Interference, Cooperation and Connectivity – A Degrees of Freedom Perspective

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***Abstract*—THIS PAPER IS ELIGIBLE FOR THE STUDENT PAPER AWARD. We explore the interplay between interference, cooperation and connectivity in heterogeneous wireless interference networks. Specifically, we consider a 4-user locally-connected interference network with pairwise clustered decoding and show that its degrees of freedom (DoF) are bounded above by $\frac{12}{5}$. Interestingly, when compared to the corresponding fully connected setting which is known to have $\frac{8}{3}$ DoF, the locally connected network is only missing interference-carrying links, but still has lower DoF, i.e., eliminating these interference-carrying links reduces the DoF. The $\frac{12}{5}$ DoF outer bound is obtained through a novel approach that translates insights from interference alignment over linear vector spaces into corresponding sub-modularity relationships between entropy functions.**

I. Introduction

The broadcast nature of the wireless medium gives rise to three fundamental aspects of wireless networks.

1) *Interference* – among concurrent transmissions. This is the greatest challenge faced by a wireless network.
2) *Cooperation* – among nodes, e.g., by relaying simultaneously overheard transmissions to their desired destinations. This is the greatest opportunity present in a wireless network.
3) *Local Connectivity* – enforced by wireless propagation path loss, it limits the range over which signals can be heard. Therefore, it limits both the harmful impact of interference and the benefits of cooperation.

Understanding the complex interplay between these three factors is essential to understanding the capacity limits of wireless networks. In this work we seek to illuminate a few interesting aspects of the problem by focusing on a specific network topology that involves all three elements.

A. The Problem

The network we consider is a K user interference network where transmitters $1, 2, \cdots, K$ wish to send independent messages $W_1, W_2, \cdots, W_K$ to their respective destination decoders $1, 2, \cdots, K$ over a heterogeneous two-hop network comprised of an intermediate stage of K receive nodes. Each source is heard by three receive nodes in its neighborhood through noisy wireless channels. Specifically, Source k is heard by Receivers $k-1, k, k+1$. Destination decoder k is able to access each of the received signals from Receivers k and $k+1$ through orthogonal, noiseless channels, which provides the destination decoder the opportunity for *pairwise clustered decoding*, i.e., the joint processing of the two received signals to decode its desired message. Note that clustered processing is also of practical interest because, by definition, it limits the scope of cooperation. The heterogeneous nature of the network lies in the assumption of noise-free, orthogonal communication links from the first hop receivers to the final destination decoders, e.g., wired links of much higher capacity than the wireless channels of the first hop, so that the bottleneck remains the first hop. For ease of exposition – i.e., in order to keep the network size small and to avoid edge effects at the same time – we assume a "wrap-around" model as the number of users is restricted to $K = 4$ and the user indices are interpreted *modulo 4* as shown in Fig. 1 (User 4 can be alternatively thought of as User 0, but we prefer to number the users as 1,2,3,4). In the full paper [8], we show that the results reported here remain valid in an extended network of K users without the wrap-around assumption, when K is large enough to ignore edge-effects.

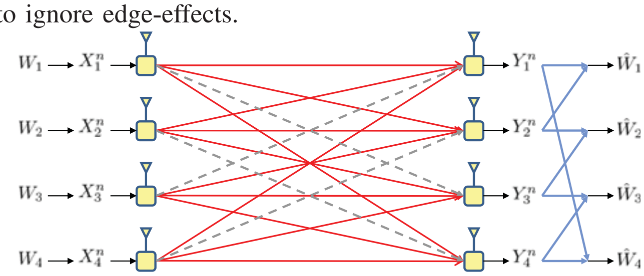

Fig. 1. 4 User Interference Channel with Pairwise Clustered Decoding

Fig. 1 shows the channel model with two kinds of links, indicated with solid red and dashed grey lines. We distinguish between two different connectivity settings.

1) **Locally Connected:** The network of Fig. 1 is locally connected if and only if all the channel coefficients corresponding to the solid red lines take non-zero values while the channel coefficients for the dashed grey lines are set to zero.
2) **Fully Connected:** The network of Fig. 1 is fully connected if and only if all the channel coefficients take non-zero values.

Problem Statement: Our goal in this paper is to explore the DoF of the network shown in Fig. 1 — to introduce new achievability and outer bounding techniques that might find use in other settings as well, and to distill interesting insights into the impact of connectivity on the DoF of the cooperative network.

B. Prior Work

Of the vast amount of literature on cooperative wireless networks, the most closely related to this work are references [1], [2], [3], [4], [5]. In particular, the fully connected setting of Fig. 1 is shown to have $\text{DoF} = \frac{8}{3}$ in [5]. This is identical to the 4-user SIMO (single input multiple output) interference channel setting without clustered decoding, where each user has one transmit and two receive antennas, and for which it is also established that the $\text{DoF} = 8/3$ in [1]. While the outer bound and achievable precoding scheme extend from [1] to [5] in a relatively straightforward manner, the proof of achievability of $\frac{8}{3}$ DoF in the fully connected clustered decoding setting of Fig. 1 is highly non-trivial, because of the strong spatial dependencies among desired and interference carrying channels (due to the sharing of receive antennas among decoders). Moving beyond the fully connected setting, clustered decoding with *local connectivity* is studied in [2], [3], [4], under a variety of Wyner-type connectivity patterns and decoding-cluster formations. However, the locally connected setting of Fig. 1 does not fall under any of the models addressed in these works. Furthermore, the tools used to obtain the DoF inner bounds and outer bounds in [2], [3], [4] appear to be insufficient to find the DoF of the pairwise clustered decoding model of Fig. 1.

C. Contribution

Our main contribution is an outer bound on the DoF of the locally connected setting of Fig. 1. Specifically, we show that *for every locally connected channel realization*, the interference network shown in Fig. 1, has $\text{DoF} \leq \mathbf{\frac{12}{5}}$.

Further, we show that this is *the best possible DoF outer bound* that could be valid for every locally connected channel realization, by constructing an explicit example where the outer bound is achieved.

D. Significance

We believe the contribution is interesting for two reasons.

First, as mentioned earlier in this section, the interplay between interference, cooperation and connectivity is of fundamental interest. This work illuminates some surprising aspects of the impact of connectivity on the DoF of cooperative interference networks. Specifically, our outer bound ($\text{DoF} \leq \frac{12}{5}$) for the locally connected setting (with only solid red channels present) is strictly *lower* than the result of [5] that $\text{DoF} = \frac{8}{3}$ for the fully connected setting. This is surprising because the missing links in the locally connected setting apparently carry only undesired interference. Yet, removing these interference-carrying links *reduces* the capacity, to the extent that even the DoF are strictly reduced. For instance, consider the dashed link from Transmitter 1 to Receiver 3. Since the signals from Receiver 3 are only available to Destination decoders 2 and 3, neither of which is interested in the message W_1 originating at Transmitter 1, it is somewhat surprising that removing this dashed channel reduces the DoF. Note that in the standard interference channel, removing interference carrying links cannot reduce capacity at all, much less reduce the DoF. A naive (and incorrect) explanation for the DoF loss could be that removing an interfering link reduces DoF because with two interfering links from the same undesired transmitter the decoder had the opportunity to cancel interference, which it cannot do with only one interfering link. Indeed, this is not the case. As we show in Section V, there is no DoF loss from removing the interfering links in a similar setting where each decoder has access to two receive antennas with the same local connectivity pattern but without the spatial dependencies introduced by *shared* receive antennas due to pairwise clustered decoding. Thus, the DoF outer bound is an illuminating indicator of the complex manner in which local connectivity impacts not only the capacity, but also the degrees of freedom of an interference network with collaborating nodes.

Second, the derivation of the outer bound itself follows a novel approach relative to prior work on related problems. While previously obtained DoF outer bounds in [2], [3], [5] follow from a common genie-aided multiple-access channel argument, the outer bound derived here fundamentally relies on the sub-modularity property of entropy functions, more commonly exploited in network coding converses [6], [7]. While the sub-modularity of entropy functions is an elementary property by itself, it is the manner in which this property is applied that is quite insightful. Specifically, first, linear vector space dimension counting arguments are formulated based on the understanding of the role of interference alignment in this problem, and then these arguments are translated into information theoretic inequalities based on the equivalent sub-modularity properties of vector spaces and entropy functions[1]. We expect that the general insights obtained in this work will be useful beyond the network of Fig. 1.

II. System model

We begin by specifying the assumptions for the 4-user locally connected interference network with pairwise clustered decoding, as shown in Fig. 1 and described in the previous section. We assume that channel coefficients corresponding to the solid red links in Fig. 1 are allowed to vary over time, can take *any non-zero values*, and that global channel state information (CSI) is available at all nodes. The channel coefficients for all the dashed grey links in Fig.1 are zero, i.e., these channels are not present.

The symbol received at Receiver k over the n^{th} channel use, $Y_k(n)$, is expressed as:

$$\begin{aligned} Y_k(n) &= H_{kk}(n)X_k(n) + H_{k,k-1}(n)X_{k-1}(n) \\ &\quad + H_{k,k-1}X_{k-1}(n) + Z_k(n) \end{aligned}$$

where $H_{ij}(n)$ is the channel coefficient from Transmitter j to first hop Receiver i, X_i is the symbol sent from Transmitter i and Z_k is the zero mean unit variance additive white Gaussian

[1]Due to space limitations, a description of the linear vector space alignment arguments that provide the intuition for the information theoretic converse is omitted here, and only the information theoretic converse is presented by itself. A detailed description of the insights behind this approach is relegated to the full paper [8].

noise (AWGN) signal experienced by Receiver k. The input signals are assumed to have power ρ. Because we are interested primarily in DoF, we use ρ and SNR interchangeably. The user index is interpreted in a cyclic wrap-around fashion.

There is pairwise clustered decoding so that the signals $Y_1^n,\ Y_2^n$ are jointly processed to decode message W_1; $Y_2^n,\ Y_3^n$ to decode W_2; $Y_3^n,\ Y_4^n$ to decode W_3; and $Y_4^n,\ Y_1^n$ to decode W_4.

The capacity region $\mathcal{C}(\rho)$ of this network is a set of achievable rate tuples $\mathbf{R}(\rho) = (R_1(\rho), \ldots, R_4(\rho))$ such that each user can simultaneously decode its own message with arbitrarily small error probability. The maximum sum rate of this channel is defined as $R(\rho) = \max_{\mathbf{R}(\rho)\in\mathcal{C}(\rho)} \sum_{k=1}^{4} R_k(\rho)$. The capacity in the high SNR regime can be characterized through DoF, i.e., DoF$= \lim_{\rho\to\infty} R(\rho)/\log\rho$.

Notation: We use the notation $o(x)$ to represent any function $f(x)$ such that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$.

III. DoF Outer Bound

Theorem 1: The 4-user locally connected interference channel has DoF $\leq \frac{12}{5}$ for all non-zero channel realizations.

Proof: Consider the achievable rate of User 1:

$$\begin{aligned}
nR_1 &\leq I(W_1; Y_1^n, Y_2^n) + o(n) && (1)\\
&\leq I(W_1; Y_1^n, Y_2^n|W_2) + o(n) && (2)\\
&= h(Y_1^n, Y_2^n|W_2) - h(Y_1^n, Y_2^n|W_1, W_2) + o(n) && (3)\\
&= h(Y_1^n, Y_2^n|W_2) - n(R_3 + R_4 + o(\log\rho)) + o(n) && (4)\\
&\leq h(Y_1^n|W_2) + h(Y_2^n|W_2)\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (5)\\
&\leq h(Y_1^n|W_1) + h(Y_1^n|W_4) - h(Y_1^n|W_1, W_4)\\
&\quad +h(Y_2^n|W_2) - n(R_3 + R_4 + o(\log\rho)) + o(n) && (6)\\
&= h(Y_1^n|W_1) + h(Y_1^n|W_4) - n(R_2 + o(\log\rho))\\
&\quad +h(Y_2^n|W_2) - n(R_3 + R_4 + o(\log\rho)) + o(n). && (7)
\end{aligned}$$

Here, (1) follows from Fano's inequality. (4) follows from the invertibility of upper/lower triangular channel matrices (regardless of the values of the channel coefficients as long as they are all non-zero), which implies that from the two output signals Y_1^n, Y_2^n, once we remove the signals due to W_1, W_2, we obtain an interference-free 2×2 MIMO channel to transmitters $3, 4$ which can be inverted to reconstruct the signals X_3^n, X_4^n with noise distortion that will depend on the channel coefficients but is independent of SNR. From these noisy inputs, one can construct signals statistically equivalent to the outputs (Y_3^n, Y_4^n, Y_1^n), again within noise tolerance that does not depend on SNR, and from these outputs, possibly by reducing noise by an amount independent of SNR, one can decode messages W_3, W_4. All these operations only have an $o(\log(\text{SNR}))$ impact on rate, and so we obtain $h(Y_1^n, Y_2^n|W_1, W_2) = h(Y_3^n, Y_4^n, Y_1^n|W_1, W_2) + n\ o(\log(\text{SNR})) = n(R_3 + R_4 + o(\log(\text{SNR}))) + o(n)$ as in (4). (5) follows from chain rule of differential entropy and because dropping conditioning cannot decrease differential entropy. (6) follows from Lemma 1 shown in the Appendix. Note that it is the use of Lemma 1 that invokes the submodularity property of entropy functions. The intuition that Lemma 1 should be applied in this manner comes from an interference alignment perspective, to be elaborated upon in the full paper. Finally, (7) follows from the observation that from Y_1^n, once all interference due to W_1, W_4 is removed, we obtain an interference-free AWGN channel to transmitter 2, from which W_2 can be decoded subject to noise reduction operations that only have an $o(\log(\text{SNR}))$ impact. Therefore $h(Y_1^n|W_1, W_4) = n(R_2 + o(\log(\text{SNR}))) + o(n)$ as in (7).

What we have so far is the first set of bounds:

$$\begin{aligned}
&n(R_1 + R_2 + R_3 + R_4 + o(\log\rho)) \leq\\
&\qquad h(Y_1^n|W_1) + h(Y_1^n|W_4) + h(Y_2^n|W_2) + o(n) && (8)\\
&n(R_2 + R_3 + R_4 + R_1 + o(\log\rho)) \leq\\
&\qquad h(Y_2^n|W_2) + h(Y_2^n|W_1) + h(Y_3^n|W_3) + o(n) && (9)\\
&n(R_3 + R_4 + R_1 + R_2 + o(\log\rho)) \leq\\
&\qquad h(Y_3^n|W_3) + h(Y_3^n|W_2) + h(Y_4^n|W_4) + o(n) && (10)\\
&n(R_4 + R_1 + R_2 + R_3 + o(\log\rho)) \leq\\
&\qquad h(Y_4^n|W_4) + h(Y_4^n|W_3) + h(Y_1^n|W_1) + o(n) && (11)
\end{aligned}$$

where (8) is obtained by rearranging the terms in (7) and the remaining inequalities (9) to (11) follow by symmetry by simply cyclically advancing the user indices.

Next we obtain the second set of bounds:

$$\begin{aligned}
nR_1 &\leq I(Y_1^n, Y_2^n; W_1) + o(n) && (12)\\
&= h(Y_1^n, Y_2^n) - h(Y_1^n, Y_2^n|W_1) + o(n) && (13)\\
&= h(Y_1^n, Y_2^n) - I(Y_1^n, Y_2^n; W_2|W_1)\\
&\quad -h(Y_1^n, Y_2^n|W_1, W_2) + o(n) && (14)\\
&\leq h(Y_1^n, Y_2^n) - I(Y_1^n; W_2|W_1)\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (15)\\
&= h(Y_1^n, Y_2^n) - h(Y_1^n|W_1) + h(Y_1^n|W_1, W_2)\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (16)\\
&= h(Y_1^n, Y_2^n) - h(Y_1^n|W_1) + n(R_4 + o(\log\rho))\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (17)\\
&\leq 2n(\log\rho + o(\log\rho)) - h(Y_1^n|W_1)\\
&\quad -n(R_3 + o(\log\rho)) + o(n) && (18)
\end{aligned}$$

and again starting from (14) in an alternative fashion:

$$\begin{aligned}
nR_1 &\leq h(Y_1^n, Y_2^n) - I(Y_1^n, Y_2^n; W_2|W_1)\\
&\quad -h(Y_1^n, Y_2^n|W_1, W_2) + o(n) && (19)\\
&\leq h(Y_1^n, Y_2^n) - I(Y_2^n; W_2|W_1)\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (20)\\
&= h(Y_1^n, Y_2^n) - h(Y_1^n|W_1) + h(Y_2^n|W_1, W_2)\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (21)\\
&= h(Y_1^n, Y_2^n) - h(Y_1^n|W_1) + n(R_3 + o(\log\rho))\\
&\quad -n(R_3 + R_4 + o(\log\rho)) + o(n) && (22)\\
&\leq 2n(\log\rho + o(\log\rho)) - h(Y_2^n|W_1)\\
&\quad -n(R_4 + o(\log\rho)) + o(n). && (23)
\end{aligned}$$

Remark: In arriving at (17) we use the substitution $h(Y_1^n|W_1, W_2) = n(R_4 + o(\log(\mathrm{SNR}))) + o(n)$. This is derived explicitly in the Appendix as Lemma 2. Similar substitution is made in arriving at (22) as well. These substitutions make use of the assumption that the mapping from messages W_i to the codewords X_i^n is deterministic and invertible. It is evident that this assumption does not incur any loss of generality in our setting, where it can be argued that the best deterministic codebook will perform at least as well as a randomized coding scheme.

Thus, we have the second set of inequalities:

$$h(Y_1^n|W_1) \leq n(2\log\rho - R_1 - R_3 + o(\log\rho)) + o(n) \quad (24)$$
$$h(Y_2^n|W_1) \leq n(2\log\rho - R_1 - R_4 + o(\log\rho)) + o(n) \quad (25)$$
$$h(Y_2^n|W_2) \leq n(2\log\rho - R_2 - R_4 + o(\log\rho)) + o(n) \quad (26)$$
$$h(Y_3^n|W_2) \leq n(2\log\rho - R_2 - R_1 + o(\log\rho)) + o(n) \quad (27)$$
$$h(Y_3^n|W_3) \leq n(2\log\rho - R_3 - R_1 + o(\log\rho)) + o(n) \quad (28)$$
$$h(Y_4^n|W_3) \leq n(2\log\rho - R_3 - R_2 + o(\log\rho)) + o(n) \quad (29)$$
$$h(Y_4^n|W_4) \leq n(2\log\rho - R_4 - R_2 + o(\log\rho)) + o(n) \quad (30)$$
$$h(Y_1^n|W_4) \leq n(2\log\rho - R_4 - R_3 + o(\log\rho)) + o(n) \quad (31)$$

where (24), (25) are rearranged forms of (18), (23), respectively, and the remaining inequalities (26) to (31) are the symmetric versions of (24), (25) obtained by cyclically advancing the indices.

Substituting the right-hand side of the inequalities (24) to (31) wherever the corresponding left-hand side appears in the inequalities (8) to (11), and adding up all the inequalities, we obtain:

$$4n(R+o(\log\rho)) \leq 12(2n\log\rho) - 6n(R+o(\log\rho)) + o(n) \quad (32)$$

where $R = R_1 + R_2 + R_3 + R_4$. Rearranging terms, dividing by n and taking the limit $n \to \infty$ we have:

$$4R \leq 24\log\rho - 6R + o(\log\rho). \quad (33)$$

Now applying the limit $\rho \to \infty$ on the sum rate outer bound, we have the DoF outer bound:

$$\mathrm{DoF} = \lim_{\rho\to\infty} \frac{R}{\log\rho} \leq \frac{12}{5}. \quad (34)$$

■

IV. A Locally Connected Example with 12/5 DoF

The DoF outer bound shown in the previous section holds for all locally connected channel realizations. In this section we prove that a better outer bound is not possible in the same sense, by providing an example of a locally connected channel realization where 12/5 DoF are achieved.

For the locally connected 4-user interference channel we consider in this article, consider a 5 symbol extension. So each transmitter and receiver has access to a 5×5 MIMO channel over 5 time lots. Note that these MIMO channels have a diagonal structure due to the nature of symbol extensions. In order to achieve 12/5 DoF, each user needs to send 3 symbols over 5 times slots. We artificially construct the channel matrices between each transmitter T_j and each receiver R_i as follows. The channel matrix from T_j to R_i is denoted as $\mathbf{H}_{ij}$. Here $\mathbf{I}$, $\mathbf{O}$ stand for the identity and zero matrices, respectively, and the diagonal matrix $\mathbf{G} = \mathrm{diag}([1\ 2\ 3\ 4\ 5])$.

	T_1	T_2	T_3	T_4
R_1	$\mathbf{I}$	$\mathbf{G}$	$\mathbf{O}$	$\mathbf{G}^2$
R_2	$\mathbf{G}^2$	$\mathbf{I}$	$\mathbf{G}$	$\mathbf{O}$
R_3	$\mathbf{O}$	$\mathbf{G}^2$	$\mathbf{I}$	$\mathbf{G}$
R_4	$\mathbf{G}$	$\mathbf{O}$	$\mathbf{G}^2$	$\mathbf{I}$

Fig. 2. The channel matrices for the artificial example

Each transmitter uses the same three beamforming vectors $\mathbf{w}$, $\mathbf{Gw}$ and $\mathbf{G}^2\mathbf{w}$ to send its three symbols $u_1^{[k]}$, $u_2^{[k]}$ and $u_3^{[k]}$ where $\mathbf{w} = [1\ 1\ 1\ 1\ 1]^T$ and k is the user index. In order to see how this scheme works, let us consider, without loss of generality, the received signal vector at user 1.

Besides its own output $\overline{Y}_1$ over 5 time slots, user 1 can also access its neighboring output $\overline{Y}_2$. Thus, user 1 is able to see a 10 dimensional space. The received signal of user 1 over the 5 time slots, $\mathbf{y}^{[1]} = \left[\overline{Y}_1^T\ \overline{Y}_2^T\right]^T$, is given by:

$$\mathbf{y}^{[1]} = \underbrace{\begin{bmatrix}\mathbf{H}_{11}\\ \mathbf{H}_{21}\end{bmatrix}\mathbf{x}^{[1]}}_{\text{desired signal}} + \underbrace{\begin{bmatrix}\mathbf{H}_{12}\\ \mathbf{H}_{22}\end{bmatrix}\mathbf{x}^{[2]} + \begin{bmatrix}\mathbf{H}_{13}\\ \mathbf{H}_{23}\end{bmatrix}\mathbf{x}^{[3]} + \begin{bmatrix}\mathbf{H}_{14}\\ \mathbf{H}_{24}\end{bmatrix}\mathbf{x}^{[4]}}_{\text{interference}} + \mathbf{z}^{[1]} \quad (35)$$

where $\mathbf{z}^{[1]}$ is the noise vector and $\mathbf{x}^{[k]}$ is the 5×1 transmit signal vector of user k which is given by:

$$\mathbf{x}^{[k]} = \underbrace{\left[\mathbf{w}\ \mathbf{Gw}\ \mathbf{G}^2\mathbf{w}\right]}_{\triangleq \mathbf{B}^{[k]}} \underbrace{\left[u_1^{[k]}\ u_2^{[k]}\ u_3^{[k]}\right]^T}_{\triangleq \mathbf{u}^{[k]}}. \quad (36)$$

In order to preserve a 3 dimensional space for user 1's desired signals, we need to align the 9 interference streams (3 per interferer) into a 7 dimensional space. In other words, we need to ensure that the rank of the following matrix $\mathbf{H_I}$, whose column vectors span the space occupied by the interference, is no larger than 7:

$$\mathbf{H_I} = \begin{bmatrix} \mathbf{H}_{12}\mathbf{B}^{[2]} & \mathbf{H}_{13}\mathbf{B}^{[3]} & \mathbf{H}_{14}\mathbf{B}^{[4]} \\ \mathbf{H}_{22}\mathbf{B}^{[2]} & \mathbf{H}_{23}\mathbf{B}^{[3]} & \mathbf{H}_{24}\mathbf{B}^{[4]} \end{bmatrix}. \quad (37)$$

Since the numerical values of all quantities are known, this is easily verified. It turns out that the second and third column vectors align in the space spanned by the fourth, fifth, seventh and eighth column vectors.

What remains to be shown is that the 3 dimensions carrying the three desired symbols are linearly independent with the remaining 7 column vectors carrying interference. For this, we prove the following matrix consisting of the ten column vectors (eliminating the second and third column vectors of $\mathbf{H_I}$ which align with the remaining interference) has full rank.

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_{11}\mathbf{B}^{[1]} & \mathbf{Gw} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^2\mathbf{w} & \mathbf{G}^3\mathbf{w} & \mathbf{G}^4\mathbf{w} \\ \mathbf{H}_{21}\mathbf{B}^{[1]} & \mathbf{w} & \mathbf{Gw} & \mathbf{G}^2\mathbf{w} & \mathbf{G}^3\mathbf{w} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \quad (38)$$

This is also easily verified by explicitly evaluating the determinant of this channel matrix. Similar analysis can be carried out to user 2, 3 and 4 due to cyclical symmetry in the construction of channels.

V. Channel Connectivity and Cooperation

As mentioned briefly in the introduction, we have shown that for the network shown in Fig. 1, the locally connected setting has strictly smaller DoF than the fully connected setting, even though the additional channel coefficients in the fully connected setting apparently carry only undesired interference.

In order to take a more refined look at this phenomenon, we investigate if the DoF loss is caused by the pairwise clustered cooperation at the receiver side, or local channel links connectivity. In Fig.3, we show an equivalent representation of the network in Fig.1 using the style of SIMO interference channel, but introducing spatial dependencies between some channel coefficients. Specifically, the channel coefficient associated with the second antenna of Receiver k is identical to that associated with the first antenna of Receiver $k+1$.

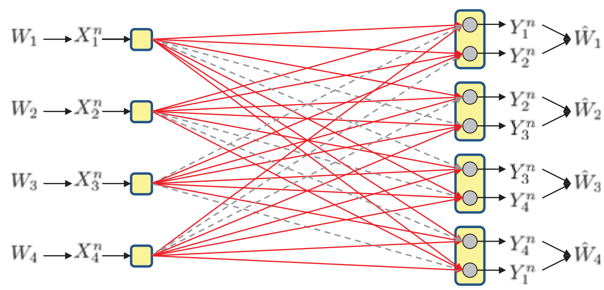

Fig. 3. 4 User Interference Channel with Cooperative Receivers

First, let us consider the impact of clustered decoding alone, by removing the locally connected assumption. Consider the network in Fig.3 in the fully connected setting, i.e., coefficients of all solid and dashed links are generic and non-zero. In this case, whether each decoder has access to two independent receive antennas (not shared with any other decoder), or the decoders share receive antennas as in Fig. 1, in both cases DoF = 8/3. Thus, the spatial dependencies between channel coefficients induced by clustered decoding (sharing of received antennas between decoders) do not not affect the DoF of this channel.

Second, we consider the role of local connectivity but without the spatial dependencies introduced by clustered decoding. Let us remove the dashed links in Fig.3. Again, if there are no spatial dependencies between any channel links, then it forms a classical SIMO interference channel with dashed links disconnected. Interestingly, we can show that for this network, the DoF is still equal to $8/3$, in spite of the local connectivity.

Thus, we find that the spatial dependencies caused by clustered decoding do not reduce the DoF if the channel is fully connected. Similarly, the local connectivity does not reduce the DoF if there are no spatial dependencies caused by clustered decoding (shared receive antennas across decoders). That is, individually, neither clustered decoding, nor local connectivity causes a loss of DoF. However, as shown by the outer bound, when taken together, the spatial dependencies caused by clustered decoding in conjunction with the local connectivity, translate into a DoF loss.

VI. Conclusion

We derived a new information theoretic outer bound on the degrees of freedom (DoF) for a 4-user locally connected interference channel with pairwise clustered decoding. Interestingly, we found that removing interference-carrying links decreases the DoF. The outer bound derivation incorporates novel elements and insights that may be useful beyond the problem considered in this work.

The DoF with generic channels (i.e., in the almost surely sense) remain open for the locally connected 4-user interference channel with pairwise clustered decoding. For more than 4 users and generic channels, the DoF remain open for both the fully connected and locally connected cases with pairwise clustered decoding, although it is known that there is a loss of DoF relative to the corresponding SIMO interference channel for the fully connected case. Interestingly, for more than 4 users there is a gap between the best outer bound valid for all non-zero channel realizations in the fully connected case (which continues to be 2/3 DoF per user and can be shown to be achievable for certain realizations) and smaller outer bounds that can be shown to be valid for *almost* all channel coefficients. For example, the fully connected K user interference network with M sized clustered decoding has a straightforward DoF outer bound of $1/2 + (M-1)/K$ per user (based on the usual multiple-access type outer bounding arguments) that is valid for almost all values of channel coefficients. Interestingly, this shows that the DoF per user in the fully connected setting lose all benefits of clustered decoding as the network size becomes large, for almost all channel realizations. Similar DoF characterizations for generic channel realizations in the locally connected setting remain a challenging open problem.

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Appendix

Lemma 1:

$$h(Y_1^n|W_2)+h(Y_1^n|W_1,W_4)\leq h(Y_1^n|W_1)+h(Y_1^n|W_4) \quad (39)$$

Proof:

$$\begin{aligned}
& h(Y_1^n|W_2)+h(Y_1^n|W_1,W_4) && (40)\\
= \ & 2h(Y_1^n)-I(W_2;Y_1^n)-I(Y_1;W_1,W_4) && (41)\\
\leq \ & 2h(Y_1^n)-I(Y_1;W_1,W_4) && (42)\\
= \ & 2h(Y_1^n)-I(Y_1^n;W_1)-I(Y_1;W_4|W_1) && (43)\\
\leq \ & 2h(Y_1^n)-I(Y_1^n;W_1)-I(Y_1;W_4) && (44)\\
= \ & h(Y_1^n|W_1)+h(Y_1^n|W_4) && (45)
\end{aligned}$$

where (44) follows from (43) because $I(A;B|C)\geq I(A;B)$ when A is independent of C. ■

Lemma 2:

$$h(Y_1^n|W_1,W_2)=n(R_4+o(\log(\text{SNR})))+o(n) \quad (46)$$

Proof: We use H_{ij} to denote the channel coefficient between Receiver i and Transmitter j. Z_i is the additive white Gaussian noise term at Receiver i.

$$\begin{aligned}
nR_4 &= H(W_4) && (47)\\
&= I(W_4;H_{14}X_4^n+Z_1^n,H_{44}X_4^n+Z_4^n)\\
&\quad +H(W_4|H_{14}X_4^n+Z_1^n,H_{44}X_4^n+Z_4^n) && (48)\\
&= I(W_4;H_{14}X_4^n+Z_1^n,H_{44}X_4^n+Z_4^n)+o(n) && (49)\\
&= I(W_4;H_{14}X_4^n+Z_1^n)\\
&\quad +I(W_4;H_{44}X_4^n+Z_4^n|H_{14}X_4^n+Z_1^n)+o(n) && (50)\\
&= I(W_4;H_{14}X_4^n+Z_1^n)+n\ o(\log(\text{SNR}))+o(n) && (51)\\
&= I(W_4;Y_1^n|W_1,W_2)+n\ o(\log(\text{SNR}))+o(n) && (52)\\
&= h(Y_1^n|W_1,W_2)-h(Y_1^n|W_1,W_2,W_4)\\
&\quad +n\ o(\log(\text{SNR}))+o(n) && (53)\\
&= h(Y_1^n|W_1,W_2)-h(Z_1^n)+n\ o(\log(\text{SNR}))+o(n) && (54)\\
&= h(Y_1^n|W_1,W_2)+n\ o(\log(\text{SNR}))+o(n) && (55)
\end{aligned}$$

where (49) follows from Fano's inequality, (52) follows from the assumption of deterministic and invertible mapping from messages to codewords. Note that $o(\log(\text{SNR}))$ terms in this set of equations can be described more tightly as $O(1)$ terms, i.e., bounded terms that do not increase with SNR. However, for our purpose, since we are primarily interested in the DoF, it suffices to highlight only their $o(\log(\text{SNR}))$ character. ■