Quantum Sensing and Information-Theoretic Foundations

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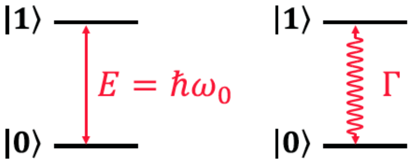
Quantum Sensing: Principles and Advantages



- Can detect changes of quantum phenomena (below atomic level)
- Changes often related to electric and magnetic fields (but also gravity, displacement, pressure)
  $\rightarrow$ imply changes in energy levels, spins, photons
- Can also detect quantum phenomena such as entaglement $\rightarrow$ requires preparation and possibly transmission of entangled states

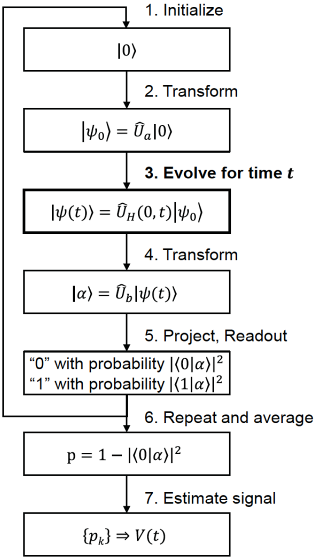
Quantum Sensing Technologies

- Atomic vapor/cold clouds (qubits: spin)
- Solid state spins: NMR sensor, P donor in Si, Semiconductor quantum dots (qubits: nuclear/electron spin)
- Elementary particles: muons, neutrons (qubits: spin)
- Supraconducting circuits (qubits: currents, charges of eigenstates)
- Interferometers (qubits: photons)

Quantum Sensing Applications

- Nuclear magnetic resonance spectrography/imaging
- Quantum magnetic encephalography
- Highresolution spectroscopy
- Material testing
- Gravimetry
- Radar
- Thermometry
- Ghost imaging

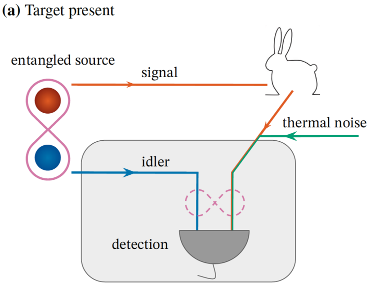
Ex 1: The Ramsey Sensing Protocol



Ramsey-protocol:

- $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
- Wish to estimate phase shift:
  $|\psi(t)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\omega_0 t}\,|1\rangle))$
- Prepare for measurement:
  $|\alpha\rangle = \frac{1}{2}(1+e^{-i\omega_0 t})\,|0\rangle + \frac{1}{2}(1-e^{-i\omega_0 t})\,|1\rangle$
- Readout probability:
  $p = \sin^2(\omega_0 t/2) = \frac{1}{2}(1 - \cos(\omega_0 t))$

Ex 2: Quantum Illumination for Radar Applications



- Detector: perform measurements to detect entanglement
- Repeat experiment many times and apply classical hypothesis test on the measurement data to decide on presence of obstacle
- Practical challenges: losses, decoherence/storage of idlers

Information-Theoretic Foundations

Basic Notions and Measurements

- The state of a system is described by a density matrix $\rho \succeq 0$ with trace 1
- Operations we can perform on a system (Kraus operators)

$$\rho_{\text{out}} = \sum_{i=1}^{d_{\text{in}} \cdot d_{\text{out}}} \mathsf{K}_i \rho_{\text{in}} \mathsf{K}_i^*, \qquad \text{for} \sum_{i=1}^{d_{\text{in}} \cdot d_{\text{out}}} \mathsf{K}_i^* \mathsf{K}_i = \mathsf{I}$$

- Measurements$\rightarrow$ output is a real value $x \in \mathcal{X}$:

$$\Pr(x) = \operatorname{tr}\left(\sum_{i=1}^{d_{\text{in}}} \mathsf{M}_{x,i} \rho_{\text{in}} \mathsf{M}_{x,i}^*\right) = \operatorname{tr}\left(\sum_{i=1}^{d_{\text{in}}} \mathsf{M}_{x,i}^* \mathsf{M}_{x,i} \cdot \rho_{\text{in}}\right) = \operatorname{tr}\left(\Lambda_x \rho_{\text{in}}\right)$$

where $\sum_x \Lambda_x = \mathsf{I}$.

The Symmetric Detection Problem

- If $\mathsf{H} = 0$ then the system has density matrix ρ
- If $\mathsf{H} = 1$ then the system has density matrix σ
- Apply measurement $\{\Lambda_0, \Lambda_1 = \mathrm{I} - \Lambda_0\}$ to determine $\hat{\mathsf{H}} \in \{0, 1\}$
- Type-I and II error prob.: $\alpha = 1 - \mathrm{tr}\,(\Lambda_0 \rho)$ and $\beta = \mathrm{tr}\,(\Lambda_0 \sigma)$
- Wish to minimize $\alpha + \beta = 1 - \mathrm{tr}\,(\Lambda_0(\rho - \sigma))$

Optimal Λ_0^*: Project on positive eigenspace of $\rho - \sigma$

$$\text{If } \rho - \sigma = \sum_i \eta_i \, |v_i\rangle\langle v_i| \qquad \text{then} \qquad \Lambda_0^* = \sum_{i:\ \eta_i > 0} |v_i\rangle\langle v_i| \, .$$

and $\qquad \alpha^* + \beta^* = 1 - \sum_{i:\ \eta_i > 0} \eta_i$

The Asymmetric Detection Problem

- If $\mathsf{H} = 0$ / $\mathsf{H} = 1$ the system has density matrix ρ / σ
- Maximize type-II error prob. under type-I error constraint $\alpha \leq \epsilon$:

$$\beta^*(\epsilon) = \inf_{\substack{\Lambda_0 \succ 0: \\ \mathsf{I}-\Lambda_0 \succeq 0 \\ \mathrm{tr}((\mathsf{I}-\Lambda_0)\rho) \leq \epsilon}} \mathrm{tr}\left(\Lambda_0 \sigma\right)$$

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- Semi-definite programme $\rightarrow$ dual problem

$$\beta^*(\epsilon) = \sup_{\substack{\mathsf{A} \succeq 0, \mu \geq 0 \\ \mathsf{A} \succeq \mu\rho - \sigma}} \left(\mathrm{tr}\left(\mu\rho\right) - \mathrm{tr}\left(\mathsf{A}\right) - \mu\epsilon\right)$$

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- Given μ, optimal A^*:

$$\mu\rho - \sigma = \sum_i \eta_i \left|v_i\right\rangle\!\left\langle v_i\right| \qquad \Rightarrow \qquad A^* = \sum_{i:\, \eta_i \geq 0} \eta_i \left|v_i\right\rangle\!\left\langle v_i\right|$$

The Quantum Neyman-Pearson Test

- Dual Problem: $\beta^*(\epsilon) = \sup_{\mu \geq 0} \left(\mathrm{tr}\left(\mu\rho\right) - \mathrm{tr}\left(\mathsf{A}^*\right) - \mu\epsilon \right)$

  for $\mathsf{A}^* = \sum_{i:\, \eta_i \geq 0} \eta_i \, |v_i\rangle\langle v_i|$ for (η_i, v_i) EVs of $(\mu\rho - \sigma)$

- Complementary Slackness Conditions: $\mu^* \mathrm{tr}\left((\mathsf{I} - \Lambda_0^*)\rho\right) = \mu^*\epsilon$ and

$$\mathsf{A}^*(\mathsf{I} - \Lambda_0^*) = 0 \qquad \text{and} \qquad (\mu^*\rho - \sigma - \mathsf{A}^*)\Lambda_0^* = 0$$

Solution:

Λ_0^* projects onto non-negative eigenspace of $(\mu^*\rho - \sigma)$ for a $\mu^* \geq 0$ that is adjusted in function of type-I constraint $\epsilon : \mathrm{tr}\left((\mathsf{I} - \Lambda_0^*)\rho\right) = \epsilon$

The Asymptotic Regime: Stein Exponent

- The asymptotic case with many observations
- If $\mathsf{H} = 0$ / $\mathsf{H} = 1$ the system has density matrix $\rho^{\otimes n}$ / $\sigma^{\otimes n}$

Stein exponent (with Umegaki's relative entropy

$$\lim_{n\to\infty} -\frac{1}{n} \log \beta_n^*(\epsilon) = D(\rho\|\sigma) := \operatorname{tr}\left(\rho(\log\rho - \log\sigma)\right)$$

[1] F. Hiai and D. Petz, "The Proper Formula for Relative Entropy and its Asymptotics in Quantum Probability," Commun. Math. Phys., 1991.

[2] T. Ogawa and H. Nagaoka, "Strong converse and Stein's lemma in the quantum hypothesis testing," Trans. IT 2000.

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- Consider spectral decomposition

$$\rho^{\otimes n} - e^{nt}\sigma^{\otimes n} = \sum_i \eta_i \underbrace{|v_i\rangle\langle v_i|}_{E_i}$$

- Optimal test is projection on nonnegative eigenspaces $\{E_i\}_{i:\ \eta_i \geq 0}$

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Sketch of the Converse Proof for Stein's Exponent

- **Relate to classical test between:** $\{p_i = \mathrm{tr}\left(\rho^{\otimes n} E_i\right)\}$ **and** $\{q_i = \mathrm{tr}\left(\rho^{\otimes n} E_i\right)\}$

- Define $K_n = \sum_{i:\ \eta_i > 0} E_i$

- Notice: $\mathrm{tr}\left(\rho^{\otimes n} K_n\right) = \sum_{i:\ \eta_i > 0} p_i \leq e^{-tsn} \sum_i p_i^{s+1} q_i^{-s}$

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- $$\begin{aligned} \forall t > 0 : \qquad \underbrace{\mathrm{tr}\left(\rho^{\otimes n}\Lambda_{0,n}\right)}_{\ge 1-\epsilon} &= \mathrm{tr}\left((\rho^{\otimes n} - e^{nt}\sigma^{\otimes n})\Lambda_{0,n}\right) + \mathrm{tr}\left(e^{nt}\sigma^{\otimes n}\Lambda_{0,n}\right) \\ &\le \mathrm{tr}\left((\rho^{\otimes n} - e^{nt}\sigma^{\otimes n})K_n\right) + \mathrm{tr}\left(e^{nt}\sigma^{\otimes n}\Lambda_{0,n}\right) \\ &\le \mathrm{tr}\left(\rho^{\otimes n}K_n\right) + e^{nt}\underbrace{\mathrm{tr}\left(\sigma^{\otimes n}\Lambda_{0,n}\right)}_{\beta_n} \end{aligned}$$

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$$\le e^{-tsn}\left(\mathrm{tr}\left(\rho^{1+s}\sigma^{-s}\right)\right)^n$$

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$$\begin{aligned}
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&\le \mathrm{tr}\left((\rho^{\otimes n} - e^{nt}\sigma^{\otimes n})K_n\right) + \mathrm{tr}\left(e^{nt}\sigma^{\otimes n}\Lambda_{0,n}\right) \\
&\le \mathrm{tr}\left(\rho^{\otimes n}K_n\right) + e^{nt}\underbrace{\mathrm{tr}\left(\sigma^{\otimes n}\Lambda_{0,n}\right)}_{\beta_n}
\end{aligned}$$

- Thus, finally: $-\frac{1}{n}\log\beta_n \le t - \frac{1}{n}\log(1-\epsilon-e^{-n\varphi(t)})$

for $\varphi(t) := \sup_{0\le s\le 1}\left(ts - \log\mathrm{tr}\left(\rho^{1+s}\sigma^{-s}\right)\right)$ and $\phi(t) > 0$ for $t \ge D(\rho\|\sigma)$

The Asymptotic Regime: Chernoff-Hoeffding Exponents

- If $\mathsf{H} = 0$ / $\mathsf{H} = 1$ the system has density matrix $\rho^{\otimes n}$ / $\sigma^{\otimes n}$

Chernoff exponent: Maximum symmetric exponent

$$\sup_{\{\Lambda_{0,n}\}} \lim_{n\to\infty} -\frac{1}{n}\log(\alpha_n + \beta_n) = \sup_{0\le s\le 1} -\log \operatorname{tr}\left(\sigma^s \rho^{1-s}\right)$$

[3] H. Nagaoka, "The converse part of the theorem for quantum Hoeffding bound," Arxiv, Nov. 2006
[4] K.M.R. Audenaert et al., "The Quantum Chernoff Bound," Phys. Review Letters 2007
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- Asymmetric Hoeffding exp: $\beta_{\mathrm{CH},n}(r) = \inf_{\substack{\Lambda_0: \\ \mathrm{tr}\left((\mathsf{I}-\Lambda_0)\rho^{\otimes n}\right)\le e^{-rn}}} \mathrm{tr}\left(\Lambda_0\sigma^{\otimes n}\right)$

Hoeffding exponent

$$\theta^*(r) = \lim_{n\to\infty} -\frac{1}{n}\log\beta^*_{\mathrm{CH},n}(r) = \sup_{0\le s\le 1} \frac{-rs-\log\mathrm{tr}\left(\sigma^s\rho^{1-s}\right)}{1-s}.$$

[3] H. Nagaoka, "The converse part of the theorem for quantum Hoeffding bound," Arxiv, Nov. 2006
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Proof Sketch of Converse to the Chernoff Bound

- Spectral decompositions $\rho = \sum_i \eta_i |x_i\rangle\langle x_i|$ and $\sigma = \sum_j \nu_j |y_j\rangle\langle y_j|$

- For any projectors Λ, (i.e., $\Lambda\Lambda = \Lambda$):

$$\begin{aligned}\beta = \quad & \operatorname{tr}(\Lambda\sigma) = \operatorname{tr}(\Lambda \mathsf{I} \Lambda\sigma) = \textstyle\sum_i \operatorname{tr}(\Lambda |x_i\rangle\langle x_i| \Lambda\sigma) \\ = & \sum_{j,i} \nu_j \operatorname{tr}(\Lambda |x_i\rangle\langle x_i| \Lambda |y_j\rangle\langle y_j|) = \sum_{j,i} \nu_j \left|\langle x_i| \Lambda |y_j\rangle\right|^2\end{aligned}$$

- Similarly: $\alpha = \sum_{j,i} \eta_i \left|\langle x_i| (\mathsf{I} - \Lambda) |y_j\rangle\right|^2$

- So:

$$\alpha + \beta = \sum_{j,i} \eta_i \left|\langle x_i| (\mathsf{I} - \Lambda) |y_j\rangle\right|^2 + \sum_{j,i} \nu_j \left|\langle x_i| \Lambda |y_j\rangle\right|^2$$

Proof Sketch of Converse to the Chernoff Bound II

Reduce to classical test using $\eta|u-v|^2+\nu|v|^2 \geq \frac{1}{2}|u|^2\min\{\eta,\nu\}$:

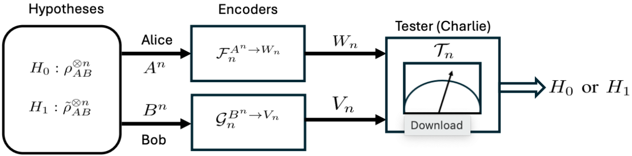
$$\begin{aligned}
\alpha+\beta &= \sum_{j,i}\eta_i\left|\langle x_i|\,\mathrm{I}\,|y_j\rangle - \langle x_i|\,\Lambda\,|y\rangle_j\right|^2 + \sum_{j,i}\nu_j\,|\langle x_i|\,\Lambda\,|y_j\rangle|^2 \\
&\geq \frac{1}{2}\sum_{j,i}\min\{\eta_i,\nu_j\}|\,\langle x_i|y_j\rangle\,|^2 = \frac{1}{2}\sum_{j,i}\min\{p_{ij},q_{ij}\}
\end{aligned}$$

- Cannot do better than optimal classical test between $\{p_{ij}=\eta_i\,\langle x_i|y_j\rangle\}$ and $\{q_{ij}=\nu_i\,\langle x_i|y_j\rangle\}$:

$$\Rightarrow \alpha_n+\beta_n \geq \frac{1}{2}P^*_{\text{class-HT}}(\boldsymbol{p}^n,\boldsymbol{q}^n)$$

- Use equality $-\log\operatorname{tr}\left(\sigma^s\rho^{1-s}\right) = -\log\sum_{i,j}q_{ij}^s p_{ij}^{1-s}$

Distributed Detection with Positive Rates (Rouzé et al.)



- Quantum testing against independence:

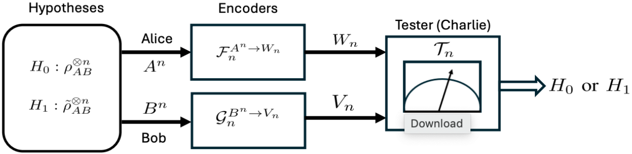
$$\begin{aligned} \mathsf{H} = 0: &\qquad \rho_{AB}^{\otimes n} \\ \mathsf{H} = 1: &\qquad \tilde{\rho}_{AB}^{\otimes n} = \rho_A^{\otimes n} \otimes \rho_B^{\otimes n} \end{aligned}$$

- Communication rate constraint $\frac{1}{n} \log |\mathcal{W}_n| \leq R$; ($B_n$ local obs.)

- Largest Stein exponent: $\sup_{\{T_n\}} \lim_{n \to \infty} -\frac{1}{n} \log \Pr[\hat{\mathsf{H}} = 0 | \mathsf{H} = 1]$

s.t. $\lim_{n\to\infty} \alpha_n = \Pr[\hat{\mathsf{H}} = 1 | \mathsf{H} = 0] = 0$ $\quad$ (*vanishing*)

Distributed Detection with Positive Rates (Rouzé et al.)



- Classical-quantum setup:

$$\mathsf{H} = 0: \qquad \left(\sum_a P(a) \, |a\rangle\langle a| \otimes \rho_{B,a} \right)^{\otimes n}$$

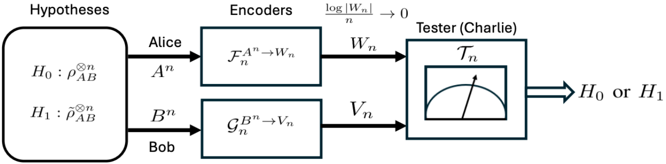
$$\mathsf{H} = 1: \qquad \left(\sum_a \tilde{P}(a) \, |a\rangle\langle a| \right)^{\otimes n} \otimes \rho_B^{\otimes n}$$

- Largest Stein exponent (for all ϵ):

$$\sup_{\{T_n\}} \lim_{n \to \infty} -\frac{1}{n} \log \Pr[\hat{\mathsf{H}} = 0 | \mathsf{H} = 1]$$

s.t. $\lim_{n\to\infty} \alpha_n = \Pr[\hat{\mathsf{H}} = 1 | \mathsf{H} = 0] = \epsilon \quad \in [0, 1)$

Distributed Detection with Zero Rates (Sreekumar et al.)

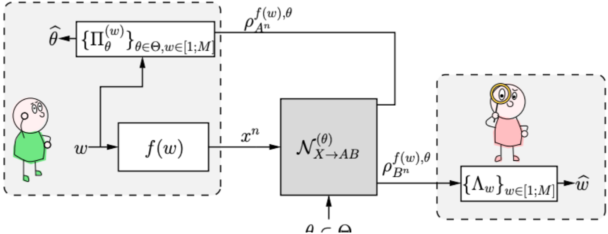


- Stein exponent at the receiver when B^n is observed locally
- Solved in terms of single-letter expression only in very special cases
- When only classical communication is allowed, a single bit suffices
- Classical communication can be infinitely worse than sending a single qubit

Distributed Detection: Some Open Problems

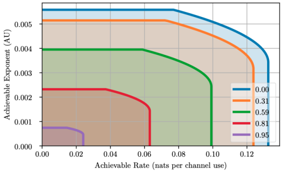
- Optimal exponents in fully quantum case?
- Multiple sensors or network scenarios?
- Noisy channels? (Dichotomy in the classical setup)
- Variable-rate coding?

Integrated Sensing and Communications (ISAC)



- Sensing: Detection exponent for θ

- Input signal influences both decoding and sensing performance



- Recent works: other sensing performances (estimation) and benefits of entanglement

[6] S.-Y. Wang et al., "Joint Quantum Communication and Sensing," ITW 2022.

A Few Words on Parameter Estimation

- There exists a Quantum Cramer-Rao Bound (QCRB) :

$$\mathrm{MSE}(\hat{\theta}) \geq \frac{1}{\mathrm{tr}\left(\mathcal{R}_{\rho_\theta}^{-1}\left(\frac{\delta\rho_\theta}{\delta\theta}\right)\rho_\theta\mathcal{R}_{\rho_\theta}^{-1}\left(\frac{\delta\rho_\theta}{\delta\theta}\right)\right)}$$

for $\mathcal{R}_\rho^{-1}(O) = \sum_{j,k}\frac{2}{\eta_j+\eta_k}\langle j|\, O\,|k\rangle\,|j\rangle\langle k|$ and $\rho = \sum_i \eta_i\,|i\rangle\,\langle i|$

- Challenge is then to prepare a good state ρ and provide measurements "exhausting" the QCRB

- Requirement for preparing entangled states

- In certain scenarios, measurements can depend on parameter $\rightarrow$ Precision beyond the QCRB...

Summary and Conclusions

- Quantum sensing/quantum metrology is an exciting field with decades of great success and huge interest at the moment
- New applications seem more imminent than for other quantum fields
- Theoretical foundations still bear lots of fundamental open questions and challenges
- Quantum nature with states in different eigenbases make information theoretic problems hard

  $\rightarrow$ still constant progress on the topic

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