Powers of rationals modulo 1 and rational base number systems

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Based on a paper with the same title:

Powers of rationals modulo 1 and rational base number systems

Israel J. Math., to appear

by *Shigeki Akiyama, Christiane Frougny & Jacques Sakarovitch*
Univ. Niigata, Paris VIII and LIAFA, ENST/CNRS

Part I

A short view on Mahler's problem

The fractional part of the powers of rational numbers

Notation

$\theta \in \mathbb{R}$ $\{\theta\}$ fractional part of θ

Problem

$\theta \in \mathbb{R}\,,\ \theta > 1$ Distribution of $S(\theta) = \{\theta^n\}_{n \in \mathbb{N}}$ **?**

Theorem

For almost all θ, $S(\theta)$ is uniformly distributed.

The fractional part of the powers of rational numbers

Very few results are known for specific values of θ.

Proposition

θ Pisot $\implies$ 0 is the only limit point of $S(\theta)$ (in $\mathbb{R}/\mathbb{Z}$).

Experimental results show that $S(\theta)$ looks :

- *uniformly distributed* for transcendental θ,
- *very chaotic* for rational θ.

Theorem (Pisot ?? — Vijayaraghavan 40)

θ rational $\implies$ $S(\theta)$ has infinitely many limit points.

Parametrization of the problem

Fix the rational $\frac{p}{q}$, $p > q \geqslant 2$ coprime integers.

New problem

$\xi \in \mathbb{R}$ Distribution of $M_{\frac{p}{q}}(\xi) = \left\{ \xi \left(\frac{p}{q} \right)^n \right\}_{n \in \mathbb{N}}$ **?**

Theorem

For almost all ξ, $M_{\frac{p}{q}}(\xi)$ *is uniformly distributed.*

The (generalized) Mahler approach

Notation

$I \subsetneq [0,1[$ I will be a finite union of semi-closed intervals.

$$\mathbf{Z}_{\frac{p}{q}}(I) = \{\xi \in \mathbb{R} \mid \quad M_{\frac{p}{q}}(\xi) \quad \text{is eventually contained in } I \ \} \ .$$

Two directions of research:

Look for I as large as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is empty.

Look for I as small as possible such that $\mathbf{Z}_{\frac{p}{q}}(I)$ is non empty.

Theorem (Mahler 68)

$\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ *is at most countable.*

Open problem Is $\mathbf{Z}_{\frac{3}{2}}\left([0, \frac{1}{2}[\right)$ non empty?

The search for big I with empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Flatto, Lagarias, Pollington 95)

The set of reals s

such that $\mathbf{Z}_{\frac{p}{q}}\left([s, s+\frac{1}{p}[\right)$ *is empty*

is dense in $[0, 1-\frac{1}{p}]$.

Theorem (Bugeaud 04)

The same set is of Lebesgue measure $1-\frac{1}{p}$.

The search for small I with non empty $\mathbf{Z}_{\frac{p}{q}}(I)$

Theorem (Pollington 81)

$\mathbf{Z}_{\frac{3}{2}}\left(\left[\frac{4}{65}, \frac{61}{65}\right[\right)$ *is non empty.*

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Theorem (Pollington 81)

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Theorem (A.-F.-S. 05)

Let $p \geqslant 2q-1$. *There exists* $Y_{\frac{p}{q}} \subset [0,1[$ *of measure* $\frac{q}{p}$

such that $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right)$ *is (countable) infinite.*

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Indeed $\mathbf{Z}_{\frac{p}{q}}\left(Y_{\frac{p}{q}}\right) = \{\xi \in \mathbb{R}_+ \mid \xi \text{ has two } \frac{p}{q}\text{-expansions } \}$.

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What this means is what this talk is about.

Part II

Number systems and finite automata

Number systems (integer base p)

$$N \in \mathbb{N}$$

Representation of N in base p : *word* in $\{0, 1, \ldots, p-1\}^*$

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$$\langle N \rangle_p = a_k \, a_{k-1} \ldots a_1 \, a_0 \qquad\qquad N = \sum_0^k a_i \, p^i$$

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$$L_p = \{\langle N \rangle_p \mid N \in \mathbb{N}\} = A^* \setminus 0\, A^*$$

The base 3 number system

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

	0
1	1
2	2
10	3
11	4
12	5
20	6
21	7
22	8
100	9
101	10
102	11
110	12

111	13
112	14
120	15
121	16
122	17
200	18
201	19
202	20
210	21
211	22
212	23
220	24
221	25

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0000	0	0111	13
0001	1	0112	14
0002	2	0120	15
0010	3	0121	16
0011	4	0122	17
0012	5	0200	18
0020	6	0201	19
0021	7	0202	20
0022	8	0210	21
0100	9	0211	22
0101	10	0212	23
0102	11	0220	24
0110	12	0221	25

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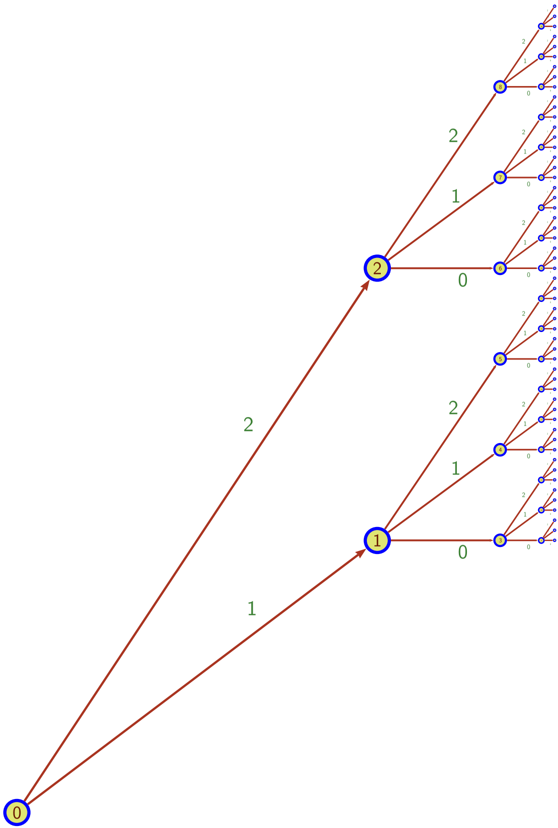
0
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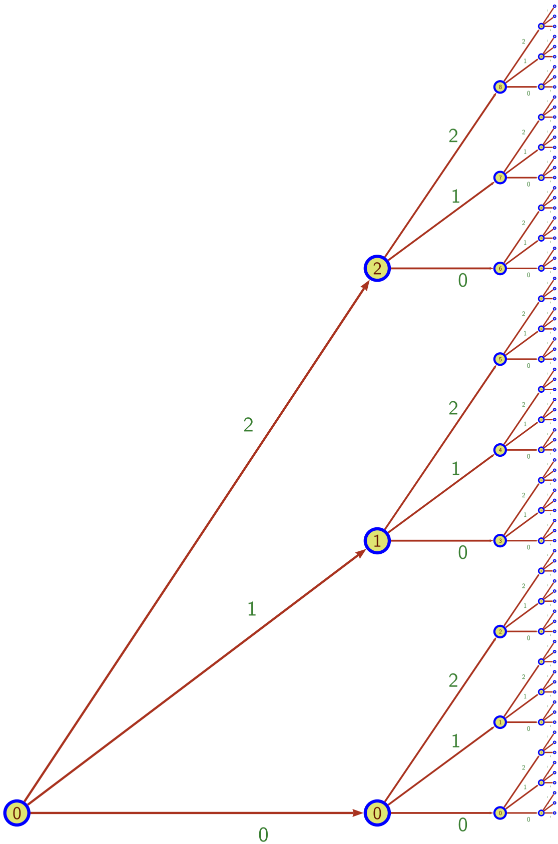
8



0
0
1
2
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1
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0
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2
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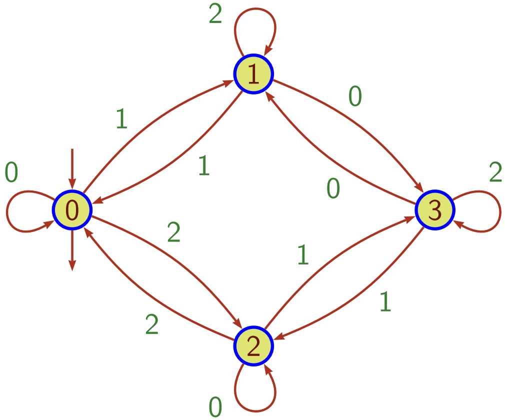
1
2
0
1
2
3
4
5
6
7
8

Number systems and finite automata (1)

Blaise Pascal in *De numeris multiplicibus* ~1650

The p-representations of the integers divisible by k is recognised by a finite automaton.



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$$X \subseteq \mathbb{N}$$

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Definition

X *recognisable*	$\stackrel{\Delta}{\Leftrightarrow}$	X	$=$ finite union of arithmetic prog.
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X *recognisable* $\Longrightarrow$ X p-*recognisable — in every base* p

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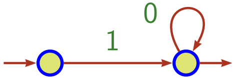

$L(\mathcal{A}) = 10^*$ $\quad$ $\pi(L(\mathcal{A})) = \{3^n \mid n \in \mathbb{N}\}$

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X *recognisable* $\Longrightarrow$ X p*-recognisable — in every base* p

Fact

$$X \;\; p\text{-recognisable} \;\; \not\Longleftarrow \;\; X \;\; \text{recognisable}$$

Theorem (Cobham 69)

p, q *two multiplicatively independent integers.*

X p*-recognisable and* q*-recognisable* $\Longrightarrow$ X *recognisable.*

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Theorem (Cobham 69)

p, q *two multiplicatively independent integers.*

$$X \ p\text{-recognisable and } q\text{-recognisable} \implies X \ \text{recognisable.}$$

Theorem (Honkala 88; Muchnik 91-03, Leroux 05)

It is decidable whether a p*-recognisable set is recognisable.*

Number systems and finite automata (2)

Addition is a normalisation

	2	1	1	1	0
	2	0	1	2	1
1	1	2	0	0	1

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	2	1	1	1	0
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2	0	1	2	1
4	1	2	3	1

Number systems and finite automata (2)

Addition is a normalisation

	2	1	1	1	0
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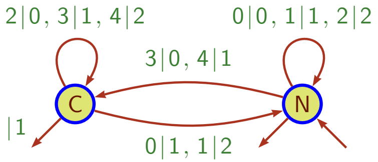
$$\nu\colon \{0,1,2,3,4\}^* \longrightarrow \{0,1,2\}^*$$



Number systems and finite automata (2)

Addition is a normalisation

$$\begin{array}{cccccc} & 2 & 1 & 1 & 1 & 0 \\ & 2 & 0 & 1 & 2 & 1 \\ \hline 1 & 1 & 2 & 0 & 0 & 1 \end{array} \qquad \begin{array}{ccccc} 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 1 \\ \hline 4 & 1 & 2 & 3 & 1 \end{array}$$

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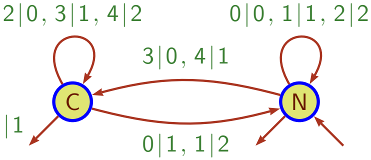

$$\xleftarrow[1]{} \mathsf{C} \xleftarrow[1]{4} \mathsf{N} \xleftarrow[2]{1} \mathsf{C} \xleftarrow[0]{2} \mathsf{C} \xleftarrow[0]{3} \mathsf{N} \xleftarrow[1]{1} \mathsf{N} \longleftarrow$$

Number systems and finite automata (2)

Proposition

In any integer base,
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by a letter-to-letter right sequential transducer.

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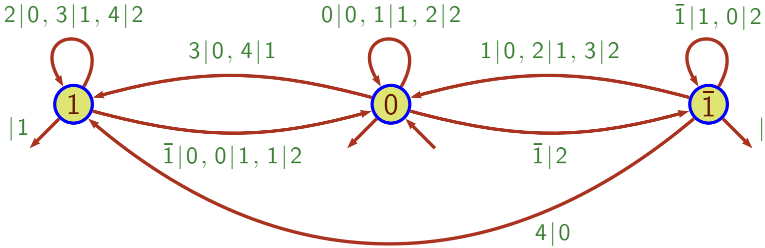

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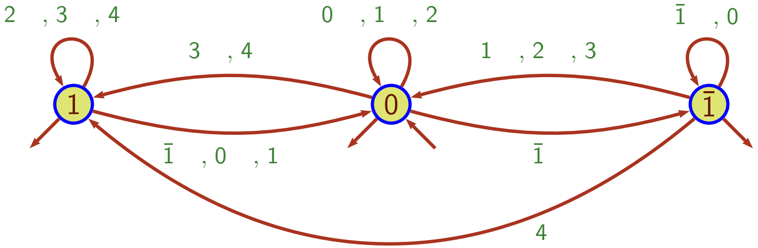

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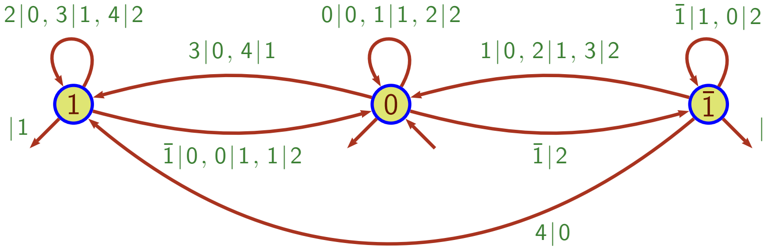

How are the representations in base 3 computed ?

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Greedy algorithm $N \in \mathbb{N}$ $\exists k$ $3^{k+1} > N \geqslant 3^k$

$$\begin{aligned} N_k &= N & & \\ N_{k-1} &= N_k - a_k\,3^k & a_k &\in A\,, & 3^k &> N_{k-1} \\ N_{k-2} &= N_{k-1} - a_{k-1}\,3^{k-1} & a_{k-1} &\in A\,, & 3^{k-1} &> N_{k-2} \\ &\ldots & &\ldots \end{aligned}$$

$$N = \sum_0^k a_i\,3^i \qquad \langle N \rangle_3 = a_k\,a_{k-1}\ldots a_1\,a_0$$

How are the representations in base 3 computed ?

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

Greedy algorithm $17 \in \mathbb{N}$ $3^{2+1} > 17 \geqslant 3^2$

$N_2 = 17$ $k = 2$

$N_1 = 17 - 1 \cdot 3^2 = 8$ $a_2 = 1 \in A\,,$ $3^2 > 8$

$N_0 = 8 - 2 \cdot 3^1 = 2 = a_0$ $a_1 = 2 \in A\,,$ $3^1 > 2$

$17 = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0$ $\langle 17 \rangle_3 = 1\,2\,2$

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Greedy algorithm $x \in \mathbb{R}$ $\exists k$ $3^{k+1} > x \geqslant 3^k$

$$
\begin{aligned}
x_k &= x \\
x_{k-1} &= x_k - a_k\,3^k && a_k \in A\,, \quad 3^k > x_{k-1} \\
x_{k-2} &= x_{k-1} - a_{k-1}\,3^{k-1} && a_{k-1} \in A\,, \quad 3^{k-1} > x_{k-2} \\
&\dots && \dots
\end{aligned}
$$

$$x = \sum_{-\infty}^{k} a_i\,3^i \qquad \langle x \rangle_3 = a_k\,a_{k-1} \dots a_1\,a_0 \centerdot a_{-1}\,a_{-2} \dots$$

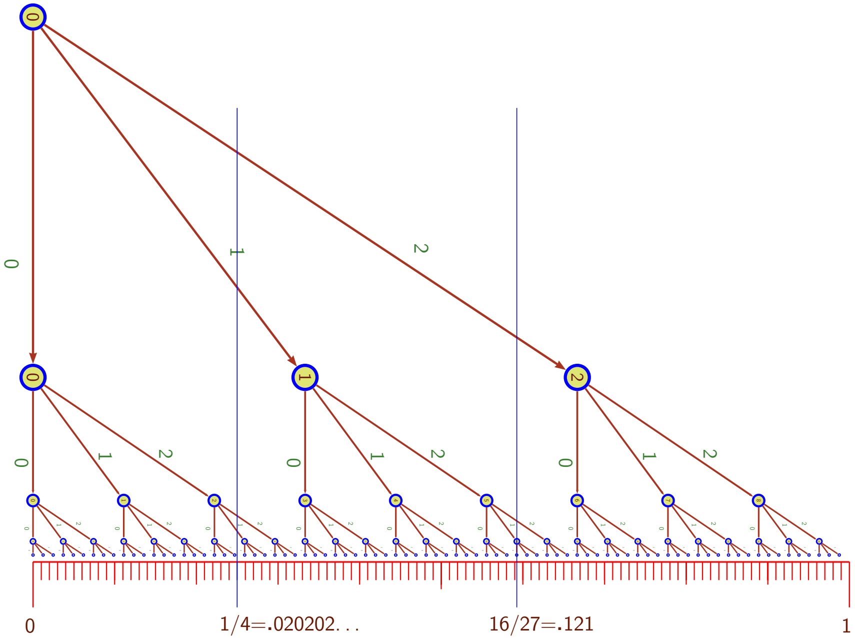
How are the representations in base 3 computed ?

$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, 2\}$

Greedy algorithm $x \in [0, 1[$

$$x_1 = x \qquad a_i = \lfloor 3\, x_i \rfloor \qquad x_{i+1} = \{3\, x_i\}$$

$$x = \sum_{1}^{\infty} a_i\, 3^{-i} \qquad \langle x \rangle_3 = .\, a_1\, a_2\, a_3 \ldots$$


0
1/4=.020202. . .
16/27=.121
1

The base β number system

$V = \{v_i = (3)^i \mid i \in \mathbb{Z}\}$ together with $A = \{0, 1, 2\}$

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$\beta > 1$ any real number

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$$x = \sum_{1}^{\infty} a_i\, \beta^{-i} \qquad \langle x \rangle_\beta = .\, a_1\, a_2\, a_3 \ldots$$

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$\beta > 1$ any real number

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β Pisot $\Longrightarrow$ Parry Theorem

$$L_\beta = \{\langle x \rangle_\beta \mid x \in \mathbb{R}\} \in \mathrm{Rat}\, A^{\mathbb{N}}$$

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β Pisot $\Longrightarrow$ Parry Theorem

$$L_\beta = \{\langle x\rangle_\beta \mid x \in \mathbb{R}\} \in \mathrm{Rat}\, A^{\mathbb{N}}$$

Theorem (Berend-Frougny)
β *is Pisot iff in base* β *,*
normalisation from any alphabet of digits
is realised by a letter-to-letter (finite) transducer.

Part III

Representation of integers in a rational base

The base $\frac{3}{2}$ number system – first approach

$$W = \{w_i = \left(\frac{3}{2}\right)^i \mid i \in \mathbb{Z}\} \qquad A = \{0, \ldots, \left\lfloor \frac{3}{2} \right\rfloor\} = \{0, 1\}$$

Greedy algorithm $\quad x \in \mathbb{R} \quad \exists k \quad \left(\frac{3}{2}\right)^{k+1} > x \geqslant \left(\frac{3}{2}\right)^k$

$$x_k = x$$

$$x_{k-1} = x_k - a_k \left(\frac{3}{2}\right)^k \qquad a_k \in A\,, \quad \left(\frac{3}{2}\right)^k > x_{k-1}$$

$$x = \sum_{-\infty}^{k} a_i \left(\frac{3}{2}\right)^i \qquad \langle x \rangle_W = a_k\, a_{k-1} \ldots a_1\, a_0 \mathbf{.} a_{-1}\, a_{-2} \ldots$$

$$\langle 2 \rangle_W = 10\mathbf{.}010000010\ \ldots$$

The base 3 number system – another look

$V = \{v_i = (3)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

Division algorithm $N \in \mathbb{N}$

$$N'_0 = N$$
$$N'_0 = 3\,N'_1 + a_0 \qquad a_0 \in A$$
$$N'_1 = 3\,N'_2 + a_1 \qquad a_1 \in A$$
$$\dots$$

$$N = \sum_0^k a_i\,3^i \qquad \langle N \rangle_3 = a_k\,a_{k-1} \dots a_1\,a_0$$

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Division algorithm $17 \in \mathbb{N}$

$$N_0' = 17$$

$$17 = N_0' = 3 \cdot 5 + 2 \qquad a_0 = 2 \in A$$

$$5 = N_1' = 3 \cdot 1 + 2 \qquad a_1 = 2 \in A$$

$$1 = N_2' = 3 \cdot 0 + 1 \qquad a_2 = 1 \in A$$

$$17 = ((1) \cdot 3 + 2) \cdot 3 + 2 \qquad \langle 17 \rangle_3 = 1\,2\,2$$

The base $\frac{3}{2}$ number system – second version

$$U = \{u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\}$$ together with $$A = \{0, 1, 2\}$$

Modified division algorithm $N \in \mathbb{N}$

$$N_0 = N$$

$$2\,N_0 = 3\,N_1 + a_0 \qquad a_0 \in A$$

$$2\,N_1 = 3\,N_2 + a_1 \qquad a_1 \in A$$

$$\ldots$$

$$N = \sum_0^k a_i \frac{1}{2}\left(\frac{3}{2}\right)^i \qquad \langle N \rangle_{\frac{3}{2}} = a_k\, a_{k-1} \ldots a_1\, a_0$$

The base $\frac{3}{2}$ number system – second version

$$U = \{u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Modified division algorithm $\qquad 5 \in \mathbb{N}$

$$N_0 = 5$$

$$2\,N_0 = 2 \cdot 5 = 3 \cdot 3 + 1 \qquad 1 \in A$$

$$2\,N_1 = 2 \cdot 3 = 3 \cdot 2 + 0 \qquad 0 \in A$$

$$2\,N_2 = 2 \cdot 2 = 3 \cdot 1 + 1 \qquad 1 \in A$$

$$2\,N_3 = 2 \cdot 1 = 3 \cdot 0 + 2 \qquad 2 \in A$$

$$5 = \frac{1}{2}\left[\,\left(\left((2) \cdot \frac{3}{2} + 1\right) \cdot \frac{3}{2} + 0\right) \cdot \frac{3}{2} + 1\right] \qquad \langle 5 \rangle_{\frac{3}{2}} = 2\,1\,0\,1$$

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$U = \{u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\}$ together with $A = \{0, 1, 2\}$

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Theorem

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N .

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Theorem

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N .

We call this representation the $\frac{3}{2}$-expansion of N

and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

The base $\frac{3}{2}$ number system

$$U = \{u_i = \frac{1}{2}\left(\frac{3}{2}\right)^i \mid i \in \mathbb{N}\} \quad \text{together with} \quad A = \{0, 1, 2\}$$

Theorem

Every N in $\mathbb{N}$ has an integer representation in the $\frac{3}{2}$-system.

It is the unique finite $\frac{3}{2}$-representation of N .

We call this representation the $\frac{3}{2}$-expansion of N

and we denote it by $\langle N \rangle_{\frac{3}{2}}$.

$$L_{\frac{3}{2}} = \{\langle N \rangle_{\frac{3}{2}} \mid N \in \mathbb{N}\} = ????$$

	0		212211	17
2	1		2101100	18
21	2		2101102	19
210	3		2101121	20
212	4		2120010	21
2101	5		2120012	22
2120	6		2120201	23
2122	7		2120220	24
21011	8		2120222	25
21200	9		2122111	26
21202	10		21011000	27
21221	11		21011002	28
210110	12		21011021	29
210112	13		21011210	30
212001	14		21011212	31
212020	15		21200101	32
212022	16		21200120	33

The set $L_{\frac{3}{2}}$ of $\frac{3}{2}$-expansions

The restriction to $L_{\frac{3}{2}}$ of the coarsest right regular equivalence that saturates $L_{\frac{3}{2}}$ is the identity.

$\Longrightarrow$ $L_{\frac{3}{2}}$ is not rational (regular).

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$$\forall m, n \in \mathbb{N}\,,\ \forall t \in A^*$$

$$\left.\begin{array}{r} \langle m\rangle_{\frac{3}{2}}\, t \quad \in L_{\frac{3}{2}} \\ \langle n\rangle_{\frac{3}{2}}\, t \quad \in L_{\frac{3}{2}} \end{array}\right\} \quad \Longleftrightarrow \quad m \equiv n \mod 2^{|t|}$$

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Every word of A^* is a suffix of a word of $L_{\frac{3}{2}}$.

$$\forall w \in A^k \qquad \exists! n \in \mathbb{N} \qquad 0 \leqslant n < 3^k \qquad \exists v \in A^*$$

$$\langle n \rangle_{\frac{3}{2}} = v\, w$$

00000000	0
00000002	1
00000021	2
00000210	3
00000212	4
00002101	5
00002120	6
00002122	7
00021011	8
00021200	9
00021202	10
00021221	11
00210110	12
00210112	13
00212001	14
00212020	15
00212022	16

00212211	17
02101100	18
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02120010	21
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0 0 0 0 0 0 0 0	0	0 0 2 1 2 2 1 1	17
0 0 0 0 0 0 0 2	1	0 2 1 0 1 1 0 0	18
0 0 0 0 0 0 2 1	2	0 2 1 0 1 1 0 2	19
0 0 0 0 0 2 1 0	3	0 2 1 0 1 1 2 1	20
0 0 0 0 0 2 1 2	4	0 2 1 2 0 0 1 0	21
0 0 0 0 2 1 0 1	5	0 2 1 2 0 0 1 2	22
0 0 0 0 2 1 2 0	6	0 2 1 2 0 2 0 1	23
0 0 0 0 2 1 2 2	7	0 2 1 2 0 2 2 0	24
0 0 0 2 1 0 1 1	8	0 2 1 2 0 2 2 2	25
0 0 0 2 1 2 0 0	9	0 2 1 2 2 1 1 1	26
0 0 0 2 1 2 0 2	10	2 1 0 1 1 0 0 0	27
0 0 0 2 1 2 2 1	11	2 1 0 1 1 0 0 2	28
0 0 2 1 0 1 1 0	12	2 1 0 1 1 0 2 1	29
0 0 2 1 0 1 1 2	13	2 1 0 1 1 2 1 0	30
0 0 2 1 2 0 0 1	14	2 1 0 1 1 2 1 2	31
0 0 2 1 2 0 2 0	15	2 1 2 0 0 1 0 1	32
0 0 2 1 2 0 2 2	16	2 1 2 0 0 1 2 0	33

000000 00	0
000000 02	1
000000 21	2
000002 10	3
000002 12	4
000021 01	5
000021 20	6
000021 22	7
000210 11	8
000212 00	9
000212 02	10
000212 21	11
002101 10	12
002101 12	13
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002122 11	17
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021011 02	19
021011 21	20
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021202 20	24
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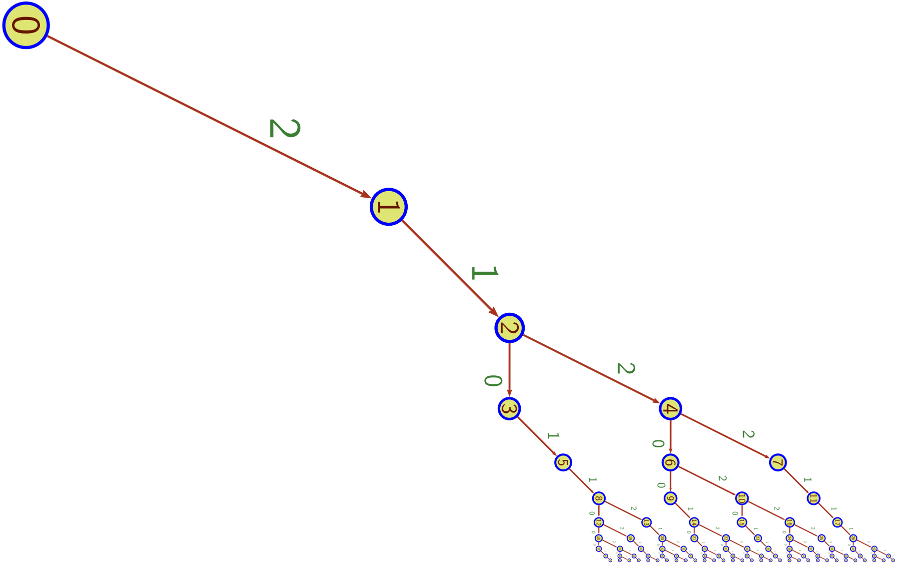
The tree $I_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$L_{\frac{3}{2}}$ prefix-closed $\Longrightarrow$ $L_{\frac{3}{2}}$ spans the edges of a subtree $I_{\frac{3}{2}}$ of the full 3-ary tree.

The tree $I_{\frac{3}{2}}$



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Any two distinct subtrees of $I_{\frac{3}{2}}$ are not isomorphic.

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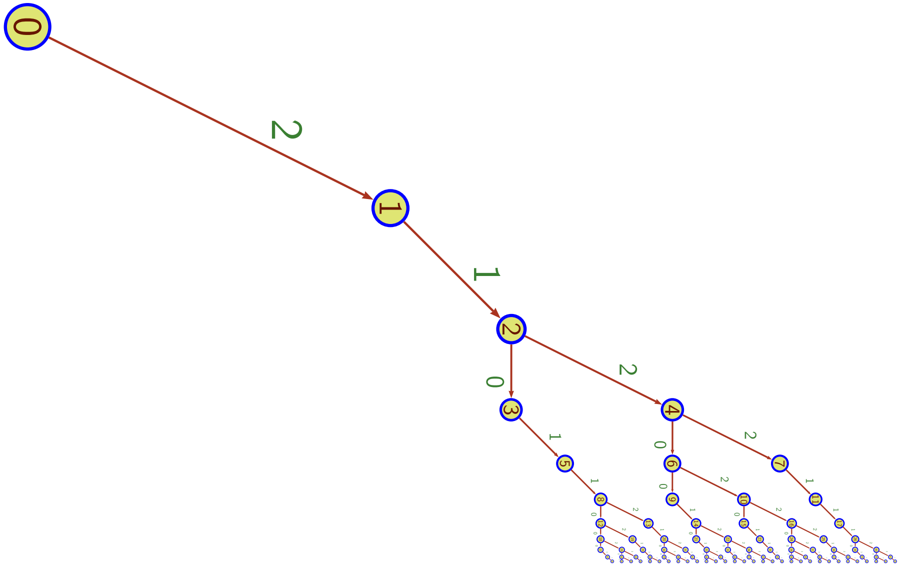


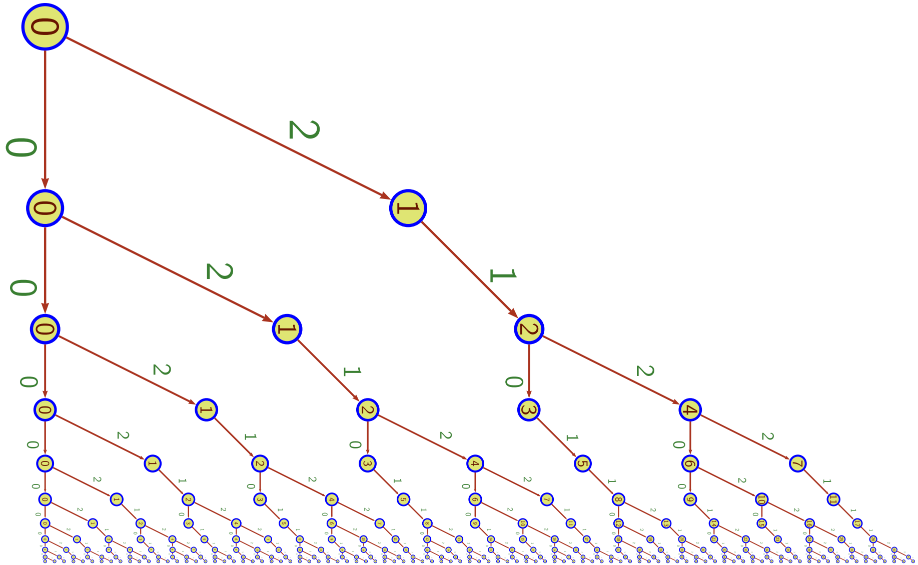
The tree $T_{\frac{3}{2}}$ of the $\frac{3}{2}$-expansions

$I_{\frac{3}{2}}$ can be "completed" into a tree $T_{\frac{3}{2}}$.

There exists a sequence of integers M_k such that
the nodes of depth k in $T_{\frac{3}{2}}$
are labeled by integers from 0 to M_k .

These labels give the lexicographic ordering
on the nodes of depth k .

The tree $T_{\frac{3}{2}}$



Digit conversion

D finite digit alphabet, that contains A .

$$\chi_D \colon D^* \to A^* \qquad \forall w \in D^* \qquad \pi(\chi_D(w)) = \pi(w) \ .$$

Proposition

For every D , χ_D *is realised*
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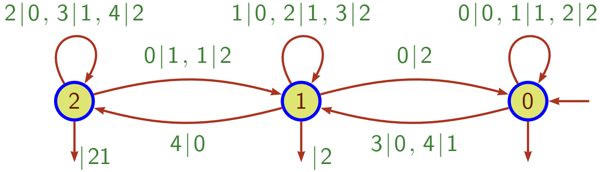

Part IV

Representation of reals in a rational base

Representation of reals in base 3 : the tree T_3

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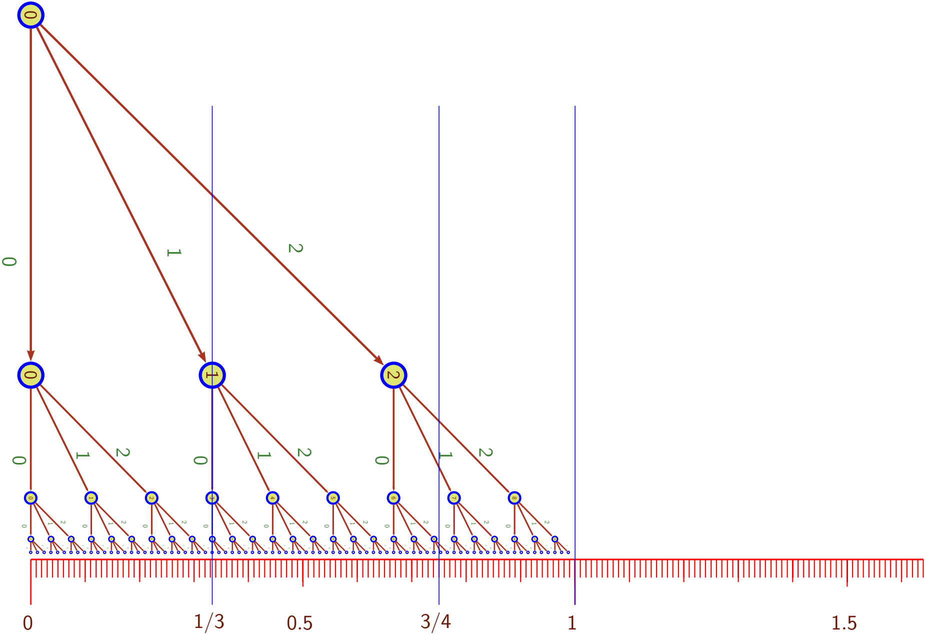

Representation of reals in base 3 : the tree T_3

$A^{\mathbb{N}}$ = labels of the *infinite paths* in T_3 $\qquad$ $\mathbf{a} = \{a_i\}_{i \geqslant 1} \in A^{\mathbb{N}}$

Definition

$\mathbf{a}$ is *an expansion in base* 3 of the real $x \in [0, 1]$ defined by:

$$x = \pi(.\mathbf{a}) = \sum_{i \geqslant 1} a_i \left(\frac{1}{3}\right)^i .$$

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The set of reals in $[0, 1]$ which have two expansions is infinite countable.

Representation of reals in base $\frac{3}{2}$: the tree $T_{\frac{3}{2}}$