Three-User MIMO MACs with Cooperation

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***Abstract*—We study the three-user multi-antenna Gaussian multiple-access channel (MAC) where prior to the transmission over the MAC the transmitters can communicate with each other over noise-free broadcast pipes of given capacities. We present the capacity region of this channel. Additionally, we also study the three-user multi-antenna Gaussian MAC with common messages and present its capacity region.**

The main step in deriving these two capacity results consists in proving that Gaussian distributions maximize certain mutual information expressions under multiple Markov constraints. Towards this end, a tool previously used in [3], [6], [7] is extended to the vector case and to multiple Markov conditions.

I. Introduction

We study the three-user multi-antenna (MIMO) multiple-access channel (MAC), i.e., a scenario where three transmitters simultaneously wish to communicate with a common receiver and where the transmitters and the receiver are equipped with multiple antennas. We focus on the Gaussian MAC where the signal observed at the receiver is corrupted by additive white Gaussian noise. In this paper we consider two different such setups.

The first setup represents a generalization of the two-user MAC with conferencing encoders in [1], [6] to three transmitters and multiple antennas at the transmitters/receiver. More specifically, in this first setup the three transmitters wish to transmit independent messages over a MIMO Gaussian MAC. Prior to each block of transmission the transmitters can hold a *conference*, i.e., they can communicate with each other over noise-free bit-pipes of given capacities. We assume *broadcast* pipes, i.e., the symbols a given transmitter feeds to a pipe are received at both other transmitters. Such broadcast pipes model orthogonal wireless links (e.g., blue-tooth links occupying separate frequency bands) when all transmitters have approximately the same distance to each other. We refer to the described setup as the *three-user MIMO Gaussian MAC with broadcast conferencing*.

A three-user extension of the MAC with conferencing encoders [1] (for single-antenna transmitters/receiver) has already been studied in [8]. However, their model differs from ours in that in [8] the pipes are assumed to form a ring and the outputs can only be observed by a single intended transmitter.

Our second setup represents a Gaussian multi-antenna version of the three-user MAC with common messages in [5, Section 7]. In this setup, the three transmitters wish to communicate seven independent messages to the receiver where the messages are known to the transmitters as follows. The first message is known only to Transmitter 1 (but not to Transmitters 2 and 3), the second only to Transmitter 2, the third only to Transmitter 3, the fourth only to Transmitters 1 and 2 (but not to Transmitter 3), the fifth only to Transmitter 1 and 3, the sixth only to Transmitter 2 and 3, and finally the last messages is known to all transmitters. We refer to this setup as the *three-user MIMO Gaussian MAC with common messages*.

In this paper we determine the capacity region for both these described setups. The achievability part for the first setup with broadcast conferencing is based on the idea in [1] but extended to three users. The achievability part for the second setup with common messages is based on the multi-layer superposition coding in [5], [8]. The converse in the first setup is similar to [6], [7] and the converse in the second setup similar to [5]. However, both converses require extending a tool used in [3], [6], [7] to the vector-case and to multiple Markov conditions. That is, the converses require proving that Gaussian vector-distributions maximize certain mutual information expressions under multiple Markov constraints. For such maximization problems the traditional approach of proving the optimality of Gaussian distributions by employing the Max-Entropy Theorem [2, Theorem 12.1.1.] or a conditional version thereof [4] fails, because replacing a non-Gaussian vector satisfying given Markovity conditions by a Gaussian vector of the same covariance matrix may result in a Gaussian vector that violates the Markovity conditions.

Before defining the two setups in more detail and presenting their capacity regions (Sections II and III ahead), we define the three-user MIMO Gaussian MAC.

We assume that Transmitter 1 is equipped with t_1 transmit antennas, Transmitter 2 with t_2 transmit antennas, Transmitter 3 with t_3 transmit antennas, and the receiver with r receive antennas. The time-t channel input at Transmitters 1, 2, and 3 is then described by the t_1-dimensional random vector $\mathbf{X}_{1,t}$, the t_2-dimensional random vector $\mathbf{X}_{2,t}$, and the t_3-dimensional random vector $\mathbf{X}_{3,t}$. Similarly, the time-t channel output observed at the receiver is described by the r-dimensional real random vector $\mathbf{Y}_t$. In this paper all quantities are assumed to be real.

To describe the MAC we introduce the fixed, time-invariant, channel matrices H_1, H_2, and H_3, where H_ν, for $\nu \in \{1,2,3\}$, is of dimensions $r \times t_\nu$. Then, for given time-t channel inputs $\mathbf{x}_{1,t}$, $\mathbf{x}_{2,t}$, and $\mathbf{x}_{3,t}$ at Transmitters 1, 2, and 3, respectively, the time-t channel output $\mathbf{Y}_t$ can be described by

$$\mathbf{Y}_t \quad = \quad \mathsf{H}_1\mathbf{x}_{1,t} + \mathsf{H}_2\mathbf{x}_{2,t} + \mathsf{H}_3\mathbf{x}_{3,t} + \mathbf{Z}_t, \tag{1}$$

where the sequence $\{\mathbf{Z}_t\}$ consists of independent and identically distributed (IID) r-dimensional zero-mean Gaussian vectors of covariance matrix equal to the identity matrix I_r.

We impose average block power constraints $P_1, P_2, P_3 \geq 0$ on the channel input sequences, i.e., when n denotes the blocklength of transmission it is required that

$$\frac{1}{n}\sum_{t=1}^{n} \|\mathbf{x}_{\nu,t}\|^2 \leq P_\nu, \qquad \nu \in \{1,2,3\}. \tag{2}$$

II. Three-User MIMO Gaussian MAC with Broadcast Conferencing

A. Setting

In this first setup, Transmitters 1, 2, and 3 wish to communicate their messages M_1, M_2, and M_3 over the MIMO Gaussian MAC described in Section I. The messages are assumed to be independent of each other, and Message M_ν, for $\nu \in \{1,2,3\}$, is assumed to be uniformly distributed over the discrete finite set $\mathcal{M}_\nu = \{1, \ldots, \lfloor e^{nR_\nu} \rfloor\}$. Here R_1, R_2, and R_3 denote the rates of transmission in nats per channel use.

Prior to each block of n channel uses, the three transmitters hold a conference. That means, they exchange information over k uses of three broadcast pipes, one broadcast pipe from each transmitter to both other transmitters. The three pipes are assumed to be

- perfect in the sense that any input symbol to a pipe is available immediately and error-free at the two outputs of the pipe; and
- of limited throughputs C_1, C_2, and C_3, in the sense that when the k inputs to the pipe from Transmitter ν take values in the sets $\mathcal{V}_{\nu,1}, \ldots, \mathcal{V}_{\nu,k}$, for $\nu \in \{1,2,3\}$ then

$$\sum_{\ell=1}^{k} \log |\mathcal{V}_{\nu,\ell}| \leq nC_\nu, \qquad \nu \in \{1,2,3\}. \tag{3}$$

Here and throughout all logarithms are natural logarithms.

Note that the communication over the pipes is assumed to be held in a conferencing way, so that the ℓ-th inputs $V_{1,\ell} \in \mathcal{V}_{1,\ell}, V_{2,\ell} \in \mathcal{V}_{2,\ell}$, and $V_{3,\ell} \in \mathcal{V}_{3,\ell}$ can depend on the respective messages as well as on the past observed pipe-outputs at the corresponding transmitter, i.e.,

$$V_{1,\ell} = f_{1,\ell}\left(M_1, V_{2,1}, \ldots, V_{2,\ell-1}, V_{3,1}, \ldots, V_{3,\ell-1}\right), \tag{4}$$
$$V_{2,\ell} = f_{2,\ell}\left(M_2, V_{1,1}, \ldots, V_{1,\ell-1}, V_{3,1}, \ldots, V_{3,\ell-1}\right), \tag{5}$$
$$V_{3,\ell} = f_{3,\ell}\left(M_3, V_{1,1}, \ldots, V_{1,\ell-1}, V_{2,1}, \ldots, V_{2,\ell-1}\right), \tag{6}$$

for some given sequences of encoding functions $\{f_{1,\ell}\}_{\ell=1}^{k}, \{f_{2,\ell}\}_{\ell=1}^{k}$, and $\{f_{3,\ell}\}_{\ell=1}^{k}$.

Define an (n, C_1, C_2, C_3)-*conference* to be the collection of:

- an integer number k,
- three sets of input alphabets $\{\mathcal{V}_{1,1}, \ldots, \mathcal{V}_{1,k}\}$, $\{\mathcal{V}_{2,1}, \ldots, \mathcal{V}_{2,k}\}$, and $\{\mathcal{V}_{3,1}, \ldots, \mathcal{V}_{3,k}\}$,
- and three sets of encoding functions $\{f_{1,1}, \ldots, f_{1,k}\}$, $\{f_{2,1}, \ldots, f_{2,k}\}$, and $\{f_{3,1}, \ldots, f_{3,k}\}$

such that the parameters n, k, C_1, C_2, C_3, and the sets $\{\mathcal{V}_{1,1}, \ldots, \mathcal{V}_{1,k}\}$, $\{\mathcal{V}_{2,1}, \ldots, \mathcal{V}_{2,k}\}$, and $\{\mathcal{V}_{3,1}, \ldots, \mathcal{V}_{3,k}\}$ satisfy (3).

Define the sequences $\mathbf{V}_1 = (V_{1,1}, \ldots, V_{1,k})$, $\mathbf{V}_2 = (V_{2,1}, \ldots, V_{2,k})$, and $\mathbf{V}_3 = (V_{3,1}, \ldots, V_{3,k})$. After the conference Transmitter 1 is cognizant of the sequences $\mathbf{V}_2$ and $\mathbf{V}_3$, Transmitter 2 of $\mathbf{V}_1$ and $\mathbf{V}_3$, and Transmitter 3 of $\mathbf{V}_1$ and $\mathbf{V}_2$. Thus, the time-t channel inputs are generated as $\mathbf{X}_{1,t} = \varphi_{1,t}^{(n)}(M_1, \mathbf{V}_2, \mathbf{V}_3)$, $\mathbf{X}_{2,t} = \varphi_{2,t}^{(n)}(M_2, \mathbf{V}_1, \mathbf{V}_3)$, and $\mathbf{X}_{3,t} = \varphi_{3,t}^{(n)}(M_3, \mathbf{V}_1, \mathbf{V}_2)$, for some sequences of encoding functions $\left\{\varphi_{1,t}^{(n)}\right\}_{t=1}^{n}$, $\left\{\varphi_{2,t}^{(n)}\right\}_{t=1}^{n}$, and $\left\{\varphi_{3,t}^{(n)}\right\}_{t=1}^{n}$ satisfying the power constraints (2).

Based on the output sequence $(\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$ the decoder applies a decoding function $\phi^{(n)}$,

$$\phi^{(n)}: \quad \mathbb{R}^{n\times r} \to \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3, \tag{7}$$

to produce the message estimates $\hat{M}_1, \hat{M}_2$, and $\hat{M}_3$, i.e.,

$$(\hat{M}_1, \hat{M}_2, \hat{M}_3) = \phi^{(n)}(\mathbf{Y}_1, \ldots, \mathbf{Y}_n). \tag{8}$$

An error occurs whenever $(M_1, M_2, M_3) \neq (\hat{M}_1, \hat{M}_2, \hat{M}_3)$.

A rate triple (R_1, R_2, R_3) is said to be *achievable* over the three-user MIMO Gaussian MAC with broadcast conferencing if there exist a sequence of $\{(n, C_1, C_2, C_3)\}$-conferences, a sequence of encoding functions $\left\{\{\varphi_{1,t}^{(n)}\}_{t=1}^{n}, \{\varphi_{2,t}^{(n)}\}_{t=1}^{n}, \{\varphi_{3,t}^{(n)}\}_{t=1}^{n}\right\}$ satisfying the power constraints (2), and a sequence of decoding functions $\{\phi^{(n)}\}$ such that the probability of error tends to 0 as the blocklength n tends to infinity, i.e.,

$$\lim_{n\to\infty} \Pr\Big[(M_1, M_2, M_3) \neq (\hat{M}_1, \hat{M}_2, \hat{M}_3)\Big] = 0. \tag{9}$$

The *capacity region* $\mathsf{C}_{\text{Conf}}(P_1, P_2, P_3; C_1, C_2, C_3)$ is defined as the closure of the set of all achievable rate triples.

B. Results

Definition 1: Given a t_1-dimensional random vector $\mathbf{X}_1$, a t_2-dimensional random vector $\mathbf{X}_2$, a t_3-dimensional random vector $\mathbf{X}_3$, and a finite-dimensional random vector $\mathbf{U}$, define the rate region

$$\begin{aligned}
&\mathcal{R}_{\text{Conf}}(\mathbf{U}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \\
&\triangleq \Big\{(R_1, R_2, R_3): \\
&\qquad R_1 \leq I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2\mathbf{X}_3\mathbf{U}) + C_1, \\
&\qquad R_2 \leq I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1\mathbf{X}_3\mathbf{U}) + C_2, \\
&\qquad R_3 \leq I(\mathbf{X}_3; \mathbf{Y}|\mathbf{X}_1\mathbf{X}_2\mathbf{U}) + C_3, \\
&\qquad R_1 + R_2 \leq I(\mathbf{X}_1\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_3\mathbf{U}) + C_1 + C_2, \\
&\qquad R_1 + R_3 \leq I(\mathbf{X}_1\mathbf{X}_3; \mathbf{Y}|\mathbf{X}_2\mathbf{U}) + C_1 + C_3, \\
&\qquad R_2 + R_3 \leq I(\mathbf{X}_2\mathbf{X}_3; \mathbf{Y}|\mathbf{X}_1\mathbf{U}) + C_2 + C_3, \\
&\qquad R_1 + R_2 + R_3 \leq I(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3; \mathbf{Y}|\mathbf{U}) + C_1 + C_2 + C_3, \\
&\qquad R_1 + R_2 + R_3 \leq I(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3; \mathbf{Y}) \Big\}
\end{aligned}$$

where $\mathbf{Y} \triangleq \mathsf{H}_1\mathbf{X}_1 + \mathsf{H}_2\mathbf{X}_2 + \mathsf{H}_3\mathbf{X}_3 + \mathbf{Z}$ and $\mathbf{Z}$ is zero-mean Gaussian of covariance matrix I_r and independent of the tuple $(\mathbf{U}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$.

Definition 2: Given powers $P_1, P_2, P_3 \geq 0$ and pipe capacities $C_1, C_2, C_3 \geq 0$, define the rate region

$$\mathcal{C}_{\text{Conf},\mathcal{G}}(P_1, P_2, P_3; C_1, C_2, C_3) \triangleq \bigcup \mathcal{R}_{\text{Conf}}\big(\mathbf{U}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\big),$$

where the union on the right-hand side is taken over all *jointly Gaussian* random vectors $\mathbf{U}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ such that $\mathbf{U}$ is of dimension $(t_1 + t_2 + t_3)$ and such that the Markov conditions

$$\mathbf{X}_\nu \multimap \mathbf{U} \multimap \mathbf{X}_{\nu'}, \qquad \nu, \nu' \in \{1,2,3\} \text{ and } \nu \neq \nu',$$

and the trace constraints

$$\operatorname{tr}(\mathsf{K}_{\mathbf{X}_\nu}) \leq P_\nu, \qquad \nu \in \{1,2,3\},$$

are satisfied, when $\mathsf{K}_{\mathbf{X}_\nu}$ denotes the covariance matrix of the vector $\mathbf{X}_\nu$.

Theorem 1: Given powers $P_1, P_2, P_3 \geq 0$ and pipe capacities $C_1, C_2, C_3 \geq 0$, the capacity region $\mathcal{C}_{\text{Conf}}(P_1, P_2, P_3; C_1, C_2, C_3)$ of the three-user MIMO Gaussian MAC with broadcast conferencing coincides with the region $\mathcal{C}_{\text{Conf},\mathcal{G}}(P_1, P_2, P_3; C_1, C_2, C_3)$. Thus,

$$\begin{aligned}
&\mathcal{C}_{\text{Conf}}(P_1, P_2, P_3; C_1, C_2, C_3) \\
&= \bigcup_{\mathsf{A}_1,\mathsf{A}_2,\mathsf{A}_3,\mathsf{B}_1,\mathsf{B}_2,\mathsf{B}_3} \Bigg\{ (R_1, R_2, R_3) : \\
&R_1 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}|\right) + C_1, \\
&R_2 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}|\right) + C_2, \\
&R_3 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_3\mathsf{B}_3\mathsf{B}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T}|\right) + C_3, \\
&R_1 + R_2 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}|\right) \\
&\qquad + C_1 + C_2, \\
&R_1 + R_3 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_3\mathsf{B}_3\mathsf{B}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T}|\right) \\
&\qquad + C_2 + C_3, \\
&R_2 + R_3 \leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} + \mathsf{H}_3\mathsf{B}_3\mathsf{B}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T}|\right) \\
&\qquad + C_2 + C_3, \\
&R_1 + R_2 + R_3 \\
&\leq \frac{1}{2}\log\left(|\mathsf{I} + \mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} + \mathsf{H}_3\mathsf{B}_3\mathsf{B}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T}|\right) \\
&\qquad + C_1 + C_2 + C_3, \\
&R_1 + R_2 + R_3 \\
&\leq \frac{1}{2}\log\big(\big|\mathsf{I} + \mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} + \mathsf{H}_3\mathsf{A}_3\mathsf{A}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T} \\
&\qquad + \mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} + \mathsf{H}_3\mathsf{B}_3\mathsf{B}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T} \\
&\qquad + \mathsf{H}_1\mathsf{A}_1\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} + \mathsf{H}_1\mathsf{A}_1\mathsf{A}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T} + \mathsf{H}_2\mathsf{A}_2\mathsf{A}_3^\mathsf{T}\mathsf{H}_3^\mathsf{T} \\
&\qquad + \mathsf{H}_2\mathsf{A}_2\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_3\mathsf{A}_3\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} + \mathsf{H}_3\mathsf{A}_3\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}\big|\big) \Bigg\},
\end{aligned}$$

where the union is over all $t_1 \times (t_1 + t_2 + t_3)$ matrices $\mathsf{A}_1, \mathsf{B}_1$, all $t_2 \times (t_1 + t_2 + t_3)$ matrices $\mathsf{A}_2, \mathsf{B}_2$, and all $t_3 \times (t_1 + t_2 + t_3)$ matrices $\mathsf{A}_3, \mathsf{B}_3$ such that the trace constraint $\operatorname{tr}(\mathsf{A}_\nu\mathsf{A}_\nu^\mathsf{T} + \mathsf{B}_\nu\mathsf{B}_\nu^\mathsf{T}) \leq P_\nu$ is satisfied, for $\nu \in \{1,2,3\}$.

III. Three-User MIMO Gaussian MAC with Common Messages

A. Setting

In this second setup, the goal is to communicate Messages M_0, M_1, M_2, M_3, M_{12}, M_{13}, and M_{23} over the three-user MIMO Gaussian MAC described in Section I. Messages M_0, M_1, M_2, M_3, M_{12}, M_{13}, M_{23} are assumed to be independent of each other and uniformly distributed over the discrete finite sets $\mathcal{M}_0 = \{1, \ldots, \lfloor e^{nR_0} \rfloor\}$, $\mathcal{M}_1 = \{1, \ldots, \lfloor e^{nR_1} \rfloor\}$, $\mathcal{M}_2 = \{1, \ldots, \lfloor e^{nR_2} \rfloor\}$, $\mathcal{M}_3 = \{1, \ldots, \lfloor e^{nR_3} \rfloor\}$, $\mathcal{M}_{12} = \{1, \ldots, \lfloor e^{nR_{12}} \rfloor\}$, $\mathcal{M}_{13} = \{1, \ldots, \lfloor e^{nR_{13}} \rfloor\}$, $\mathcal{M}_{23} = \{1, \ldots, \lfloor e^{nR_{23}} \rfloor\}$, respectively.

Transmitter 1 is cognizant of Messages M_0, M_1, M_{12}, M_{13}, Transmitter 2 is cognizant of Messages M_0 ,M_2, M_{12}, M_{23}, and Transmitter 3 is cognizant of Messages M_0, M_3, M_{13}, M_{23}. Thus, the time-t channel inputs are generated as

$$\begin{aligned}
\mathbf{X}_{1,t} &= \varphi_{1,t}^{(n)}(M_0, M_1, M_{12}, M_{13}), \\
\mathbf{X}_{2,t} &= \varphi_{2,t}^{(n)}(M_0, M_2, M_{12}, M_{23}), \\
\mathbf{X}_{3,t} &= \varphi_{3,t}^{(n)}(M_0, M_3, M_{23}, M_{13}),
\end{aligned}$$

for some sequences of encoding functions $\left\{\varphi_{1,t}^{(n)}\right\}_{t=1}^n$, $\left\{\varphi_{2,t}^{(n)}\right\}_{t=1}^n$, and $\left\{\varphi_{3,t}^{(n)}\right\}_{t=1}^n$ satisfying the power constraints (2).

The receiver decodes the messages by applying a decoding function $\phi^{(n)}$ on the output sequence $(\mathbf{Y}_1, \ldots, \mathbf{Y}_n)$, i.e., it produces the message estimates

$$(\hat{M}_0, \hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_{12}, \hat{M}_{13}, \hat{M}_{23}) = \phi^{(n)}(\mathbf{Y}_1, \ldots, \mathbf{Y}_n),$$

where

$$\phi^{(n)} : \quad \mathbb{R}^{r \times n} \to \mathcal{M}_0 \times \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \times \mathcal{M}_{12} \times \mathcal{M}_{13} \times \mathcal{M}_{23}.$$

An error occurs in the transmission whenever

$$(\hat{M}_0, \hat{M}_1, \hat{M}_2, \hat{M}_3, \hat{M}_{12}, \hat{M}_{13}, \hat{M}_{23}) \neq (M_0, M_1, M_2, M_3, M_{12}, M_{13}, M_{23}).$$

A rate tuple $(R_0, R_1, R_2, R_3, R_{12}, R_{13}, R_{23})$ is said to be achievable over the three-user MIMO Gaussian MAC with common messages if there exist a sequence of encoding functions $\left\{\left\{\varphi_{1,t}^{(n)}\right\}_{t=1}^n, \left\{\varphi_{2,t}^{(n)}\right\}_{t=1}^n, \left\{\varphi_{3,t}^{(n)}\right\}_{t=1}^n\right\}$ satisfying (2) and a sequence of decoding functions $\{\phi^{(n)}\}$ such that the average probability of error tends to 0 as the blocklength n tends to infinity. The capacity region $\mathcal{C}_{\text{3MAC}}(P_1, P_2, P_3)$ is defined as the closure of the set of all achievable rate tuples.

B. Results

Definition 3: Given a t_1-dimensional random vector $\mathbf{X}_1$, a t_2-dimensional random vector $\mathbf{X}_2$, a t_3-dimensional random vector $\mathbf{X}_3$, and finite-dimensional random vectors $\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}$, define the rate region

$$
\begin{aligned}
&\mathcal{R}_{\text{3MAC}}(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \\
&\triangleq \Big\{(R_0, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}) : \\
&\qquad R_1 \le I(\mathbf{X}_1; \mathbf{Y}|\mathbf{X}_2, \mathbf{X}_3, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}) \\
&\qquad R_2 \le I(\mathbf{X}_2; \mathbf{Y}|\mathbf{X}_1, \mathbf{X}_3, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{23}) \\
&\qquad R_3 \le I(\mathbf{X}_3; \mathbf{Y}|\mathbf{X}_1, \mathbf{X}_2, \mathbf{U}_0, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_1 + R_2 \le I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}|\mathbf{X}_3, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_1 + R_3 \le I(\mathbf{X}_1, \mathbf{X}_3; \mathbf{Y}|\mathbf{X}_2, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_2 + R_3 \le I(\mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{X}_1, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_1 + R_2 + R_3 \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_{12} + R_1 + R_2 \le I(\mathbf{X}_1, \mathbf{X}_2; \mathbf{Y}|\mathbf{X}_3, \mathbf{U}_0, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_{12} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{13}, \mathbf{U}_{23}) \\
&\qquad R_{13} + R_1 + R_3 \le I(\mathbf{X}_1, \mathbf{X}_3; \mathbf{Y}|\mathbf{X}_2, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{23}) \\
&\qquad R_{13} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{23}) \\
&\qquad R_{23} + R_2 + R_3 \le I(\mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{X}_1, \mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}) \\
&\qquad R_{23} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}) \\
&\qquad R_{12} + R_{13} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{23}) \\
&\qquad R_{12} + R_{23} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{13}) \\
&\qquad R_{13} + R_{23} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0, \mathbf{U}_{12}) \\
&\qquad R_{12} + R_{13} + R_{23} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}|\mathbf{U}_0) \\
&\qquad R_0 + R_{12} + R_{13} + R_{23} + R_1 + R_2 + R_3 \\
&\qquad\qquad \le I(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3; \mathbf{Y}), \qquad \Big\} \qquad (10)
\end{aligned}
$$

where $\mathbf{Y} \triangleq \mathsf{H}_1\mathbf{X}_1 + \mathsf{H}_2\mathbf{X}_2 + \mathsf{H}_3\mathbf{X}_3 + \mathbf{Z}$ and $\mathbf{Z}$ is zero-mean Gaussian of covariance matrix I_r and independent of the seven-tuple $(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$.

Definition 4: Given powers $P_1, P_2, P_3 \ge 0$, define the region

$$
\begin{aligned}
&\mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3) \\
&\qquad \triangleq \bigcup \mathcal{R}_{\text{3MAC}}(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3),
\end{aligned}
$$

where the union on the right-hand side is taken over all *jointly Gaussian* seven-tuples $(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ such that $\mathbf{U}_0$ is of dimension $(t_1+t_2+t_3)$, $\mathbf{U}_{12}$ of dimension (t_1+t_2), $\mathbf{U}_{13}$ of dimension (t_1+t_3), $\mathbf{U}_{23}$ of dimension (t_2+t_3), and $\mathbf{X}_\nu$ is of dimension t_ν, for $\nu \in \{1,2,3\}$, and such that the random vectors $\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}$ are independent of each other and the three Markov chains

$$
\begin{aligned}
\mathbf{X}_1 -\!\!\circ\!\!- (\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}) -\!\!\circ\!\!- (\mathbf{X}_2, \mathbf{X}_3, \mathbf{U}_{23}), \qquad (11)\\
\mathbf{X}_2 -\!\!\circ\!\!- (\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{23}) -\!\!\circ\!\!- (\mathbf{X}_1, \mathbf{X}_3, \mathbf{U}_{13}), \qquad (12)\\
\mathbf{X}_3 -\!\!\circ\!\!- (\mathbf{U}_0, \mathbf{U}_{13}, \mathbf{U}_{23}) -\!\!\circ\!\!- (\mathbf{X}_1, \mathbf{X}_2, \mathbf{U}_{12}), \qquad (13)
\end{aligned}
$$

and the power constraints

$$
\operatorname{tr}(\mathsf{K}_{\mathbf{X}_\nu}) \le P_\nu, \qquad \nu \in \{1,2,3\}, \qquad (14)
$$

are satisfied.

Theorem 2: Given powers $P_1, P_2, P_3 \ge 0$, the capacity region of the three-user MIMO Gaussian MAC with common messages $\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3)$ coincides with the rate region $\mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3)$, i.e.,

$$
\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3) = \mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3).
$$

IV. Proof of Theorem 2

The achievability of $\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3)$, i.e.,

$$
\mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3) \subseteq \mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3),
$$

follows by applying a multi-layer superposition scheme as in [5], [8, Section 7] and using vector-valued Gaussian distributions in the code construction. The details are omitted.

To prove the converse, i.e.,

$$
\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3) \subseteq \mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3),
$$

we first outer bound $\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3)$ by $\mathsf{C}_{\text{Out}}(P_1, P_2, P_3)$ (Lemma 1). The converse is then established by showing that

$$
\mathsf{C}_{\text{Out}}(P_1, P_2, P_3) = \mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3) \qquad (15)
$$

for all powers $P_1, P_2, P_3 \ge 0$. Notice that the regions $\mathsf{C}_{\text{Out}}(P_1, P_2, P_3)$ and $\mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3)$ differ only with respect to the tuples of random vectors over which the unions are taken: for $\mathsf{C}_{\text{Out}}(P_1, P_2, P_3)$ the union is taken over *all* seven-tuples satisfying the independence conditions, the Markov chains, and the trace constraints, and for $\mathsf{C}_{\text{3MAC},\mathcal{G}}(P_1, P_2, P_3)$ the union is only over those that are *Gaussian* and have appropriate dimensions. Thus, (15) can be shown by proving that for $\mathsf{C}_{\text{Out}}(P_1, P_2, P_3)$ it is sufficient to take the union only over those seven-tuples that are Gaussian and have appropriate dimensions. Hence, Lemma 2 ahead establishes the proof.

Definition 5: Given powers $P_1, P_2, P_3 \ge 0$, define the region

$$
\begin{aligned}
&\mathsf{C}_{\text{Out}}(P_1, P_2, P_3) \\
&\qquad \triangleq \bigcup \mathcal{R}_{\text{3MAC}}(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3),
\end{aligned}
$$

where the union on the right-hand side is taken over all (not necessarily Gaussian) seven-tuples of finite-dimensional independent random vectors $\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}$, t_1-dimensional random vectors $\mathbf{X}_1$, t_2-dimensional random vectors $\mathbf{X}_2$, and t_3-dimensional random vectors $\mathbf{X}_3$ satisfying the Markov chains (11)–(13) and the trace constraints (14).

Lemma 1: The region $\mathsf{C}_{\text{Out}}(P_1, P_2, P_3)$ includes the capacity region of the three-user MIMO Gaussian MAC with common messages:

$$\mathsf{C}_{\text{3MAC}}(P_1, P_2, P_3) \subseteq \mathsf{C}_{\text{Out}}(P_1, P_2, P_3).$$

Proof: Requires only a slight modification of the converse in [5] to account for the power constraints. ■

Lemma 2: Let a seven-tuple $(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ be given where $\mathbf{X}_\nu$ is of dimension t_ν, for $\nu \in \{1, 2, 3\}$, and where the random vectors $\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}$ are independent of each other and Conditions (11)–(14) are satisfied. There exists a jointly Gaussian tuple $(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}})$, where $\mathbf{V}_0^{\mathcal{G}}$ is of dimension $(t_1 + t_2 + t_3)$, $\mathbf{V}_{12}^{\mathcal{G}}$ of dimension $(t_1 + t_2)$, $\mathbf{V}_{13}^{\mathcal{G}}$ of dimension $(t_1 + t_3)$, $\mathbf{V}_{23}^{\mathcal{G}}$ of dimension $(t_2 + t_3)$, and $\mathbf{X}_\nu^{\mathcal{G}}$ of dimension t_ν, for $\nu \in \{1, 2, 3\}$ and where the following four conditions are satisfied:

1.) the random vectors $\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}$ are independent of each other;

2.) the three Markov chains

$$\mathbf{X}_1^{\mathcal{G}} \multimap (\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}) \multimap (\mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}), \quad (16)$$
$$\mathbf{X}_2^{\mathcal{G}} \multimap (\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}) \multimap (\mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}), \quad (17)$$
$$\mathbf{X}_3^{\mathcal{G}} \multimap (\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}) \multimap (\mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}), \quad (18)$$

hold;

3.) the random vectors $\mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}$ satisfy

$$\operatorname{tr}\left(\mathsf{K}_{\mathbf{X}_\nu^{\mathcal{G}}}\right) \leq P_\nu, \qquad \nu \in \{1, 2, 3\}; \quad (19)$$

4.) the following Inclusion (20) holds:

$$\mathcal{R}_{\text{3MAC}}(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \subseteq \mathcal{R}_{\text{3MAC}}(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}). \quad (20)$$

Proof of Lemma 2: Define

$$\mathbf{V}_0 \triangleq \begin{pmatrix} \mathsf{E}[\mathbf{X}_1|\mathbf{U}_0] \\ \mathsf{E}[\mathbf{X}_2|\mathbf{U}_0] \\ \mathsf{E}[\mathbf{X}_3|\mathbf{U}_0] \end{pmatrix},$$

and for all pairs $(\nu, \nu') \in \{(1, 2), (1, 3), (2, 3)\}$:

$$\mathbf{V}_{\nu\nu'} \triangleq \begin{pmatrix} \mathsf{E}[\mathbf{X}_\nu|\mathbf{U}_{\nu\nu'}, \mathbf{U}_0] - \mathsf{E}[\mathbf{X}_\nu|\mathbf{U}_0] \\ \mathsf{E}[\mathbf{X}_{\nu'}|\mathbf{U}_{\nu\nu'}, \mathbf{U}_0] - \mathsf{E}[\mathbf{X}_{\nu'}|\mathbf{U}_0] \end{pmatrix}.$$

Further, define the tuple $(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}})$ to be zero-mean jointly Gaussian of the same covariance matrix as the tuple $(\mathbf{V}_0, \mathbf{V}_{12}, \mathbf{V}_{13}, \mathbf{V}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$. The proof of the lemma is established by showing that the tuple $(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}})$ satisfies the desired Conditions 1.)–4.) in the lemma. This is sketched in the following.

The vectors $\mathbf{V}_0$, $\mathbf{V}_{12}$, $\mathbf{V}_{13}$, and $\mathbf{V}_{23}$ are deterministic functions of $\mathbf{U}_0$, of $(\mathbf{U}_{12}, \mathbf{U}_0)$, of $(\mathbf{U}_{13}, \mathbf{U}_0)$, and of $(\mathbf{U}_{23}, \mathbf{U}_0)$, respectively. Due to these functional relations the right-hand sides of the constraints in (10) can only increase when replacing the tuple $(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23})$ with $(\mathbf{V}_0, \mathbf{V}_{12}, \mathbf{V}_{13}, \mathbf{V}_{23})$. Consequently,

$$\mathcal{R}_{\text{3MAC}}(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \subseteq \mathcal{R}_{\text{3MAC}}(\mathbf{V}_0, \mathbf{V}_{12}, \mathbf{V}_{13}, \mathbf{V}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3). \quad (21)$$

Moreover, by the conditional max-entropy theorem in [4]:

$$\mathcal{R}_{\text{3MAC}}(\mathbf{V}_0, \mathbf{V}_{12}, \mathbf{V}_{13}, \mathbf{V}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) \subseteq \mathcal{R}_{\text{3MAC}}(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}),$$

which together with Inclusion (21) establishes Condition (20).

Since the triple $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ satisfies the trace constraints (14) and the triple $\mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}}$ is chosen zero-mean and of the same covariance matrix, we can further conclude that Constraints (19) in the lemma are satisfied.

We next prove that the random vectors $\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}$ are independent of each other. Since the tuple $(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}})$ is jointly Gaussian, it suffices to show pair-wise orthogonality. Indeed, e.g.,

$$\begin{aligned} \mathsf{E}\left[\mathbf{V}_{12}^{\mathcal{G}} \left(\mathbf{V}_{13}^{\mathcal{G}}\right)^{\mathsf{T}}\right] &= \mathsf{E}[\mathbf{V}_{12}\mathbf{V}_{13}^{\mathsf{T}}] \\ &= \mathsf{E}[\mathsf{E}[\mathbf{V}_{12}\mathbf{V}_{13}^{\mathsf{T}}|\mathbf{U}_0]] \\ &= \mathsf{E}[\mathsf{E}[\mathbf{V}_{12}|\mathbf{U}_0]\,\mathsf{E}[\mathbf{V}_{13}^{\mathsf{T}}|\mathbf{U}_0]] \\ &= 0, \end{aligned}$$

where in the first equality we used the fact that the pairs $(\mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}})$ and $(\mathbf{V}_{12}, \mathbf{V}_{13})$ have the same covariance matrix; in the third equality we used that $\mathbf{V}_{12}$ and $\mathbf{V}_{13}$ are independent conditional on $\mathbf{U}_0$; and in the fourth inequality we used that $\mathsf{E}[\mathbf{V}_{12}|\mathbf{U}_0] = \mathsf{E}[\mathbf{V}_{13}|\mathbf{U}_0] = 0$.

It remains to prove the Markov conditions (16)–(18) in the lemma. They can be shown using the fact that the tuples $(\mathbf{V}_0, \mathbf{V}_{12}, \mathbf{V}_{13}, \mathbf{V}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ and $(\mathbf{V}_0^{\mathcal{G}}, \mathbf{V}_{12}^{\mathcal{G}}, \mathbf{V}_{13}^{\mathcal{G}}, \mathbf{V}_{23}^{\mathcal{G}}, \mathbf{X}_1^{\mathcal{G}}, \mathbf{X}_2^{\mathcal{G}}, \mathbf{X}_3^{\mathcal{G}})$ have the same covariance matrix and the fact that for the original seven-tuple $(\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ the random vectors $\mathbf{U}_0, \mathbf{U}_{12}, \mathbf{U}_{13}, \mathbf{U}_{23}$ are independent of each other and Markov conditions (11)–(13) are satisfied. The details are omitted. ■

Acknowledgment

We would like to thank A. Lapidoth and S. I. Bross for helpful discussions.

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