Strong and epsilon-Dependent Converses for Source Coding, Channel Coding, and Hypothesis Testing

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Strong and ϵ-Dependent Converses

- In IT we typically have constraints of the form:

  A performance criteria (an expectation) needs to be below $\epsilon > 0$ asymptotically as $n \to \infty$

- Strong converse: Show that ultimate limits of other performance measures (rate etc.) do not depend on $\epsilon > 0$.

- If other performances depend on ϵ, we call it an ϵ-dependent converse

Message and Outline of the Talk

Take-Away Message

Converse proofs based on a “typical” change of measure
(inspired by Guo-Effros ’09 and Tyagi-Watanabe ’19)
and asymptotic Markov chains

In this talk:

- Source Coding
- Distributed Hypothesis Testing
- Channel Coding

Change of Measure on the Typical Set and a Useful Lemma

- $\{(X_i, Y_i)\}$ i.i.d. $\sim P_{XY}$ and independent thereof $T \sim \mathcal{U}\{1, \ldots, n\}$

- Strongly typical set $\mathcal{T}^{(n)}(P_{XY})$

$$\left\{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \colon |\pi_{x^n y^n}(a,b) - P_{XY}(a,b)| < n^{-1/3}\right\}.$$

- $\mathcal{D}_n \subseteq \mathcal{T}^{(n)}(P_{XY})$ so that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[\mathcal{D}_n] = 0.$

- Change of measure:

$$(\tilde{X}^n, \tilde{Y}^n) \sim P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n) = \frac{P_{XY}^{\otimes}(x^n, y^n)}{\mathbb{P}[\mathcal{D}_n]} \cdot \mathbb{1}\{(x^n, y^n) \in \mathcal{D}_n\}$$

Lemma

$$\left|P_{\tilde{X}_T \tilde{Y}_T} - P_{XY}\right| \to 0$$

$$\left|\frac{1}{n} H(\tilde{X}^n \tilde{Y}^n) - H(\tilde{X}_T \tilde{Y}_T)\right| \to 0, \quad \left|\frac{1}{n} H(\tilde{X}^n | \tilde{Y}^n) - H(\tilde{X}_T | \tilde{Y}_T)\right| \to 0$$

Proof of the Lemma

1. $P_{\tilde{X}_T \tilde{Y}_T}(x, y) - P_{XY}(x, y) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$

   Proof:

$$\begin{aligned} P_{\tilde{X}_T \tilde{Y}_T}(x, y) &= \frac{1}{n}\sum_{t=1}^{n} P_{\tilde{X}_t \tilde{Y}_t}(x, y) = \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}\{\tilde{X}_t = x, \tilde{Y}_t = y\}\right] \\ &= \underbrace{\mathbb{E}[\pi_{\tilde{X}^n \tilde{Y}^n}(x, y)]}_{=P_{XY}(x,y) \pm n^{-1/3}} \end{aligned}$$

Proof of the Lemma

1. $P_{\tilde{X}_T \tilde{Y}_T}(x, y) - P_{XY}(x, y) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$
2. $D(P_{\tilde{X}_T \tilde{Y}_T} \| P_{XY}) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$

Proof of the Lemma

1. $P_{\tilde{X}_T \tilde{Y}_T}(x,y) - P_{XY}(x,y) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$

2. $D(P_{\tilde{X}_T \tilde{Y}_T} \| P_{XY}) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$

3. $\frac{1}{n} D(P_{\tilde{X}^n \tilde{Y}^n} \| P_{XY}^{\otimes n}) \to 0$ because $\frac{1}{n} \log \mathbb{P}[\mathcal{D}_n] \to 0$

   Proof:

$$
\begin{aligned}
0 &\leq \frac{1}{n} D(P_{\tilde{X}^n \tilde{Y}^n} \| P_{XY}^{\otimes n}) = \frac{1}{n} \sum_{(x^n, y^n) \in \mathcal{D}_n} P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n) \log \frac{P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n)}{P_{XY}^{\otimes n}(x, y)} \\
&= -\frac{1}{n} \sum_{(x^n, y^n) \in \mathcal{D}_n} P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n) \log \mathbb{P}[\mathcal{D}_n] = -\frac{1}{n} \log \mathbb{P}[\mathcal{D}_n] \to 0.
\end{aligned}
$$

Proof of the Lemma

1. $P_{\tilde{X}_T \tilde{Y}_T}(x, y) - P_{XY}(x, y) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$
2. $D(P_{\tilde{X}_T \tilde{Y}_T} \| P_{XY}) \to 0$ because $\mathcal{D}_n \subseteq \mathcal{T}(P_{XY})$
3. $\frac{1}{n} D(P_{\tilde{X}^n \tilde{Y}^n} \| P_{XY}^{\otimes n}) \to 0$ because $\frac{1}{n} \log \mathbb{P}[\mathcal{D}_n] \to 0$
4. $\frac{1}{n} H(\tilde{X}^n \tilde{Y}^n) + \frac{1}{n} D(P_{\tilde{X}^n \tilde{Y}^n} \| P_{XY}^{\otimes n}) = H(\tilde{X}_T \tilde{Y}_T) + D(P_{\tilde{X}_T \tilde{Y}_T} \| P_{XY})$

Back to the Useful Lemma

- (X_i, Y_i) i.i.d. $\sim P_{XY}$ and independent thereof $T \sim \mathcal{U}\{1, \ldots, n\}$

- Strongly typical set $\mathcal{T}^{(n)}(P_{XY})$
$$\left\{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n \colon |\pi_{x^n y^n}(a,b) - P_{XY}(a,b)| < n^{-1/3}\right\}.$$

- $\mathcal{D}_n \subseteq \mathcal{T}^{(n)}(P_{XY})$ so that $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}[\mathcal{D}_n] = 0.$

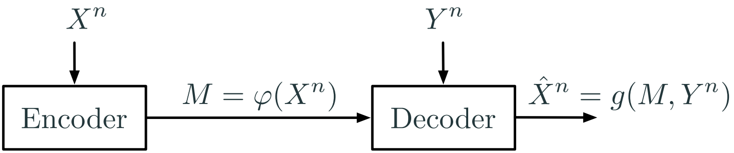
- Change of measure:
$$(\tilde{X}^n, \tilde{Y}^n) \sim P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n) = \frac{P_{XY}^{\otimes}(x^n, y^n)}{\mathbb{P}[\mathcal{D}_n]} \cdot \mathbb{1}\{(x^n, y^n) \in \mathcal{D}_n\}$$

Lemma

$$\frac{1}{n} H(\tilde{X}^n \tilde{Y}^n) \to H(\tilde{X}_T \tilde{Y}_T), \qquad \frac{1}{n} H(\tilde{X}^n | \tilde{Y}^n) \to H(\tilde{X}_T | \tilde{Y}_T), \quad P_{\tilde{X}_T \tilde{Y}_T} \to P_{XY}$$

Lossless Source Coding

Almost Lossless Source Coding with Side-Information



- $\{(X_i, Y_i)\}$ i.i.d. $\sim P_{XY}$
- $M \in \{1, \ldots, 2^{nR}\}$
- Rate $R > 0$ is $\epsilon > 0$-*achievable* if $\overline{\lim}_{n\to\infty} \mathbb{P}\left[X^n \neq \hat{X}^n\right] \leq \epsilon$.

Theorem (Slepian-Wolf '83, Oohama-Han '94)

Given $\epsilon \in [0, 1)$, rates $R > H(X|Y)$ are ϵ-achievable and rates $R < H(X|Y)$ not.

Strong Converse Proof Irrespective of $\epsilon \in [0, 1)$

- $\mathcal{D}_n := \left\{(x^n, y^n) \in \mathcal{T}^{(n)}(P_{XY}) : \quad g\left(\varphi(x^n), y^n\right) = x^n\right\}$ (no error!)

  with $\mathbb{P}[\mathcal{D}_n] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

- Change of measure:

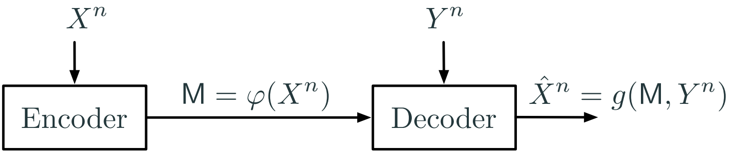
$$(\tilde{X}^n, \tilde{Y}^n) \sim P_{\tilde{X}^n \tilde{Y}^n}(x^n, y^n) = \frac{P_{XY}^{\otimes}(x^n, y^n)}{\mathbb{P}[\mathcal{D}_n]} \cdot \mathbb{1}\{(x^n, y^n) \in \mathcal{D}_n\}$$

  and $\tilde{M} = \varphi(\tilde{X}^n)$

- Rate bound:

$$R \geq \frac{1}{n} H(\tilde{\mathrm{M}}) \geq \frac{1}{n} H(\tilde{\mathrm{M}} | \tilde{Y}^n) \geq \frac{1}{n} H(\tilde{X}^n | \tilde{Y}^n) \to H(X|Y)$$

Lossless Source Coding under Variable-Length Coding



- (X^n, Y^n) i.i.d. $\sim P_{XY}$
- $\mathsf{M} \in \{0,1\}^\star$ and $\mathbb{E}[\mathrm{len}(\mathsf{M})] \leq nR$
- Rate $R > 0$ is $\epsilon > 0$-*achievable* if $\overline{\lim}_{n\to\infty} \mathbb{P}\left[X^n \neq \hat{X}^n\right] \leq \epsilon$.

Theorem

Given $\epsilon \in [0,1)$. Rates $R > (1-\epsilon)H(X|Y)$ are ϵ-achievable and rates $R < (1-\epsilon)H(X|Y)$ not.

- Achievability: With probability ϵ send a dummy bit for M

ϵ-Dependent Converse Proof

- $\mathcal{D}_n := \left\{(x^n, y^n) \in \mathcal{T}^{(n)}(P_{XY}) : \quad g\left(\varphi(x^n), y^n\right) = x^n\right\}$ (no error!)

  with $\mathbb{P}[\mathcal{D}] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

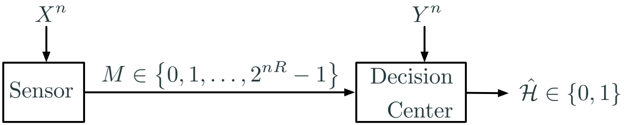
- Change of measure $(\tilde{X}^n, \tilde{Y}^n)$ on $\mathcal{D}_n$ and $\tilde{M} = \varphi(\tilde{X}^n)$

- Proof steps to incorporate expected rate constraint:

$$
\begin{aligned}
R &\geq \frac{1}{n}\mathbb{E}[\mathrm{len}(\mathrm{M})] \geq \frac{1}{n}\mathbb{E}[\mathrm{len}(\tilde{\mathrm{M}})] \cdot \mathbb{P}[\mathcal{D}_n] \geq \frac{1}{n}H(\tilde{\mathrm{M}}|\mathrm{len}(\tilde{\mathrm{M}})) \cdot \mathbb{P}[\mathcal{D}_n] \\
&= \frac{1}{n}H(\tilde{\mathrm{M}}) \cdot \mathbb{P}[\mathcal{D}_n] - \underbrace{\frac{1}{n}H(\mathrm{len}(\tilde{\mathrm{M}}))}_{\to 0}\mathbb{P}[\mathcal{D}_n] \\
&\geq \frac{1}{n}H(\tilde{X}^n|\tilde{Y}^n) \cdot \mathbb{P}[\mathcal{D}_n] - o(1) \to H(X|Y) \cdot (1 - \epsilon)
\end{aligned}
$$

Distributed Hypothesis Testing

Testing Against Independence



- $\mathcal{H} = 0 :\ (X^n, Y^n) \sim$ i.i.d. P_{XY}
- $\mathcal{H} = 1 :\ (X^n, Y^n) \sim$ i.i.d. $P_X P_Y$
- Type-I error $\overline{\lim_{n\to\infty}} \mathbb{P}[\hat{\mathcal{H}} = 1 | \mathcal{H} = 0] \leq \epsilon$
- Type-II error exponent $\theta = - \overline{\lim_{n\to\infty}} \frac{1}{n} \log \mathbb{P}[\hat{\mathcal{H}} = 0 | \mathcal{H} = 1]$

Largest Exponent independent of $\epsilon \in [0, 1)$ (Ahlswede-Csiszár 86)

$$\theta^*_\epsilon(R) = \max_{\substack{P_{S|X}: \\ R \geq I(S;X)}} I(S; Y)$$

Strong Converse: Change of Measure And Rate Constraint

- $\mathcal{D}_n := \left\{(x^n, y^n) \in \mathcal{T}^{(n)}(P_{XY}) \colon \quad g\left(\varphi(x^n), y^n\right) = 0\right\} \qquad \to \hat{\mathcal{H}} = 0$

  $P_{XY}^{\otimes}[\mathcal{D}_n] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

- Change of measure: $(\tilde{X}^n, \tilde{Y}^n)$ on $\mathcal{D}_n$ and $\tilde{M} = \varphi(\tilde{X}^n)$

- Proof steps for the rate:

$$
\begin{aligned}
R &\geq \frac{1}{n} H(\tilde{M}) = \frac{1}{n} I(\tilde{M}; \tilde{X}^n, \tilde{Y}^n) = \frac{1}{n} H(\tilde{X}^n \tilde{Y}^n) - \frac{1}{n} H(\tilde{X}^n \tilde{Y}^n | \tilde{M}) \\
&= H(\tilde{X}_T \tilde{Y}_T) + o(1) - \frac{1}{n} \sum_{t=1}^{n} H(\tilde{X}_t \tilde{Y}_t | \tilde{X}^{t-1} \tilde{Y}^{t-1} \tilde{M}) \\
&= H(\tilde{X}_T \tilde{Y}_T) + o(1) - H(\tilde{X}_T \tilde{Y}_T | \tilde{X}^{T-1} \tilde{Y}^{T-1} \tilde{M} T) \\
&= I(\tilde{X}_T \tilde{Y}_T; \underbrace{\tilde{X}^{T-1} \tilde{Y}^{T-1} \tilde{M} T}_{=:S}) \geq I(\tilde{X}_T; S)
\end{aligned}
$$

Strong Converse: Exponent

- $\hat{\mathcal{H}} = \varphi(M, Y^n)$ and $\tilde{\mathcal{H}} = \varphi(\tilde{M}, \tilde{Y}^n) = 0$
- Interesting inequality

$$D(P_{\tilde{Y}^n\tilde{M}} \| P_{\tilde{Y}^n} P_{\tilde{M}}) \geq D(P_{\tilde{Y}^n\tilde{M}}(\tilde{\mathcal{H}}) \| P_{\tilde{Y}^n} P_{\tilde{M}}(\tilde{\mathcal{H}})) = 1 \cdot \log \frac{1}{P_{\tilde{Y}^n} P_{\tilde{M}}(\tilde{\mathcal{H}} = 0)}$$

- Proof steps for exponent

$$\begin{aligned}
& -\frac{1}{n} \log P_{Y^n} P_M(\hat{\mathcal{H}} = 0) \\
\leq \quad & -\frac{1}{n} \log P_{\tilde{Y}^n} P_{\tilde{M}}(\tilde{\mathcal{H}} = 0) - \frac{2}{n} \log P_{XY}^{\otimes n}[\mathcal{D}_n] \\
\leq \quad & \frac{1}{n} D(P_{\tilde{Y}^n\tilde{M}} \| P_{\tilde{Y}^n} P_{\tilde{M}}) + o(1) = \frac{1}{n} I(\tilde{M}; \tilde{Y}^n) + o(1) \\
\leq \quad & \frac{1}{n} H(\tilde{Y}^n) - H(\tilde{Y}_T | \underbrace{\tilde{M}\tilde{X}^{T-1}\tilde{Y}^{T-1}T}_{=S}) + o(1) \\
= \quad & H(\tilde{Y}_T) + o(1) - H(\tilde{Y}_T | S) \to I(\tilde{Y}_T; S)
\end{aligned}$$

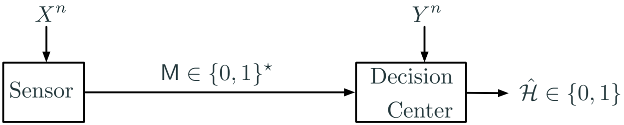
Asymptotic Markov Chain

- In the theorem we need asymptotic Markov chain $S \to \tilde{X}_T \to \tilde{Y}_T$
- For any n:

$$
\begin{aligned}
0 &= \frac{1}{n} I(\tilde{M}; \tilde{Y}^n | \tilde{X}^n) \\
&= \frac{1}{n} H(\tilde{Y}^n | \tilde{X}^n) - \frac{1}{n} H(\tilde{Y}^n | \tilde{X}^n \tilde{M}) \\
&\geq H(\tilde{Y}_T | \tilde{X}_T) + o(1) - H(\tilde{Y}_T | \tilde{X}_T \underbrace{\tilde{X}^{T-1} \tilde{Y}^{T-1} \tilde{M} T}_{S}) \\
&= I(\tilde{Y}_T; S | \tilde{X}_T) + o(1) \geq o(1).
\end{aligned}
$$

- So $I(\tilde{Y}_T; S | \tilde{X}_T) \to 0$ and the Markov chain holds asymptotically.

Testing Against Indep. Under Variable-Length Coding



- $\mathcal{H} = 0: \ (X^n, Y^n) \sim$ i.i.d. P_{XY}
- $\mathcal{H} = 1: \ (X^n, Y^n) \sim$ i.i.d. $P_X P_Y$
- Expected rate constraints $\mathbb{E}[\mathrm{len}(\mathsf{M})] \leq nR$

Largest Possible Error Exponent depends on ϵ

$$\theta^*_{\mathsf{VL},\epsilon}(R) = \max_{\substack{P_{S|X}: \\ R \geq (1-\epsilon) I(S;X)}} I(S;Y)$$

- Achievability: With probability ϵ send a dummy bit for M

Change of Measure And Rate Constraint as Before!

- $\mathcal{D}_n := \left\{(x^n, y^n) \in \mathcal{T}^{(n)}(P_{XY}) : \quad g\left(\varphi(x^n), y^n\right) = 0\right\}$
  no error under $\mathcal{H} = 0! \Rightarrow P_{XY}^{\otimes n}[\mathcal{D}_n] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

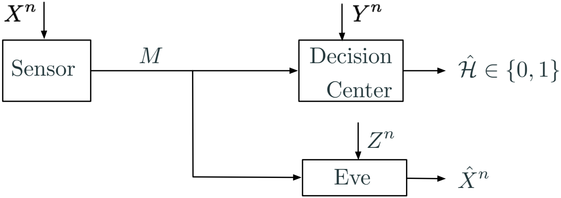
- Change of measure $(\tilde{X}^n, \tilde{Y}^n, \tilde{M})$

- Proof steps for the rate

$$
\begin{aligned}
R &\geq \frac{1}{n}\mathbb{E}[\mathrm{len}(\mathrm{M})] \geq \frac{1}{n}\mathbb{E}[\mathrm{len}(\tilde{\mathrm{M}})] \cdot P_{XY}^{\otimes n}[\mathcal{D}_n] \geq \frac{1}{n}H(\tilde{\mathrm{M}}|\mathrm{len}(\tilde{\mathrm{M}})) \cdot P_{XY}^{\otimes n}[\mathcal{D}_n] \\
&= \frac{1}{n}H(\tilde{\mathrm{M}}) \cdot P_{XY}^{\otimes n}[\mathcal{D}_n]] - \underbrace{\frac{1}{n}H(\mathrm{len}(\tilde{\mathrm{M}}))}_{\to 0} P_{XY}^{\otimes n}[\mathcal{D}_n] \\
&\vdots \\
&\geq I(\tilde{X}_T; S) \cdot P_{XY}^{\otimes n}[\mathcal{D}_n] + o(1) \to I(\tilde{X}_T; S)(1 - \epsilon)
\end{aligned}
$$

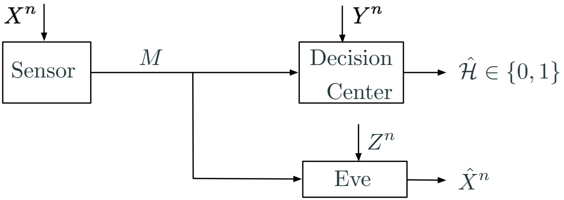
- Remaining steps as before!

Testing Against Independence with an Eavesdropper



- $\mathcal{H} = 0$: $(X^n, Y^n, Z^n) \sim$ i.i.d. P_{XYZ}
- $\mathcal{H} = 1$: $(X^n, Y^n, Z^n) \sim$ i.i.d. $P_X P_Y P_{Z|XY}$
- Equivocation constraint $\overline{\lim}_{n\to\infty} \frac{1}{n} H(X^n | M Z^n \mathcal{H} = 0) \geq \Delta$

Testing Against Independence with an Eavesdropper



- $\mathcal{H} = 0 :\ (X^n, Y^n, Z^n) \sim$ i.i.d. P_{XYZ}
- $\mathcal{H} = 1 :\ (X^n, Y^n, Z^n) \sim$ i.i.d. $P_X P_Y P_{Z|XY}$
- Equivocation constraint $\overline{\lim}_{n\to\infty} \frac{1}{n} H(X^n | M Z^n \mathcal{H} = 0) \geq \Delta$

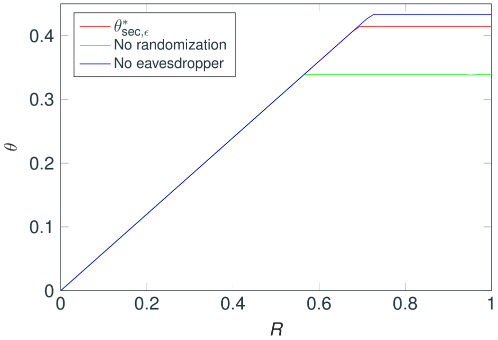
Largest Possible Error Exponent depends on ϵ

$$\theta^*_{\text{sec},\epsilon}(R) = \max_{\substack{P_{S|X}: \\ R \geq I(S;X) \\ \Delta \leq (1-\epsilon) H(X|ZS) + \epsilon H(X|Z)}} I(S;Y)$$

- Achievability: With probability ϵ send a dummy bit for M

An Example

- $X \sim \mathcal{B}(0.2)$
- $P_{Y|X}$ a BEC(0.4)
- $P_{Z|X}$ a BSC(0.2)
- $\epsilon = 0.2$
- $\Delta = 0.13$



Converse Proof

- $E = \mathbb{1}\{(X^n, Y^n) \in \mathcal{D}_n\}$

- Equivocation bound:

$$\frac{1}{n}H(X^n|M, W^n) = \sum_{t=1}^{n} H(X_t|X^{t-1}, M, W^n)$$

$$\begin{aligned}
&= \frac{1}{n}\sum_{t=1}^{n} H(X_t|X^{t-1}, Y^{t-1}, M, W^n) \\
&\leq \frac{1}{n}\sum_{t=1}^{n} H(X_t|X^{t-1}, Y^{t-1}, M, W^n, E) \\
&\quad + \underbrace{\frac{1}{n}\sum_{t=1}^{n} I(E; X_t|X^{t-1}, Y^{t-1}, M, W^n)}_{\leq 1} \\
&\leq H(\tilde{X}_T|S, \tilde{W}_T)P_E(1) + H(X_T|W_T, E = 0)P_E(0) + o(1)
\end{aligned}$$

Channel Coding

Capacity of Discrete Memoryless Channels

- M uniform over $\{1, \ldots, 2^{nR}\}$
- Memoryless channel $P_{Y^n|X^n} = P_{Y|X}^{\otimes n}$
- $R > 0$ is $\epsilon > 0$-*achievable* if $\overline{\lim}_{n \to \infty} \max_m \mathbb{P}\left[\hat{M} \neq M \middle| M = m\right] \leq \epsilon$.
- Capacity $C \triangleq \max_{P_X} I(X; Y)$

Theorem (Wolfowitz '78)

For any $\epsilon \in [0, 1)$, rates $R < C$ are ϵ-achievable and rates $R > C$ not.

Strong Converse Proof Irrespective of $\epsilon \in [0,1)$)

- $\mathcal{T}^{(n)}(P_{Y|X}, x^n) = \left\{y^n \colon |\pi_{x^n,y^n}(x,y) - \pi_{x^n}(x)P_{Y|X}(y)| < n^{-1/3}\right\}$
- $\mathcal{D}_m := \left\{y^n \in \mathcal{T}^{(n)}(P_{Y|X}, x^n(m)) \colon \quad g^{(n)}(y^n) = m\right\}$

  no decoding error when $M = m \Rightarrow \mathbb{P}[\mathcal{D}_m | M = m] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

- Change of measure:

  $$P_{\tilde{X}^n\tilde{Y}^n\tilde{M}}(x^n, y^n, m) = \frac{1}{2^{nR}} \cdot \mathbb{1}\{x^n = \varphi(m)\} \frac{P_{Y|X}^{\otimes n}(y^n|x^n)}{\mathbb{P}[\mathcal{D}_m | M = m]} \cdot \mathbb{1}\{y^n \in \mathcal{D}_m\}$$

Strong Converse Proof Irrespective of $\epsilon \in [0,1)$)

- $\mathcal{T}^{(n)}(P_{Y|X}, x^n) = \left\{ y^n \colon |\pi_{x^n,y^n}(x,y) - \pi_{x^n}(x)P_{Y|X}(y)| < n^{-1/3} \right\}$

- $\mathcal{D}_m := \left\{ y^n \in \mathcal{T}^{(n)}(P_{Y|X}, x^n(m)) \colon \quad g^{(n)}(y^n) = m \right\}$

  no decoding error when $M = m \Rightarrow \mathbb{P}[\mathcal{D}_m | M = m] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

- Change of measure:

  $$P_{\tilde{X}^n \tilde{Y}^n \tilde{M}}(x^n, y^n, m) = \frac{1}{2^{nR}} \cdot \mathbb{1}\{x^n = \varphi(m)\} \frac{P_{Y|X}^{\otimes n}(y^n|x^n)}{\mathbb{P}[\mathcal{D}_m | M = m]} \cdot \mathbb{1}\{y^n \in \mathcal{D}_m\}$$

- Rate bound: $\quad R = \frac{1}{n} H(\tilde{M}) = \frac{1}{n} I(\tilde{M}; \tilde{Y}^n) \leq H(\tilde{Y}_T) - \frac{1}{n} H(\tilde{Y}^n | \tilde{M})$

Strong Converse Proof Irrespective of $\epsilon \in [0,1))$

- $\mathcal{T}^{(n)}(P_{Y|X}, x^n) = \left\{y^n \colon |\pi_{x^n,y^n}(x,y) - \pi_{x^n}(x)P_{Y|X}(y)| < n^{-1/3}\right\}$

- $\mathcal{D}_m := \left\{y^n \in \mathcal{T}^{(n)}(P_{Y|X}, x^n(m)) \colon \quad g^{(n)}(y^n) = m\right\}$

  no decoding error when $M = m \Rightarrow \mathbb{P}[\mathcal{D}_m | M = m] \geq 1 - \epsilon - \frac{|\mathcal{X}||\mathcal{Y}|}{4n^{1/3}}$

- Change of measure:

  $$P_{\tilde{X}^n\tilde{Y}^n\tilde{M}}(x^n, y^n, m) = \frac{1}{2^{nR}} \cdot \mathbb{1}\{x^n = \varphi(m)\} \frac{P_{Y|X}^{\otimes n}(y^n|x^n)}{\mathbb{P}[\mathcal{D}_m|M=m]} \cdot \mathbb{1}\{y^n \in \mathcal{D}_m\}$$

- Rate bound: $\quad R = \frac{1}{n}H(\tilde{M}) = \frac{1}{n}I(\tilde{M}; \tilde{Y}^n) \leq H(\tilde{Y}_T) - \frac{1}{n}H(\tilde{Y}^n|\tilde{M})$

- To show:

  $$\frac{1}{n}H(\tilde{Y}^n|\tilde{M}) \quad \to \quad H_{P_X P_{Y|X}}(Y|X)$$

  $$P_{\tilde{Y}_T} \quad \to \quad \sum_x P_X(x) P_{Y|X}(y|x)$$

- Then: $R \leq H_{P_X P_{Y|X}}(Y) - H_{P_X P_{Y|X}}(Y|X) = I_{P_X P_{Y|X}}(X;Y) \leq C.$

Strong Converse for Channel Coding Continued I

$$
\begin{aligned}
P_{\tilde{Y}_T}(y) &= \frac{1}{n}\sum_{t=1}^{n} P_{\tilde{Y}_t}(y) = \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} \mathbb{1}\{\tilde{Y}_t = y\}\right] = \mathbb{E}\left[\pi_{\tilde{Y}^n}(y)\right] \\
&= \mathbb{E}\left[\sum_{x\in\mathcal{X}} \pi_{\tilde{Y}^n x^n(\tilde{M})}(y,x)\right] \\
&= \frac{1}{2^{nR}}\sum_{m=1}^{2^{nR}} \mathbb{E}\left[\sum_{x\in\mathcal{X}} \pi_{\tilde{Y}^n x^n(m)}(y,x)\middle|\tilde{M}=m\right] \\
&= \sum_{x\in\mathcal{X}} \underbrace{\frac{1}{2^{nR}}\sum_{m=1}^{2^{nR}} \pi_{x^n(m)}(x)}_{\to P_X(x)} \cdot \underbrace{\mathbb{E}\left[\pi_{\tilde{Y}^n|\tilde{x}^n(m)}(y|x)\middle|\tilde{M}=m\right]}_{=P_{Y|X}(y|x)\pm n^{-1/3}}.
\end{aligned}
$$

Strong Converse for Channel Coding Continued II

$$
\begin{aligned}
\frac{1}{n} H(\tilde{Y}^n | \tilde{M} = m) &= -\frac{1}{n} \sum_{y^n \in \mathcal{D}_m} P_{\tilde{Y}^n | \tilde{M} = m}(y^n) \log \frac{P_{Y|X}^{\otimes n}(y^n | x^n(m))}{\mathbb{P}[\mathcal{D}_m | M = m]} \\
&= -\frac{1}{n} \sum_{t=1}^{n} \sum_{y^n \in \mathcal{D}_m} P_{\tilde{Y}^n | \tilde{M} = m}(y^n) \log P_{Y|X}(y_t | x_t(m)) + \underbrace{\frac{1}{n} \log \mathbb{P}[\mathcal{D}_m | M = m]}_{\to 0} \\
&= -\frac{1}{n} \sum_{t=1}^{n} \sum_{y \in \mathcal{Y}} P_{\tilde{Y}_t | \tilde{M} = m}(y) \log P_{Y|X}(y | x_t(m)) + o(1) \\
&= -\sum_{x \in \mathcal{X}} \frac{n_x(m)}{n} \sum_{y \in \mathcal{Y}} \mathbb{E}\left[\frac{\sum_{t:\, x_t(m) = x} \mathbb{1}\{\tilde{Y}_t = y\}}{n_x(m)} \middle| \tilde{M} = m \right] \log P_{Y|X}(y|x) + o(1) \\
&= -\sum_{x \in \mathcal{X}} \pi_{x^n(m)}(x) \sum_{y \in \mathcal{Y}} \underbrace{\mathbb{E}\left[\pi_{\tilde{Y}^n | x^n(m)}(y|x) \middle| M = m \right]}_{= P_{Y|X}(y|x) \pm n^{-1/3}} \cdot \log P_{Y|X}(y|x) + o(1)
\end{aligned}
$$

- Average over $\tilde{M}$ and take $n \to \infty$.

Summary

- Strong converse proofs based on change of measure arguments and asymptotic Markov chains for source and channel coding and for hypothesis testing

- Methods well adapted also to prove ϵ-dependent converses under expectation constraints

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Proof of Equivalence of Entropy-Divergence Sums

$$
\begin{aligned}
&\frac{1}{n}H(\tilde{X}^n\tilde{Y}^n) + \frac{1}{n}D(P_{\tilde{X}^n\tilde{Y}^n}\|P_{XY}^{\otimes}) \\
&= -\frac{1}{n}\sum_{(x^n,y^n)\in\mathcal{D}_n} P_{\tilde{X}^n\tilde{Y}^n}(x^n,y^n)\log P_{XY}^{\otimes}(x^n,y^n) \\
&= -\frac{1}{n}\sum_{t=1}^{n}\sum_{(x^n,y^n)\in\mathcal{D}_n} P_{\tilde{X}^n\tilde{Y}^n}(x^n,y^n)\log P_{XY}(x_i,y_i) \\
&= -\sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}}\frac{1}{n}\sum_{t=1}^{n} P_{\tilde{X}_t\tilde{Y}_t}(x,y)\log P_{XY}(x,y) \\
&= H(\tilde{X}_T\tilde{Y}_T) + D(P_{\tilde{X}_T\tilde{Y}_T}\|P_{XY})
\end{aligned}
$$