Upper bound of loss probability for the dimensioning of an OFDMA system with multi class randomly located users

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***Abstract*—This work proposes a new analytical model for the dimensioning of OFDMA systems. It relies on a rough but easily computable upper bound for the probability of lost communications by insufficient number of sub-channels on downlink. The positions of receiving users in the system as well as the number of sub-channels dedicated to each one are randomized. Users are classified in different classes according to their throughput requirements and traffic patterns. We use recent results of the theory of point processes which reduce our calculations to that of the first and second moments of the total required number of sub-channels. The upper bound probability leads to an acceptable over dimensioning in terms of sub-channels.**

I. Introduction

Future wireless systems will widely rely on OFDMA (Orthogonal Frequency Division Multiple Access) multiple access technique. OFDMA can satisfy end user's demands in terms of throughput. It also fulfills operator's requirements in terms of capacity for high data rate services. Systems such as 802.16e and 3G-LTE (Third Generation Long Term Evolution) already use OFDMA on the downlink. For the uplink, 802.16e has also adopted OFDMA, while 3G-LTE uses SCFDMA (Single Carrier Frequency Division Multiple Access). OFDMA can also be possibly combined with multiple antenna technology to improve either quality or capacity of systems.

Dimensioning of OFDMA systems is then of the up-most importance for wireless telecommunications industry.

The model introduced in this contribution takes into account the randomness of user locations and user traffic. It provides also an upper bound of loss probability in terms of sub-channels.

The paper first provides a short introduction to OFDMA air interfaces, by providing some insights on sub-channel concepts and OFDMA jargon (see section II). Besides a review on point Poisson Process theory and concentration inequalities is provided in section III. The dimensioning analytical model is first developed for a deterministic wireless channel, taking only into account the path-loss effect (cf. section IV). Section V analyses a more realistic situation, where wireless channel also encompasses shadowing effects. Section V-B extends the results to a multi class user traffic. The accuracy of analytical model is evaluated by comparing them with simulation.

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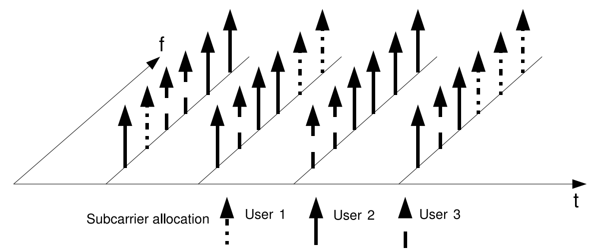

Fig. 1. OFDMA sub-carrier allocation principle

II. Introduction to OFDMA air interfaces

OFDM (Orthogonal Frequency Division Multiplex) is a multi carrier technique especially designed for high data rate services. It divides the spectrum in a large number of frequency bands called sub-carriers that overlap partially in order to reduce spectrum occupation. Overlapping is made possible because the different sub-carriers are made orthogonal to each other by choosing a sub-carrier spacing multiple of the inverse of the OFDM symbol duration.

Each sub-carrier has a small bandwidth compared to the coherence bandwidth of the channel in order to mitigate frequency selective fading. User data is then transmitted in parallel on each sub carrier.

Systems such as ADSL (Asymmetric Digital Subscriber Line), digital audio broadcasting (DAB) and digital video broadcasting (DVB-T) rely on OFDM modulation. Most recently, power line systems (Home Plug) and WiMedia (short range communications) have also adopted OFDM.

In OFDM systems, all available sub-carriers are affected to one user at a given time for transmission. OFDMA extends OFDM by making it possible to share dynamically the available sub-carriers between different users (see figure 1). In that sense, it can then be seen as multiple access technique that both combines FDMA and TDMA features.

In practical systems, such as WiMAX or 3G-LTE, the sub-carriers are not allocated individually for implementation reasons mainly inherent to the scheduler design and physical layer signaling. Several sub-carriers are then grouped in sub-channels according to different strategies specific to each system. The unit of resource allocation is the sub-channel.

For example, in WiMAX, there are three modes available for building sub-channels: FUSC (Fully Partial Usage of Sub-channels), PUSC (Partial Usage of Sub-Channels) and AMC

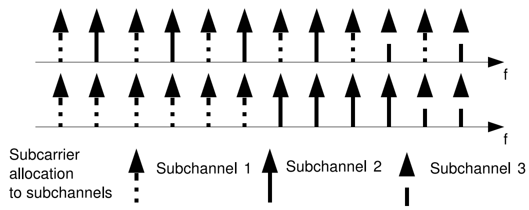

Fig. 2. OFDMA sub-channel principle

(Adaptive modulation and coding). In FUSC, sub-channels are made of sub-carriers spread over all the frequency band. In AMC, the sub-carriers of a sub-channel are adjacent instead of being uniformly distributed over the spectrum. FUSC provides an averaging effect on quality which makes it more suitable for mobile application, while AMC is more adapted for fixed users.

The sub-channel concept makes it easier to schedule radio resources. However, it becomes more difficult to assess channel quality as it is composed by different sub-carriers that can possibly span over several timeslots. An extensive literature has addressed that problem, and we will assume in the following, that whatever the sub-channelization scheme adopted, it is possible to consider an equivalent single channel gain for all the sub-carriers making part of a sub-channel (for example the average of channel gain computed on some sub-carrier pilots). We also assume that subcarrier allocation to different sub-channels is done slot by slot.

III. Poisson point processes

For details on point processes, we refer to [1, 4, 5, 6]. A configuration η in $\mathbf{R}^k$ is a set $\{x_n,\, n \ge 1\}$ where for each $n \ge 1$, $x_n \in \mathbf{R}^k$, $x_n \neq x_m$ for $n \neq m$ and each compact subset of $\mathbf{R}^k$ contains only a finite subset of η. We denote by $\Gamma_{\mathbf{R}^k}$ the set of configurations in $\mathbf{R}^k$. Equipped with the vague topology of discrete measures, $\Gamma_{\mathbf{R}^k}$ is a complete, separable metric space. A point process Φ is a random variable with values in $\Gamma_{\mathbf{R}^k}$, i.e., $\Phi(\omega) = \{X_n(\omega),\, n \ge 1\} \in \Gamma_{\mathbf{R}^k}$. For $A \subset \mathbf{R}^k$, we denote by Φ_A the random variable which counts the number of atoms of $\Phi(\omega)$ in A:

$$\Phi_A(\omega) = \sum_{n\ge 1} \mathbf{1}_{X_n(\omega)\in A} \;\in \mathbf{N} \cup \{+\infty\}.$$

Poisson point processes are particular instances of point processes such that:

Definition 1. *Let Λ be a σ finite measure on $\mathbf{R}^k$. A point process Φ is a Poisson process of intensity Λ whenever the following two properties hold.*

1 - *For any compact subset $A \in \mathbf{R}^k$, Φ_A is a Poisson random variable of parameter $\Lambda(A)$, i.e.,*

$$\mathbf{P}(\Phi_A = k) = e^{-\Lambda(A)} \frac{\Lambda(A)^k}{k!}.$$

2 - *For any disjoint subsets A and B, the random variables Φ_A and Φ_B are independent.*

The notion of point process can be extended to configurations in $\mathbf{R}^k \times X$ where X is a subset of $\mathbf{R}^m$. A configuration is then typically of the form $\{(x_n,\, y_n),\, n \ge 1\}$ where for each $n \ge 1$, $x_n \in \mathbf{R}^k$ and $y_n \in X$. We keep writing $(x_n,\, y_n)$ as a couple, though it could be thought as an element of $\mathbf{R}^{k+m}$, to stress the asymmetry between the spatial coordinate x_n and the so-called mark, y_n. For a marked point process, we denote by Φ the set of locations, i.e., $\Phi(\omega) = \{X_n,\, n \ge 1\}$ and by $\bar\Phi$ the set of both locations and marks, i.e., $\bar\Phi(\omega) = \{(X_n,\, Y_n),\, n \ge 1\}$. A marked point process with position dependent marking is a marked point process for which the law of Y_n, the mark associated to the atom located at X_n, depends only on X_n through a kernel K:

$$\mathbf{P}(Y_n \in B \,|\, \Phi) = K(X_n,\, B), \text{ for any } B \subset X.$$

If K is a probability kernel, i.e., if $K(x,\, X) = 1$ for any $x \in \mathbf{R}^k$ then it is well known that $\bar\Phi$ is a Poisson process of intensity $K(x,\, \mathrm{d}y)\mathrm{d}\Lambda(x)$ on $\mathbf{R}^k \times \mathbf{R}^m$. The Campbell formula is a well known and useful formula

Theorem 1. *Let $\bar\Phi$ be a marked Poisson process on $\mathbf{R}^k \times \mathbf{R}^m$. Let Λ be the intensity of the underlying Poisson process and K the kernel of the position dependent marking. For $f : \mathbf{R}^k \times \mathbf{R}^m \to \mathbf{R}$ a measurable non-negative function, let*

$$F = \int f \, d\bar\Phi = \sum_{n \ge 1} f(X_n,\, Y_n).$$

Then,

$$\mathbf{E}\,[F] = \int_{\mathbf{R}^k \times \mathbf{R}^m} f(x,\, y)\, K(x,\, dy) d\Lambda(x).$$

Definition 2. *For $F : \Gamma_{\mathbf{R}^k} \to \mathbf{R}$, for any $x \in \mathbf{R}^k$, we define*

$$D_x F(\omega) = F(\omega \cup \{x\}) - F(\omega).$$

Note that for $F = \int f \mathrm{d}\Phi$, $D_x F = f(x)$, for any $x \in \mathbf{R}^k$. We now quote from [3, 8] the main result on which our inequalities are based:

Theorem 2 (Concentration inequality)**.** *Assume that Φ is a Poisson process on $\mathbf{R}^k$ of intensity Λ. Let $f : \mathbf{R}^k \to \mathbf{R}^+$ a measurable non-negative function and let*

$$F(\omega) = \int f \, d\Phi = \sum_{n\ge 1} f(X_n(\omega)).$$

Assume that $|D_x F(\omega)| \le s$ for any $x \in \mathbf{R}^k$. Let

$$m_F = \mathbf{E}\,[F] = \int f(x)\, d\Lambda(x)$$

and

$$v_F = \int |D_x F(\omega)|^2 d\Lambda(x) = \int f^2(x)\, d\Lambda(x).$$

Then, for any $t \in \mathbf{R}^+$,

$$\mathbf{P}(F - m_F \ge t) \le \exp\left(-\frac{v_F}{s^2}\; g\left(\frac{t\, s}{v_F}\right)\right)$$

where $g(x) = (1+x)\ln(1+x) - x$.

IV. Deterministic gain

We state the following assumptions:

Assumption 1. *The position of each user is independent of the position of all other. The users are indistinguishable, i.e., the positions are identically distributed.*

Assumption 2. *The time between two consecutive demands of users for service in the system (or inter arrival time) is exponentially distributed.*

We define $\rho(x)$ as the surface density of inter arrival time in $\mathrm{s}^{-1}\mathrm{m}^{-2}$, constant in time. Hence, for a region $H \subseteq B$, the mean inter arrival rate is $h = \int_H \rho(x)dx$ in s^{-1}.

Assumption 3. *The service time for every user is exponentially distributed with mean $1/\nu$.*

Assumption 4. *The cell C is circular, with radius R and with the antenna in the center.*

Assumption 5. *The channel gain depends only on the distance from the transmitting antenna.*

Assumption 6. *The surface density of inter arrival time is constant.*

These assumptions are commonly done to simplify the mathematical treatment and are quite reasonable. If we show that the point process given by the location of the users is a Poisson process, then it is sufficient to have the two first moments in order to apply theorem 2 and then calculate an upper bound P_{sup} for the probability P_{loss} of loosing communications due to a lack of sub-channels. To do this, we consider the following lemma:

Lemma 1. *Considering assumptions 1, 2 and 3, the point process Φ of the active users positions is, in equilibrium, a Poisson process with intensity $d\Lambda(x) = \rho(x)\nu^{-1}dx$*

Proof: For a region H, in virtue of assumptions 2 and 3, the number of receiving (i.e., active) customers is the same as the number of customers in an M/M/∞ queue with input rate h and mean service time ν^{-1}. It is known [7] that the distribution of the number of users U in equilibrium is then

$$P(U = u) = \frac{(h/\nu)^u}{u!} e^{-h/\nu}.$$

It follows that the first condition of definition 1 is satisfied with intensity measure $\Lambda(H)$

$$\Lambda(H) = h/\nu = \int_H \frac{\rho(x)}{\nu} \, \mathrm{d}x.$$

Condition 2 of definition 1 follows straightforwardly from assumption 1. ■

Without loss of generality, we consider the cell C has its antenna located at the origin. We are looking at evaluating

$$P_{loss} = \mathbf{P}(\int N \, \mathrm{d}\Phi \geq N_0),$$

α	1.5	1.6	1.7	1.8	1.9	2
P_{sup}	0.18	0.1	0.04	0.02	0.008	0.003
Δ	0.98	0.1	1.15	1.3	1.3	1.4

TABLE I
Comparison between P_{sup} and P_{loss} for deterministic gain.

where $N(x)$ is defined by

$$N(x) = \left\lceil \frac{C_0}{W \log_2\left(1 + \frac{P_t K \bar{g}}{(I+\eta)\|x\|^\gamma}\right)} \right\rceil,$$

where $\bar{g}$ is the mean gain due to shadowing, C_0 the throughput requested by users, I the interference generated by outer cells and η the noise. We will not take into account interference generated by outer cells, so $I = 0$. Note that, with respect to x, N is increasing and piecewise constant. Let $R_j,\ j = 1, \cdots, N_{max}$ be the values such that $N(x) = j$ for $x \in [R_j,\ R_{j+1})$. We can easily determine them by

$$R_j = \left(\frac{P_t K \overline{g}}{\eta(2^{C_0/(jW)} - 1)}\right)^{1/\gamma}.$$

According to Theorem 1, it is then clear that

$$\mathbf{E}\left[\int N \, \mathrm{d}\Phi\right] = \int N \mathrm{d}\Lambda = \frac{\pi\rho}{\nu} \sum_{j=1}^{N_{max}} j(R_j^2 - R_{j-1}^2).$$

We denote by m_N the last quantity. Moreover,

$$\int N^2 \mathrm{d}\Lambda = \frac{\pi\rho}{\nu} \sum_{j=1}^{N_{max}} j^2(R_j^2 - R_{j-1}^2).$$

We denote by v_N the last quantity. We take N_0 of the form αm_N, so that according to Theorem 2:

$$\mathbf{P}(\int N \, \mathrm{d}\Phi \geq \alpha m_N) \leq P_{sup}(\alpha)$$

where

$$P_{sup}(\alpha) = \exp\left(-\frac{v_N}{N_{max}^2} \, g\left(\frac{(\alpha-1)m_N N_{max}}{v_N}\right)\right).$$

It is then natural to verify how far this bound is from the exact value of the loss probability in simple situations where simulation is available. We used here $\gamma = 2.8$, $C_0 = 200$ kb/s, $W = 250$ kHz and $P_t K/(\eta) = 1 \times 10^6$. For the surface density of inter arrival time we use $\rho = 0.0006 \ \mathrm{min}^{-1}\mathrm{m}^{-2}$ and the service time is $1/\nu = 1$ min, so, the mean number of users in the system is $\pi R^2 \rho/\nu = 18.85$ users. If we consider the shadowing with $\sigma = \sqrt{10}$ dB and $\mu = 6$ dB, we can use the mean gain g, giving $\overline{g} = 1/12$. Thus, users in the cell boundary use 3 sub-channels, so $N_{max} = 3$. For α varying from 1 to 2, which corresponds here to loss probabilities about 2% or 0.01%, we computed $\Delta = \log_{10} P_{sup}/P_{loss}$.

Though concentration inequalities are usually thought as almost optimal, the results shown in Table I seem at first glance disappointing. Note though that the computation of the bound is immediate whereas the simulation on a fast PC took several hours to get a decent confidence interval. Note also that the error is about the same order of magnitude as the error made when using a usual trick which consists in replacing infinite buffers by finite ones in Jackson networks (see [2]). The margin provided by the bounds may be viewed as a protection against errors in the modeling or in the estimates of the parameters.

V. Random gain

A. Single class of user traffic

Let us determine now the upper bound probability P_{sup} for P_{loss} without assumption 5 but holding all other assumptions of the preceding section. Lemma 1 still holds, since it is a consequence of assumptions 1, 2 and 3. We also state two other natural assumptions:

Assumption 7. *The random gain is totally described by the log-normal shadowing, with mean μ and standard deviation σ, both in dB.*

For a user at distance d from the origin, the gain is $G = 1/S$, where S follows a log-normal distribution:

$$p_S(y) = \frac{\xi}{\sqrt{2\pi}\sigma y} \exp\left[-\frac{(10\log_{10} y - \mu)^2}{2\sigma^2}\right],$$

where $\xi = 10/\ln 10$.

Assumption 8. *A user is able to receive the signal only if the signal-to-interference ratio is above some constant β_{min}.*

This means, in particular, that the number of sub-carriers needed by a transmitting user is surely bounded by

$$N_{max} = \left\lceil \frac{C_0}{W\log_2(1+\beta_{min})} \right\rceil.$$

The situation is slightly different from that of Section IV, since the functional depends on two aleas: positions and gains. Consider now that our configurations are of the form (x, s) where $x \in \mathbf{R}^2$ is still a position and $s \in \mathbf{R}$ is a gain. Since gain and positions are independent, we then have a Poisson process on $\mathbf{R}^3$ of intensity measure $d\Lambda(x) \otimes p_S(y)\mathrm{d}y$. Thus we want to evaluate an upper bound of

$$\mathbf{P}(\int N \mathrm{d}\Phi \geq \mathbf{N}_0)$$

where

$$N(x,\, y) = \left\lceil \frac{C_0}{W\log_2\left(1 + \dfrac{P_t K}{\eta y \|x\|^\gamma}\right)} \right\rceil.$$

According to Theorem 2, we must compute

$$m_N = \int N(x,\ y) p_S(y) \mathrm{d}y \ \mathrm{d}\Lambda(x)$$

and

$$v_N = \sup_\omega \int |D_{x,y}F(\omega)|^2 p_S(y)\mathrm{d}y \ \mathrm{d}\Lambda(x) = \int N^2(x,\ y) p_S(y)\mathrm{d}y \ \mathrm{d}\Lambda(x).$$

Let $\beta_0 = \infty$ and $\beta_j = 2^{C_0/(Wj)} - 1$ for $j = 1, \cdots, N_{max}-1$. For $j = 1, \cdots, N_{max}-1$, let

$$A_j = \int_{C\times\mathbf{R}^+} \mathbf{1}_{\{y\|x\|^\gamma \leq P_t K/\eta\beta_j\}} p_S(y)\, \mathrm{d}y\, \mathrm{d}x$$

and $A_0 = 0$.

Lemma 2. *For $j = 1, \cdots, N_{max}-1$,*

$$A_j = \pi R^2 Q(\alpha_j - \zeta \ln R) + \pi e^{2/\zeta^2 + 2\alpha_j/\zeta} Q(\zeta \ln R - 2/\zeta - \alpha_j),$$

where

$$\alpha_j = \frac{1}{\sigma}(10\log_{10}(P_t K/\eta\beta_j) - \mu) \text{ and } \zeta = \frac{10\gamma}{\sigma \ln 10}.$$

Proof: We can write

$$A_j = \int_C \mathbf{P}(S\|x\|^\gamma \leq \tilde{\beta}_j)\, \mathrm{d}x$$

where $\tilde{\beta}_j = P_t K/\eta\beta_j$. Remind that S is equal in distribution to $\exp(\mathcal{N}(\mu, \sigma^2)\xi)$ with $\xi = \ln(10)/10$. Thus after a few manipulations, we get

$$A_j = 2\pi \int_0^R r\, Q(\alpha_j - \zeta \ln r)\, \mathrm{d}r,$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp(-\frac{u^2}{2})\, \mathrm{d}u.$$

The final result follows by a tedious but straightforward integration by parts. ∎

Theorem 3. *For any function $\theta : \mathbf{R} \to \mathbf{R}$,*

$$\int \theta(N(x,\ y)) p_S(y) dy\ d\Lambda(x) = \sum_{j=1}^{N_{max}-1} \theta(j)(A_j - A_{j-1}) + \theta(N_{max})(\pi R^2 - A_{N_{max}-1}).$$

Proof: Since N can take only a finite number of values, we have

$$\int \theta(N(x,\ y)) p_S(y)\mathrm{d}y\ \mathrm{d}\Lambda(x) = \frac{\rho}{\nu} \sum_{j=1}^{N_{max}} \theta(j) \int_{C\times\mathbf{R}^+} \mathbf{1}_{\{(x,y),\ N(x,y)=j\}} p_S(y)\mathrm{d}y\ \mathrm{d}x.$$

Now we see that

$$N(x,\, y) = j \Longleftrightarrow \tilde{\beta}_{j-1} < y\|x\|^\gamma \leq \tilde{\beta}_j,$$

for $j = 1, \cdots, N_{max}-1$ and $N(x,\, y) = N_{max}$ when $y\|x\|^\gamma > \tilde{\beta}_{N_{max}-1}$. The proof is thus complete. ∎

We used the same set of values as for the simulation of Section IV together with assumptions 8 and 7 with $\beta_{min} = 0.2$. Results of Table II show that the theoretical bound is rather stable when gains become stochastic.

α	1.5	1.6	1.7	1.8	1.9	2
P_{sup}	0.2	0.1	0.05	0.02	0.01	0.004
Δ	1.7	1.8	2.1	2.3	2.4	2.6

TABLE II
COMPARISON BETWEEN P_{sup} AND P_{loss} FOR RANDOM GAIN.

B. Multi class user traffic

1) Upper bound of loss probability: We consider in this section, M classes of users. Class j users request a throughput of C_j. The configurations associated to each class are of the form (x, y), where $x \in \mathbf{R}^2$ is a position, $y \in \mathbf{R}$ is a gain. Since gain and positions are independent, we then have for each class of users a Poisson process on $\mathbf{R}^3$ of intensity measure $\Lambda_j(x)\mathrm{d}x \otimes p_S(y)\mathrm{d}y$, where $\Lambda_j(x) = \rho_j(x)\nu_j^{-1}$ and j is the user class.

For the sake of computational simplicity, we assume in the following, that $\rho_j(x)$ is constant with respect to x but the theory is still valid unaltered otherwise. Furthermore we consider that the random gain is totally described by the log-normal shadowing, with mean μ and standard deviation σ, both in dB. For a user at distance d from the origin, the gain is $G = 1/S$, where S follows a log-normal distribution as in section V-A. We also assume that a user is able to receive the signal only if the signal-to-interference ratio is above some constant β_{min}. This means, in particular, that the number of sub-channels needed by a transmitting user of class j is surely bounded by

$$N_j^{max} = \left\lceil \frac{C_j}{W \log_2(1 + \beta_{min})} \right\rceil .$$

Without loss of generality, we consider the cell C has its antenna located at the origin. We are then looking at evaluating

$$\mathbf{P}(\int N \mathrm{d}\Phi \geq \mathbf{N}_0)$$

where

$$N(x,\, j,\, y) = \left\lceil \frac{C_j}{W \log_2\left(1 + \dfrac{P_t K}{\eta y \|x\|^\gamma}\right)} \right\rceil .$$

The functional depends on two aleas: positions and gains. It has also an additional parameter that describes the class of the user.

Theorem 4. *With the assumptions of this Section,*

$$\mathbf{P}(\int N\, d\Phi \geq \alpha m_N) \leq P_{sup}(\alpha)$$

where

$$P_{sup}(\alpha) = \exp\left(-\frac{v}{N_{max}^2}\; g\left(\frac{(\alpha-1)mN_{max}}{v}\right)\right),$$

with $N_{max} = \max_j N_j^{max}$,

$$m = \sum_{j=1}^{M} \int N(x,\, j,\, y)\Lambda_j(x)p_S(y)dxdy,$$

and

$$v = \sum_{j=1}^{M} \int N(x,\, j,\, y)^2\Lambda_j(x)p_S(y)dxdy.$$

Proof: Let Λ_j be the intensity of the Poisson process representing class j customers and $\Lambda = \sum_{j=1}^M \Lambda_j$. Let Φ be a Poisson process on $\mathbf{R}^2$ of intensity Λ. Consider the probability kernel

$$K(x, \{j\}) = \frac{\Lambda_j(x)}{\Lambda(x)}.$$

For a configuration $\omega = \{x_n,\, n \geq 1\}$, there is thus a sequence of marks $\{u_n,\, n \geq 1\}$, $u_n \in \{1, \cdots, M\}$ for all $n \geq 1$, corresponding to the position dependent marking according to the kernel K. According to the properties of Poisson process, the process $\Phi_j = \{x_n, u_n = j\}$ is a Poisson process of intensity Λ_j. Now add to each point of Φ, an independent mark y_n, corresponding to the random gain, distributed according to a log-normal distribution. Denote by $\bar{\Phi}$ this point process which turns to be a Poisson process since the marks are independent from the positions. From section 2, we know that the process, the atoms of which are $\bar{\omega} = (x_n, u_n, y_n)$, is a Poisson process of intensity $\sum_j K(x, \{j\})\Lambda(x)p_S(y)\mathrm{d}x\mathrm{d}y\delta_j$:

$$\begin{aligned}\mathbf{E}\left[\sum_{n\geq 1} f(X_n, U_n, Y_n)\right] &= \sum_{j=1}^{M} \int f(x,j,y)\frac{\Lambda_j(x)}{\Lambda(x)}\Lambda(x)p_S(y)\mathrm{d}x\mathrm{d}y \\ &= \sum_{j=1}^{M} \int f(x,j,y)\Lambda_j(x)p_S(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We are thus in position to apply the Theorem 2 to the Poisson process $\bar{\Phi}$. The difference operator defined in Definition 2, is here equal to

$$D_{x,j,y}F(\bar{\omega}) = F(\bar{\omega} \cup \{x, j, y\}) - F(\bar{\omega}),$$

as it suffices to take $k = 2 + 1 + 1 = 4$. That is to say, we look at the impact of adding a user at position x, with class j and gain y. For $F = \int N \mathrm{d}\Phi$, we obtain

$$D_{x,j,y}F(\bar{\omega}) = N(x,\, j,\, y) \leq N_j^{max}.$$

Thus, inequality (2) holds with $s = \max_j N_j^{max}$,

$$m_N = \sum_{j=1}^{M} \int N(x,\, j,\, y)\Lambda_j(x)p_S(y)\mathrm{d}x\mathrm{d}y,$$

and

$$v_N = \sum_{j=1}^{M} \int N(x,\, j,\, y)^2\Lambda_j(x)p_S(y)\mathrm{d}x\mathrm{d}y.$$

■

Both m and v can be computed taking advantage of the fact that N is piecewise constant (see section V-A). Let $\beta_0 = \infty$ and $\beta_{j,k} = 2^{C_j/(Wk)} - 1$ for $k = 1, \cdots, N_j^{max} - 1$. For $k = 1, \cdots, N_j^{max} - 1$, let

$$A_{j,k} = \int_{C\times\mathbf{R}^+} \mathbf{1}_{\{y\|x\|^\gamma \leq P_t K/\eta\beta_k\}} p_S(y)\, \mathrm{d}y\, \mathrm{d}x$$

and $A_0 = 0$. It can proved from results of section V that for $k = 1, \cdots, N_j^{max} - 1$,

$$A_{j,k} = \pi R^2 Q(\alpha_{j,k} - \zeta \ln R) + \pi e^{2/\zeta^2 + 2\alpha_{j,k}/\zeta} Q(\zeta \ln R - 2/\zeta - \alpha_{j,k}),$$

where

$$\alpha_{j,k} = \frac{1}{\sigma}(10 \log_{10}(P_t K/\eta\beta_{j,k}) - \mu) \text{ and } \zeta = \frac{10\gamma}{\sigma \ln 10}.$$

We finally obtain the following formula.

Theorem 5. *For any function $\theta : \mathbf{R} \to \mathbf{R}$,*

$$\int \theta(N(x, j \ y)) p_S(y) dy \ d\Lambda(x) = \sum_{l=1}^{N_j^{max}-1} \theta(l)(A_l - A_{l-1}) + \theta(N_j^{max})(\pi R^2 - A_{N_j^{max}-1}).$$

2) Numerical application: In this section we will apply the upper bound calculated previously to the dimensioning of sub-channels in a OFDMA system. We consider here a cell, where two classes of users are competing to the access of available sub-channels. More precisely we consider here $M = 2$. The capacities required by each class of user is fixed to $C_1 = 200$ kb/s and $C_2 = 100kb/s$ respectively. The path-loss exponent is fixed to $\gamma = 3.8$ and the sub-channel bandwidth is equal to $W = 250$ kHz. We also consider $P_t K/\eta = 1 \times 10^{12}$. For the surface density of inter arrival time we use $\rho_1 = 0.0006$ $\mathrm{min}^{-1}\mathrm{m}^{-2}$ and $\rho_2 = 0.0006$ $\mathrm{min}^{-1}\mathrm{m}^{-2}$. The service times are $1/\nu_1 = 1$ min and $1/\nu_2 = 0.5$ min, so the mean number of users in the system is $\pi R^2 \rho_1/\nu_1 = 18.85$ for class 1 users and $\pi R^2 \rho_2/\nu_2 = 9.425$ for class 2 users. We consider the shadowing with $\sigma = \sqrt{10}$ dB and $\mu = 6$ dB. We have also considered $\beta_{min} = 0.2$

We made α varying from 1.6 to 1.8, by steps of 0.05. This corresponds here to an upper bound of loss probability varying between 0.0068 and 0.045 As the analytical expression obtained in the previous section, is an upper bound of the real loss probability, applying it to dimension an OFDMA cell will lead to an over dimensioning in terms of sub-channels. We have computed the number of sub-channels N_0 with the analytical expression of upper bound of loss probability. We have computed by simulation the number of sub-channels required if the upper bound probability is used as the loss probability to dimension the system.

Results of table III show the over dimensioning is about 20% in terms of sub-channels. At a first sight, this result can seem disappointing. We should note nevertheless that the computation of the upper bound and associated N_0 is immediate whereas the simulation on a fast PC is more tedious to get a decent confidence interval. The margin provided by the bounds may be viewed as a protection against errors in the modeling or in the estimates of the parameters.

α	1.6	1.65	1.7	1.75	1.8
P_{sup}	0.0445	0.0286	0.0180	0.0111	0.0068
N_0 obtained with the analytical upper bound	45.2	46.7	48	49.5	50.9
N_0 obtained by simulation for the same loss probability value as P_{sup}	38	39	40.4	41.6	42.8

TABLE III
DIFFERENCE IN TERMS OF SUB-CHANNELS OBTAINED BY SIMULATION AND ANALYTICALLY

VI. CONCLUDING REMARKS

Using the concentration and deviation inequalities and the difference operator on Poisson space, we have calculated the upper bound probability of overloading the system by high demand of sub-carriers, over path loss and shadow fading. To do this we have found the first and second moment of the marked Poisson point process of users. We conclude that it is possible to find an upper bound for the overloading probability, even in a relatively complex system, which is analytically computable in a very simple fashion. The method works for any functional of the configurations, possibly enriched by marks, which depends only on the positions of each user. It does not work for functionals involving relative distance between two or more users. Actually, for such a functional F, there is no bound on $D_x F(\omega)$ valid for all x and ω.

REFERENCES

[1] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. I.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2003. ISBN 0-387-95541-0. Elementary theory and methods.

[2] L. Decreusefond, H. Korezlioglu, and N. Van Dijk. An error bound for infinite approximation s of queuing networks with large finite stations. In R.O. Onvural and I.F. Akyildiz, editors, *Queuing Networks with finite capacity*, pages 239–252. North-Holland, 1993.

[3] C. Houdré and N. Privault. Concentration and deviation inequalities in infinite dimensions via covariance representations. *Bernoulli*, 8(6):697–720, 2002. ISSN 1350-7265.

[4] O. Kallenberg. *Random Measures*. Academic Press, 3rd edition, 1983.

[5] Olav Kallenberg. *Foundations of Modern Probability*. Springer-Verlag, 1998.

[6] A. F. Karr. *Point processes and their statistical inference*, volume 7 of *Probability: Pure and Applied*. Marcel Dekker Inc., New York, second edition, 1991. ISBN 0-8247-8532-0.

[7] P. Robert. *Stochastic networks and queues*, volume 52 of *Applications of Mathematics*. Springer-Verlag, Berlin, French edition, 2003. ISBN 3-540-00657-5. Stochastic Modeling and Applied Probability.

[8] L. Wu. A new modified logarithmic Sobolev inequality for Poisson point processes and several applications. *Probability Theory and Related Fields*, 118(3):427–438, 2000.