
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automaton		algorithm
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A chicken and egg problem

automaton		algorithm
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Validity of weighted automata

$\mathcal{A} = \langle \mathbb{K}, Q, A, E, I, T \rangle$ possibly with ε-transitions

E^* *free monoid* generated by E

$\mathsf{P}_{\mathcal{A}}$ *set of paths* in $\mathcal{A}$ (local) rational subset of E^*

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$\mathcal{A}$ is valid iff
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Validity implies well-definition of behaviour

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Remark

If every subfamily of a summable family in $\mathbb{K}$ *is summable,*
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Eg. $\mathbb{R}$, $\mathbb{C}$ *(and* $\mathbb{N}$, $\mathbb{Z}$, $\mathcal{N}$ *).*

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Nota Bene

We do not know yet how to decide whether
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$\mathcal{A}$ is valid iff the behaviour of every covering of $\mathcal{A}$ is well-defined

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Deciding validity

Straightforward cases

- Non starable semirings (eg. $\mathbb{N}$, $\mathbb{Z}$)

  $\mathcal{A}$ valid $\iff$ $\mathcal{A}$ acyclic
- Complete topological semirings (eg. $\mathcal{N}$) every $\mathcal{A}$ valid
- Rationally additive semirings (eg. $\mathrm{Rat}\, A^*$) every $\mathcal{A}$ valid
- Locally closed commutative semirings every $\mathcal{A}$ valid

Deciding validity

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$\mathbb{K}$ topological, ordered, positive, star-domain downward closed

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A $\mathbb{K}$-automaton is valid if, and only if,

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Hidden parts

- The removal algorithm itself:
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues

- Automata and expressions validity

- 'Infinitary' axioms : *strong*, *star-strong* semirings

- Links with the 'axiomatic' approach (Bloom–Ésik–Kuich)

- References to previous work (on removal algorithm):
  - *locally closed* srgs (Ésik–Kuich), *k-closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)

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- On 'usual' semirings,
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All's well, that ends well!

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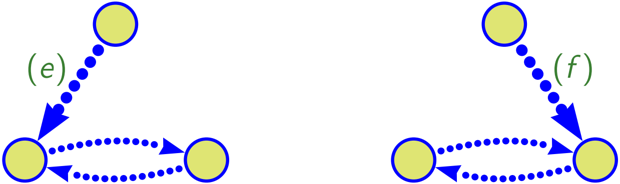
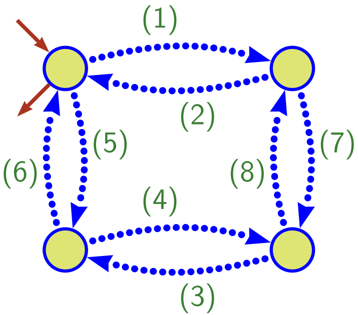
Termination issues



weighted ε-removal procedure does not terminate

if newly created ε-transitions are stored in a stack

Termination issues



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Automata and expressions validity

'Kleene' theorem

Automata	$\Longleftrightarrow$	Expressions
$\mathcal{A}$	$\Longleftrightarrow$	E
Weighted automata	$\Longleftrightarrow$	Weighted expressions

Automata and expressions validity

'Kleene' theorem

Automata $\Longleftrightarrow$ Expressions

$$\mathcal{A} \Longleftrightarrow \mathsf{E}$$

Weighted automata $\Longleftrightarrow$ Weighted expressions

Notion of a valid expression

E *valid* $\Longleftrightarrow$ $\mathsf{c(E)}$ well-defined

$\mathsf{c(E)}$ computed by a bottom-up traversal of the syntactic tree of E

Automata and expressions validity

Valid $\mathcal{A}$ yields valid E

Valid E yields valid $\mathcal{A}$ with Glushkov construction

Valid E may yield non valid $\mathcal{A}$ with Thompson construction

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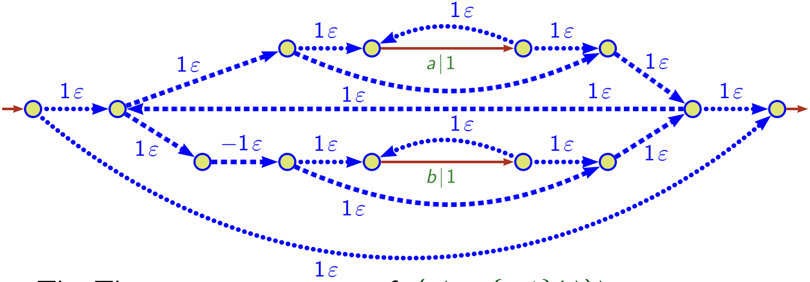
Valid E yields valid $\mathcal{A}$ with Glushkov construction

Valid E may yield non valid $\mathcal{A}$ with Thompson construction



The Thompson automaton of $(a^* + \{-1\}b^*)^*$

Hidden parts

- The removal algorithm itself:
  - Termination issues (weighted versus Boolean cases)
  - Complexity issues
- Automata and expressions validity
- 'Infinitary' axioms : *strong*, *star-strong* semirings

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Theorem

$\mathbb{K}$ *strong semiring* $\qquad s \in \mathbb{K}\langle\langle A^* \rangle\rangle$ *starable iff* $s_0 \in \mathbb{K}$ *starable*

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Definition

A topological semiring is a *star-strong* semiring if
the star of a summable family, whose sum is starable, is summable

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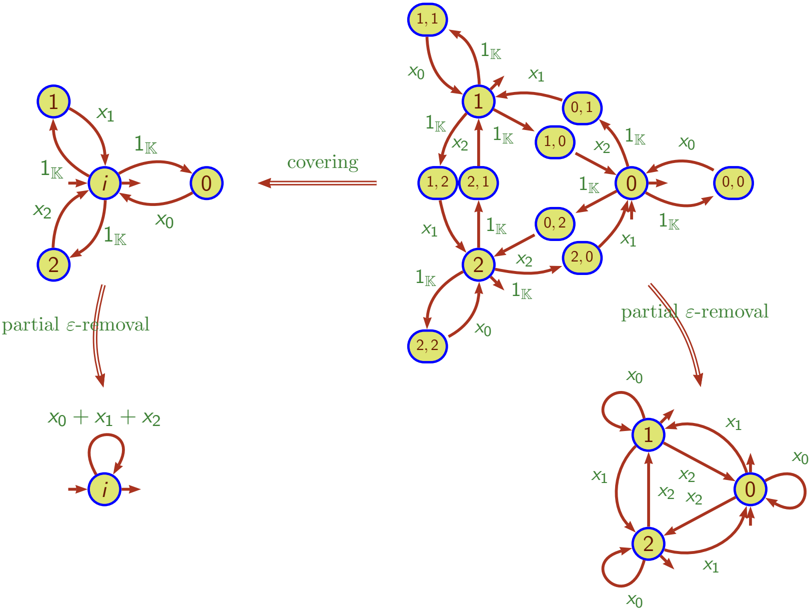
Theorem

A starable star-strong semiring is an iteration semiring

Group identities



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- Links with the ‘axiomatic’ approach (Bloom–Ésik–Kuich):
- References to previous work (on removal algorithm):
  - *locally closed* srgs (Ésik–Kuich), k*-closed* srgs (Mohri)
  - links with other algorithms:
    shortest-distance algorithm (Mohri),
    state-elimination method (Hanneforth–Higueira)