



Extrinsic Jensen-Shannon Divergence: Applications to Variable-Length Coding

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Extrinsic Jensen–Shannon Divergence: Applications to Variable-Length Coding

Mohammad Naghshvar, Tara Javidi, and Michèle Wigger

Abstract

This paper considers the problem of variable-length coding over a discrete memoryless channel (DMC) with noiseless feedback. The paper provides a stochastic control view of the problem whose solution is analyzed via a newly proposed symmetrized divergence, termed extrinsic Jensen–Shannon (EJS) divergence. It is shown that strictly positive lower bounds on EJS divergence provide non-asymptotic upper bounds on the expected code length. Strictly positive lower bound on EJS divergence, and hence non-asymptotic upper bounds on the expected code length, are obtained for the following two sequential coding schemes: posterior matching and MaxEJS coding scheme which is based on a greedy maximization of the EJS divergence.

As an asymptotic corollary of the main results, this paper also provides a rate–reliability test. Variable-length coding schemes that satisfy the condition(s) of the test, are guaranteed to achieve the capacity (and the optimal error exponent). The results are specialized for posterior matching and MaxEJS to obtain a deterministic one-phase coding scheme achieving the capacity and the optimal reliability. For the special case of symmetric binary-input channels, simpler deterministic schemes are proposed and analyzed.

Index Terms

Discrete memoryless channel, variable-length coding, sequential analysis, feedback gain, reliability function.

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I. INTRODUCTION

In his seminal paper [1], Burnashev provided upper and lower bounds on the minimum expected number of channel uses $\mathbb{E}[\tau_\epsilon^*]$ that are needed to convey a message (from a fixed message set of size M) with average probability of error smaller than some ϵ over a discrete memoryless channel (DMC) with feedback. For all code rates below the capacity of the DMC, the ratio between the upper and lower bounds approaches 1 as $\epsilon \rightarrow 0$. Therefore, the bounds yield the optimal reliability function

$$E(R) := \lim_{\epsilon \rightarrow 0} \frac{-\log \epsilon}{\mathbb{E}[\tau_\epsilon^*]} = C_1 \left(1 - \frac{R}{C}\right) \quad (1)$$

where C denotes the capacity of the channel, $R \in [0, C]$ is the expected rate of the code, and C_1 is the maximum Kullback–Leibler (KL) divergence between the conditional output distributions given any two inputs.

Burnashev proved the upper bound using a two-phase coding scheme. In the first phase, referred to as the *communication* phase, the transmitter tries to increase the decoder's belief about the true message. At the end of this phase, the message with the highest posterior probability is selected as a candidate. The second phase, referred to as the *confirmation* phase, serves to verify the correctness of the output of phase one. Subsequently, in [2], [3] alternative two-phase coding schemes attaining the optimal reliability function were provided, while it was shown in [4] that Burnashev's communication phase can be replaced with *any* capacity achieving block code. In [5], Burnashev's reliability function was shown to be attainable using a two-phase scheme for a binary symmetric channel (BSC) with an unknown crossover probability.

In [6], [7], see also [8], a sequential, one-phase scheme for transmission over a BSC with noiseless feedback was proposed. This scheme, first proposed in [6], is briefly explained next. Each message is represented as a subinterval of size $\frac{1}{M}$ of the unit interval. After each transmission and given the channel output, the posterior probability of all subintervals are updated. In the next time slot, the transmitter sends 0 if the true message's corresponding subinterval is below the current median, or 1 if it is above. If the current median lies within the true message's subinterval, then the transmitter sends 0 and 1 randomly according to weights determined by the length of the portions of the subinterval above and below the median. As the rounds of transmission proceed, the posterior probability of the true message's subinterval most likely grows larger than $\frac{1}{2}$, which pushes the median within the message's subinterval and thus leads

to a randomized encoding. This simple one-phase scheme is known to achieve the capacity of a BSC [7], and its posterior matching extension has recently been shown to achieve the capacity of general DMCs [8]. However, its corresponding error exponent is not studied and not known.

These previous results raise the question whether having two separate phases of operation and randomized encoding are necessary to achieve the optimal reliability function or not. In this paper we show that this is not the case. More generally, the main contributions of our paper are:

- This paper provides a stochastic control view of the problem of variable-length coding with feedback. This stochastic control problem, a discrete version of that suggested in [9], is analyzed via a newly proposed symmetrized divergence.
- Drawing parallels between mutual information and symmetrized L divergence [10], the *extrinsic Jensen–Shannon (EJS) divergence* of the conditional output distributions with respect to the receiver’s posterior probability is proposed as the key performance measure of any given coding scheme.
- It is shown that strictly positive lower bounds on the EJS divergence provide a *non-asymptotic* upper bound on the expected number of channel uses necessary for a coding scheme to obtain a given (arbitrarily small) error probability. Specific (strictly positive) lower bounds on the EJS divergence are derived for a variable-length version of the posterior matching scheme and for the newly proposed MaxEJS coding scheme.
- As a corollary, a rate–reliability test for variable-length coding schemes is proposed. This test is utilized to provide an alternative (simple and concise) proof that the variable-length version of posterior matching achieves capacity when $C_1 < \infty$. Furthermore, for the first time, an achievable error exponent (reliability) is obtained for posterior matching.
- A deterministic one-phase coding scheme is proposed and it is proved that this scheme achieves the optimal reliability function of the DMC with noiseless feedback.

The remainder of this paper is organized as follows. In Section II, we introduce the EJS divergence and discuss some of its properties. In Section III, we formulate the problem of channel coding with noiseless feedback. Section IV provides the main results of the paper for general DMCs: i) an EJS-divergence based non-asymptotic analysis of variable-length coding, ii) a specialization of this analysis to variable-length posterior matching, and iii) a specialization to a new deterministic one-phase coding scheme that is based on greedy maximization of the EJS

divergence. In Section V, we consider the special case of symmetric binary-input channels and propose simple deterministic schemes. Finally, in Section VI, we analyze the achievable rates and error exponents (reliability) of the coding schemes presented in the previous two sections.

We finish this section with some notation.

Notation: Let $[x]^+ = \max\{x, 0\}$. The indicator function $\mathbf{1}_{\{A\}}$ takes the value 1 whenever event A occurs, and 0 otherwise. The i^{th} element of vector \mathbf{v} is denoted by v_i . For any set \mathcal{S} , $|\mathcal{S}|$ denotes the cardinality of \mathcal{S} . All logarithms are in base 2. The entropy function on a vector $\boldsymbol{\rho} = [\rho_1, \rho_2, \dots, \rho_M] \in [0, 1]^M$ is defined as $H(\boldsymbol{\rho}) := \sum_{i=1}^M \rho_i \log \frac{1}{\rho_i}$, with the convention that $0 \log \frac{1}{0} = 0$. We denote the conditional probability $P(Y|X = x)$ by P_x .

II. PRELIMINARIES

A. Known Symmetric Divergences and Mutual Information

We first recall some well known divergences. The *Kullback–Leibler (KL) divergence* between two probability distributions P_Y and P'_Y over a finite set \mathcal{Y} is defined as $D(P_Y \| P'_Y) := \sum_{y \in \mathcal{Y}} P_Y(y) \log \frac{P_Y(y)}{P'_Y(y)}$ with the convention $0 \log \frac{a}{0} = 0$ and $b \log \frac{b}{0} = \infty$ for $a, b \in [0, 1]$ with $b \neq 0$. The KL divergence satisfies the following lemma.

Lemma 1. *For any two distributions P and Q on a set \mathcal{Y} and $\alpha \in [0, 1]$, $D(P \| \alpha P + (1 - \alpha)Q)$ is decreasing in α .*

Proof: Let $\beta \in [0, 1]$ satisfy $\beta \leq \alpha$. Then,

$$\alpha P + (1 - \alpha)Q = \gamma(\beta P + (1 - \beta)Q) + (1 - \gamma)P$$

where $\gamma = \frac{1 - \alpha}{1 - \beta} \leq 1$. By Jensen's inequality and the convexity of the KL divergence:

$$\begin{aligned} D(P \| \alpha P + (1 - \alpha)Q) &\leq \gamma D(P \| \beta P + (1 - \beta)Q) + (1 - \gamma)D(P \| P) \\ &\leq D(P \| \beta P + (1 - \beta)Q) \end{aligned} \quad (2)$$

where the last inequality follows because $D(P \| P) = 0$ and $\gamma \leq 1$. ■

The KL divergence is *not* symmetric, i.e., in general $D(P_Y \| P'_Y) \neq D(P'_Y \| P_Y)$. The *J divergence* [11] and *L divergence* [10] symmetrize the KL divergence:

$$J(P_1, P_2) := D(P_1 \| P_2) + D(P_2 \| P_1), \quad (3)$$

$$L(P_1, P_2) := D\left(P_1 \| \frac{1}{2}P_1 + \frac{1}{2}P_2\right) + D\left(P_2 \| \frac{1}{2}P_1 + \frac{1}{2}P_2\right). \quad (4)$$

The L divergence can also be related to the *Jensen difference* with respect to the Shannon entropy function [12]:

$$\frac{1}{2}L(P_1, P_2) = H\left(\frac{1}{2}P_1 + \frac{1}{2}P_2\right) - \left(\frac{1}{2}H(P_1) + \frac{1}{2}H(P_2)\right). \quad (5)$$

The *Jensen–Shannon (JS) divergence* [10], [12] is defined similarly to the L divergence but for general $M \geq 2$ probability distributions. Given M probability distributions P_1, P_2, \dots, P_M over a set \mathcal{Y} and a vector of a priori weights $\boldsymbol{\rho} = [\rho_1, \rho_2, \dots, \rho_M]$, where $\boldsymbol{\rho} \in [0, 1]^M$ and $\sum_{i=1}^M \rho_i = 1$, the JS divergence is defined as [10], [12]:

$$\begin{aligned} JS(\boldsymbol{\rho}; P_1, \dots, P_M) &:= \sum_{i=1}^M \rho_i D\left(P_i \parallel \sum_{j=1}^M \rho_j P_j\right) \\ &= H\left(\sum_{i=1}^M \rho_i P_i\right) - \sum_{i=1}^M \rho_i H(P_i). \end{aligned} \quad (6)$$

Let θ be a random variable that takes values in $\{1, 2, \dots, M\}$ and has probability mass function $\boldsymbol{\rho}$ and $Y \sim P_\theta$ (which implies that $\Pr(Y = y) = \sum_{i=1}^M \rho_i P_i(y)$). From (6),

$$JS(\boldsymbol{\rho}; P_1, \dots, P_M) = H(Y) - H(Y|\theta) = I(\theta; Y) \quad (7)$$

where $I(\theta; Y)$ is the *mutual information* between θ and Y .

B. A New Divergence: Extrinsic Jensen–Shannon Divergence

We introduce the *extrinsic Jensen–Shannon (EJS) divergence* which extends the J divergence for general $M \geq 2$ probability distributions P_1, \dots, P_M and for an M -dimensional weight vector $\boldsymbol{\rho}$:

$$EJS(\boldsymbol{\rho}; P_1, \dots, P_M) := \sum_{i=1}^M \rho_i D\left(P_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) \quad (8a)$$

when $\rho_i < 1$ for all $i \in \{1, \dots, M\}$, and as

$$EJS(\boldsymbol{\rho}; P_1, \dots, P_M) := \max_{j \neq i} D(P_i \parallel P_j) \quad (8b)$$

when $\rho_i = 1$ for some $i \in \{1, \dots, M\}$.

Let $U(\cdot)$ denote the average log-likelihood function:

$$U(\boldsymbol{\rho}) := \sum_{i=1}^M \rho_i \log \frac{1 - \rho_i}{\rho_i}. \quad (9)$$

Lemma 2 (Properties of EJS Divergence). *The EJS divergence $EJS(\boldsymbol{\rho}; P_1, \dots, P_M)$ as defined in (8) satisfies the following three properties.*

1) *It is lower bounded by the JS divergence:*

$$EJS(\boldsymbol{\rho}; P_1, \dots, P_M) \geq JS(\boldsymbol{\rho}; P_1, \dots, P_M). \quad (10)$$

2) *It can be expressed as*

$$EJS(\boldsymbol{\rho}; P_1, \dots, P_M) = U(\boldsymbol{\rho}) - \sum_{y \in \mathcal{Y}} P_{\boldsymbol{\rho}}(y) U\left(\left[\frac{\rho_1 P_1(y)}{P_{\boldsymbol{\rho}}(y)}, \dots, \frac{\rho_M P_M(y)}{P_{\boldsymbol{\rho}}(y)}\right]\right) \quad (11)$$

where $P_{\boldsymbol{\rho}}(y) = \sum_{i=1}^M \rho_i P_i(y)$.

3) *It is convex in the distributions P_1, \dots, P_M .*

The proof of Lemma 2 is given in Appendix I.

Remark 1. The EJS divergence defined in this paper is not the unique generalization of the J divergence. There exist other M -dimensional generalizations of the J divergence such as $\sum_{i=1}^M \rho_i \sum_{j=1}^M \rho_j J(P_i, P_j)$ which was studied in [13]. However, as will be discussed in details later in the paper, properties of EJS such as the one provided by (11) above makes it a suitable measure of information for our applications of interest.

Remark 2. The EJS divergence is equivalent to the *full anthropic correction* proposed in the context of mutual information estimation [14]. In particular, the authors in [14] used the notion of anthropic correction as an estimator of the mutual information between signals acquired in neurophysiological experiments where only a small number of stimuli can be tested.

III. CODING OVER DMC WITH NOISELESS FEEDBACK

A. The Problem Setup

Consider the problem of coding over a discrete memoryless channel (DMC) with noiseless feedback as depicted in Fig. 1. The DMC is described by finite input and output sets \mathcal{X} and \mathcal{Y} , and a collection of conditional probabilities $P(Y|X)$. To simplify notation, and without loss of generality, we assume that

$$\mathcal{X} = \{0, 1, \dots, |\mathcal{X}| - 1\}, \quad (12)$$

$$\mathcal{Y} = \{0, 1, \dots, |\mathcal{Y}| - 1\}. \quad (13)$$

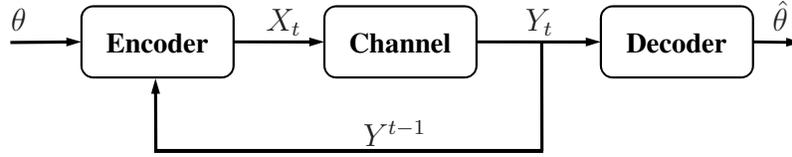


Fig. 1. A noisy memoryless channel with a noiseless causal feedback link.

Let C denote the *Shannon capacity* of the DMC $P(Y|X)$ [15, p. 184]:

$$C = \max_{P_X} I(X; Y), \quad (14)$$

and let $(\pi_0^*, \pi_1^*, \dots, \pi_{|\mathcal{X}|-1}^*)$ be the maximizer of (14), the so-called *capacity-achieving input distribution*. The operational meaning of the Shannon capacity is discussed in Section VI.

The following result will be used in our proofs.

Fact 1 (Theorem 4.5.1 in [16]). *Consider a DMC with capacity-achieving input distribution $\pi_0^*, \pi_1^*, \dots, \pi_{|\mathcal{X}|-1}^*$. For each $k \in \{0, 1, \dots, |\mathcal{X}| - 1\}$, if $\pi_k^* > 0$,*

$$D\left(P(Y|X = k) \parallel \sum_{l=0}^{|\mathcal{X}|-1} \pi_l^* P(Y|X = l)\right) = C.$$

Let C_1 be the KL divergence between the two most distinguishable inputs of the DMC:

$$C_1 = \max_{x, x' \in \mathcal{X}} D(P(Y|X = x) \parallel P(Y|X = x')). \quad (15)$$

We also denote

$$C_2 = \max_{y \in \mathcal{Y}} \frac{\max_{x \in \mathcal{X}} P(Y = y|X = x)}{\min_{x \in \mathcal{X}} P(Y = y|X = x)}. \quad (16)$$

In this paper, we assume C, C_1, C_2 are positive and finite.¹

Let τ denote the total transmission time (or equivalently the total length of the code). The transmitter wishes to communicate a message θ to the receiver, where the message is uniformly distributed over a message set

$$\Omega := \{1, 2, \dots, M\}. \quad (17)$$

¹It can be easily shown that $C \leq C_1 \leq \log C_2 \leq C_2$. Furthermore, if $C_1 < \infty$, then the transition probability $P(Y = y|X = x)$ is positive for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, which implies that $C_2 < \infty$ as well. Therefore, $C > 0$ and $C_1 < \infty$ are sufficient to ensure that C, C_1, C_2 are positive and finite.

To this end, the transmitter produces channel inputs X_t for $t = 0, 1, \dots, \tau - 1$, which it can compute as a function of the message θ and (thanks to the feedback) also of the past channel outputs $Y^{t-1} := (Y_0, Y_1, \dots, Y_{t-1})$:

$$X_t = e_t(\theta, Y^{t-1}), \quad t = 0, 1, \dots, \tau - 1, \quad (18)$$

for some encoding function $e_t: \Omega \times \mathcal{Y}^t \rightarrow \mathcal{X}$.

After observing the τ channel outputs $Y_0, Y_1, \dots, Y_{\tau-1}$, the receiver guesses the message θ as

$$\hat{\theta} = d(Y^{\tau-1}), \quad (19)$$

for some decoding function $d: \mathcal{Y}^\tau \rightarrow \Omega$. The probability of error of the scheme is thus

$$\text{Pe} := \Pr(\hat{\theta} \neq \theta).$$

In contrast to fixed-length coding where the total transmission time τ is deterministic and known before the transmission starts, in this paper, our focus is on variable-length coding, i.e., the case where τ is a random stopping time decided at the receiver as a function of the observed channel outputs. Thanks to the noiseless feedback, the transmitter is also informed of the channel outputs and hence of the stopping time.

For a fixed DMC and for a given $\epsilon > 0$, the goal is to find encoding and decoding rules as in (18) and (19), and a stopping time τ_ϵ such that the probability of error satisfies $\text{Pe} \leq \epsilon$ and the expected number of channel uses $\mathbb{E}[\tau_\epsilon]$ is minimized. Let $\mathbb{E}[\tau_\epsilon^*]$ be the minimum expected number of channel uses that can be achieved by coding schemes with the stopping rule τ_ϵ .

We shall often use the functions $\{\gamma_{y^{t-1}}\}$ for $y^{t-1} \in \mathcal{Y}^t$ and $t \in \{0, 1, \dots, \tau - 1\}$ where

$$\gamma_{y^{t-1}}: \Omega \rightarrow \mathcal{X} \quad (20a)$$

$$i \mapsto e_t(i, y^{t-1}) \quad (20b)$$

to describe the encoding process. To simplify notation and where it is clear from the context, we shall often omit the subscript y^{t-1} and simply write γ .

In some examples we also allow for *randomized* encoding rules. In this case the encoding is described by the *random* encoding functions $\{\Gamma_{y^{t-1}}\}$ whose realizations $\gamma_{y^{t-1}}$ are of the form in (20). Again, for notational convenience we shall omit the subscript y^{t-1} where it is clear from the context.

Note that a variable-length code is more than a single encoding function but instead is an adaptive and sequential rule that dictates the choice of (random) encoding functions depending on the past channel observations and past selected encoding functions prior to the stopping time. In this paper, we refer to this adaptive and sequential rule as an encoding scheme, \mathfrak{c} , which together with the particular realization of channel outputs $y_0, y_1, \dots, y_{\tau-2}$, dictates the encoding functions $\Gamma_{y^0}^{\mathfrak{c}}, \Gamma_{y^1}^{\mathfrak{c}}, \dots, \Gamma_{y^{\tau-2}}^{\mathfrak{c}}$.

B. Asymptotic Bounds on Minimum Expected Length

In [1], Burnashev provided the following lower and upper bounds on the minimum expected number of channel uses, $\mathbb{E}[\tau_\epsilon^*]$, for a large class of DMCs and arbitrary $\epsilon > 0$.

Fact 2 (Theorems 1 and 2 in [1]). *For any DMC with $C > 0$ and $C_1 < \infty$:*

$$\mathbb{E}[\tau_\epsilon^*] \geq \left(\frac{\log M}{C} + \frac{\log \frac{1}{\epsilon}}{C_1} \right) (1 - o(1)), \quad (21)$$

and

$$\mathbb{E}[\tau_\epsilon^*] \leq \left(\frac{\log M}{C} + \frac{\log \frac{1}{\epsilon}}{C_1} \right) (1 + o(1)) \quad (22)$$

where $o(1) \rightarrow 0$ as $\epsilon \rightarrow 0$.²

Inequality (21) was proved in [1] using a Martingale argument, and it was reproved in alternative ways in [17], [18].

Burnashev proved the upper bound (22) using the following two-phase scheme [1]. While in the first phase (*communication phase*) the transmitter iteratively refines the receiver's belief about the true message, in the second phase (*confirmation phase*) it simply confirms whether the receiver's highest belief after the first phase corresponds to the true message. As shown in [2], [4] the specific sequential scheme in the first phase can be exchanged by any capacity achieving block coding schemes.

C. Stochastic Control View

The problem of variable-length coding with noiseless feedback is a decentralized team problem with two agents (the encoder and the decoder) and non-classical information structure [19].

²If $\epsilon \rightarrow 0$, then $o(1) \rightarrow 0$ regardless of M being fixed or $M \rightarrow \infty$. For fixed ϵ , $\mathbb{E}[\tau_\epsilon^*] \approx \frac{(1-\epsilon)\log M}{C}$ and hence, the positive term $o(1) \not\rightarrow 0$ even if $M \rightarrow \infty$ (see [17] for more details).

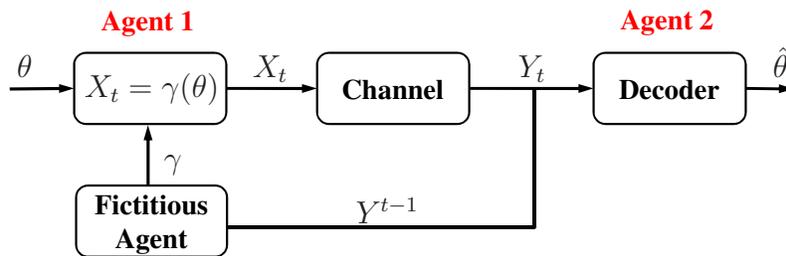


Fig. 2. Two-agent problem with common and private observations from the point of view of the fictitious agent.

Appealing to [20], the problem can be interpreted as a special case of active hypothesis testing [21] in which a (fictitious) Bayesian decision maker is responsible to enhance his information about the correct message in a speedy manner by sequentially sampling from conditionally independent observations at the output of the channel (given the input). Here the (fictitious) decision maker has access to the channel output symbols causally (common observations) and is responsible to control the conditional distribution of the observations given the true message (private observation) by selecting encoding functions for the encoder which map the message θ to the input symbols of the channel. In other words, as also observed in [9], the problem can be viewed as a (centralized) partially observable Markov decision problem (POMDP) with (static) state space Ω and the observation space \mathcal{Y} . Let $\mathcal{E} := \{\gamma(\cdot) : \Omega \rightarrow \mathcal{X}\}$ be the set of all mappings from Ω to \mathcal{X} . The action space (for the fictitious agent) becomes $\mathcal{E} \cup \{T\}$ where T denotes the termination of the transmission phase, hence the realization of the stopping time τ .

Casting the problem as a POMDP allows for the structural characterization of the information state, also known as sufficient statistics: Let the decision maker's belief about each possible message $i \in \Omega$, updated after each channel use (observation) for $t = 0, 1, \dots, \tau - 1$, be

$$\rho_i(t) := \Pr(\theta = i | Y^{t-1}). \quad (23)$$

The decision maker's posteriors about the messages collectively,

$$\boldsymbol{\rho}(t) := [\rho_1(t), \rho_2(t), \dots, \rho_M(t)], \quad (24)$$

form a sufficient statistics for our (fictitious) Bayesian decision maker. Furthermore, this (fictitious) decision maker's posterior at any time t coincides with the receiver's posterior and, thanks

to the perfect feedback, is available to the transmitter. (Notice that $\rho_i(0) = \Pr(\theta = i) = \frac{1}{M}$ denotes the receiver's initial belief of $\theta = i$ before the transmission starts.) In other words, the selection of encoding and decoding rules as a function of this posterior does not incur any loss of optimality [22]. In particular, the optimal receiver produces as its guess the message with the highest posterior at time τ , i.e.,

$$\hat{\theta} = \arg \max_{i \in \Omega} \rho_i(\tau). \quad (25)$$

We also note that the dynamics of the information state, i.e. the posterior, follows Bayes' rule. More specifically, given an encoding function γ at time t and an information state ρ , the conditional distribution of the next channel output Y_t , given the past observation Y^{t-1} , is

$$P_\rho(y) = \sum_{i=1}^M \rho_i P(Y = y | X = \gamma(i)).$$

Similarly, given also the output symbol $Y_t = y$, according to Bayes' rule, the posterior at time $t + 1$ is:

$$\rho(t + 1) = \left[\frac{\rho_1 P_{\gamma(1)}(y)}{P_\rho(y)}, \dots, \frac{\rho_M P_M(y)}{P_\rho(y)} \right],$$

Taking cue from the seminal work of DeGroot on statistical decision theory [23], the above stochastic control view of the variable-length coding has been used in [24], to characterize the performance of any given coding scheme using the information utility provided by the channel output. Information utility, here, generalizes the Shannon theoretic notion of mutual information [23], [24]. More specifically, consider any given measure of the uncertainty of the posterior vector; information utility is defined as the expected reduction in the uncertainty of the posterior at time $t + 1$ relative to that at time t . The result in [24], as also manifested in Lemma 2, implies a characterization of the performance of a given coding scheme in terms of the symmetric divergences JS and EJS between the conditional output distributions of the channel induced by the encoding function.

In the sections that follow, we utilize this connection and analytical tool in our achievability analysis. In particular, in Section IV we particularize the approach in [24] with respect to the EJS divergence induced by the encoding mapping. This allows us to provide achievability analysis for two sequential one-phase coding schemes, namely posterior matching and MaxEJS. These

schemes are based on the suboptimal stopping rule described in the next section. Furthermore, we show that MaxEJS coding scheme provably achieves Burnashev's asymptotic optimal performance given by (22).

D. A Suboptimal Stopping Rule

In this paper we focus on the following (possibly suboptimal) stopping rule. For any given coding scheme \mathfrak{c} , the transmission is only stopped when one of the posteriors becomes larger than $1 - \epsilon$, where $\epsilon > 0$ is the desired probability of error:

$$\tilde{\tau}_\epsilon := \min\{t : \max_{i \in \Omega} \rho_i(t) \geq 1 - \epsilon\}. \quad (26)$$

Let $\mathbb{E}[\tilde{\tau}_\epsilon^*]$ denote the optimal expected length of the code with the stopping rule as given in (26).

Lemma 3. Consider stopping times defined earlier with scalars $\iota \geq \epsilon > 0$. We have

$$\mathbb{E}[\tilde{\tau}_\iota^*] \left(1 - \frac{\epsilon}{\iota}\right) \leq \mathbb{E}[\tau_\epsilon^*] \leq \mathbb{E}[\tilde{\tau}_\epsilon^*]. \quad (27)$$

The proof of Lemma 3 is given in Appendix IV-A.

Furthermore,

Lemma 4. For any $\iota \in (0, 1)$, and for any $\delta \in (0, 1/2)$,

$$\mathbb{E}[\tilde{\tau}_\iota^*] \geq \left[\frac{\log M - F_M(\delta) - F_M(\iota)}{C} + \frac{\log \frac{1-\iota}{\iota} - \log \frac{1-\delta}{\delta} - \log C_2 - 1}{C_1} \right]^+ \quad (28)$$

where $F_M(z) := H([z, 1-z]) + z \log(M-1)$ for $0 \leq z \leq 1$.

The proof of Lemma 4 utilizes the dynamic programming characterization of the above stochastic control problem and is given in Appendix IV-B.

Note that combining (27) with (28) when $\iota = \frac{\epsilon}{2} \log \frac{4}{\epsilon}$ and $\delta = \frac{1}{\log \frac{4}{\epsilon}}$ provides an alternative proof for Burnashev's converse (21). In fact, by some algebraic manipulations and simple upper bounds, we obtain the inequalities (29), as shown below.

$$\begin{aligned} \mathbb{E}[\tau_\epsilon^*] &\geq \left(1 - \frac{2}{\log \frac{4}{\epsilon}}\right) \left[\frac{\left(1 - \frac{1}{\log \frac{4}{\epsilon}} - \frac{\epsilon}{2} \log \frac{4}{\epsilon}\right) \log M - 2}{C} + \frac{\log \frac{1 - \frac{\epsilon}{2} \log \frac{4}{\epsilon}}{\frac{\epsilon}{2} \log \frac{4}{\epsilon}} - \log \log \frac{2}{\epsilon} - \log C_2 - 1}{C_1} \right]^+ \\ &\geq \left[\frac{\left(1 - \frac{2}{\log \frac{4}{\epsilon}}\right) \left(1 - \frac{\epsilon}{2} \log \frac{4}{\epsilon}\right) \log M - \frac{\log M}{\log \frac{4}{\epsilon}} - 2}{C} \right]^+ \end{aligned}$$

$$\begin{aligned}
& + \frac{\left(1 - \frac{2}{\log \frac{4}{\epsilon}}\right) \log \frac{1}{\frac{\epsilon}{2} \log \frac{4}{\epsilon}} - \log \frac{1}{1 - \frac{\epsilon}{2} \log \frac{4}{\epsilon}} - \log \log \frac{2}{\epsilon} - \log C_2 - 1 \Big]^+}{C_1} \\
& \geq \left[\frac{\left(1 - \frac{2}{\log \frac{4}{\epsilon}} - \frac{\epsilon}{2} \log \frac{1}{\epsilon}\right) \log M - \frac{\log M}{\log \frac{4}{\epsilon}} - 2}{C} \right. \\
& \quad \left. + \frac{\log \frac{1-\epsilon}{\epsilon} - \log \log \frac{4}{\epsilon} - 1 - \log \frac{1-\epsilon}{1 - \frac{\epsilon}{2} \log \frac{4}{\epsilon}} - \log \log \frac{2}{\epsilon} - \log C_2 - 1 \Big]^+}{C_1} \right. \\
& \geq \left[\frac{\left(1 - \frac{3}{\log \frac{4}{\epsilon}} - \frac{\epsilon}{2} \log \frac{1}{\epsilon}\right) \log M - 2}{C} + \frac{\log \frac{1-\epsilon}{\epsilon} - 2 \log \log \frac{4}{\epsilon} - \log C_2 - 4}{C_1} \right]^+ \\
& \geq \left(\frac{\log M}{C} + \frac{\log \frac{1}{\epsilon}}{C_1} \right) \left(1 - \frac{\epsilon}{2} \log \frac{1}{\epsilon} + \frac{\log \frac{1}{1-\epsilon} + 2 \log \log \frac{4}{\epsilon} + \log C_2 + 4 + \frac{2C_1}{C}}{\log \frac{1}{\epsilon}} \right). \quad (29)
\end{aligned}$$

IV. MAIN RESULT AND APPLICATIONS

In this section, we first characterize the performance of an encoding scheme in terms of its corresponding extrinsic Jensen–Shannon (EJS) divergence obtained. To make this precise we first introduce some further notation.

Given a DMC $P(Y|X)$ and a (deterministic) encoding function $\gamma: \Omega \rightarrow \mathcal{X}$ together with a set of time- t posteriors $\boldsymbol{\rho}(t)$, we use the short hand notation

$$EJS(\boldsymbol{\rho}(t), \gamma) := EJS(\boldsymbol{\rho}(t); P_{\gamma(1)}, \dots, P_{\gamma(M)}). \quad (30)$$

For a (possibly) randomized encoding function Γ , we use

$$EJS(\boldsymbol{\rho}(t), \Gamma) := \sum_{\gamma \in \mathcal{E}} \Pr(\Gamma = \gamma | Y^{t-1} = y^{t-1}) EJS(\boldsymbol{\rho}(t), \gamma) \quad (31)$$

where recall that \mathcal{E} denotes the set of all possible encoding functions.

A. Main Theorem

Let

$$\tilde{\rho} := 1 - \frac{1}{1 + \max\{\log M, \log \frac{1}{\epsilon}\}}. \quad (32)$$

Theorem 1. Consider a (possibly randomized) encoding scheme \mathfrak{c} under which at each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and for each y^{t-1} the encoding function $\Gamma^{\mathfrak{c}}$ satisfies

$$EJS(\boldsymbol{\rho}(t), \Gamma^{\mathfrak{c}}) \geq R_{\min}, \quad (33a)$$

and furthermore,

$$EJS(\boldsymbol{\rho}(t), \Gamma^c) \geq \tilde{\rho} E_{\min} \quad \text{if } \max_{i \in \Omega} \rho_i(t) \geq \tilde{\rho}, \quad (33b)$$

for some $E_{\min} \geq R_{\min} > 0$. Then,

$$\mathbb{E}_c[\tilde{\tau}_\epsilon] \leq \frac{\log M + \log \log \frac{M}{\epsilon}}{R_{\min}} + \frac{\log \frac{1}{\epsilon} + 1}{E_{\min}} + \frac{6(4C_2)^2}{R_{\min} E_{\min}} \quad (34)$$

where C_2 is defined in (16).

Corollary 1. Under the assumptions of Theorem 1,

$$\mathbb{E}_c[\tilde{\tau}_\epsilon] \leq \left(\frac{\log M}{R_{\min}} + \frac{\log \frac{1}{\epsilon}}{E_{\min}} \right) (1 + o(1)) \quad (35)$$

where $o(1) \rightarrow 0$ as $\epsilon \rightarrow 0$ or $M \rightarrow \infty$.

The proof of Theorem 1 is given in Appendix II. Here we provide a brief sketch of the proof. Let $\mathcal{F}(t)$ denote the history of the receiver's knowledge up to time t , i.e., $\mathcal{F}(t) = \sigma\{Y^{t-1}\}$, and let

$$\tilde{U}(t) := \sum_{i=1}^M \rho_i(t) \log \frac{\rho_i(t)}{1 - \rho_i(t)} - \log \frac{\tilde{\rho}}{1 - \tilde{\rho}}.$$

Since,

$$\mathbb{E}_c \left[\tilde{U}(t+1) | \mathcal{F}(t) \right] = \tilde{U}(t) + EJS(\boldsymbol{\rho}(t), \Gamma^c), \quad (36)$$

the sequence $\{\tilde{U}(t)\}$ forms a submartingale. The assertion of the theorem directly results from the following fact about submartingales: For any submartingale $\{\xi(t)\}$, $t = 0, 1, 2, \dots$, if there exist positive constants K_1 and K_2 such that

$$\mathbb{E}[\xi(t+1) | \mathcal{F}(t)] \geq \xi(t) + K_1 \quad \text{if } \xi(t) < 0,$$

$$\mathbb{E}[\xi(t+1) | \mathcal{F}(t)] \geq \xi(t) + K_2 \quad \text{if } \xi(t) \geq 0,$$

then, under certain technical conditions, the stopping time $v = \min\{t : \xi(t) \geq B\}$, $B > 0$ can be approximately upper bounded as

$$\mathbb{E}[v] \lesssim \frac{B - \xi(0)}{K_2} + \xi(0) \mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right).$$

B. Application I: Variable-Length Posterior Matching

We consider a variable-length version of the coding schemes in [6]–[8]. At each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$, if $\theta = i$ and given the posterior vector $\boldsymbol{\rho}(t)$, the input $X(t)$ takes value in the set

$$\mathcal{X}_i(t) := \left\{ x \in \mathcal{X} : \sum_{i'=1}^{i-1} \rho_{i'}(t) < \sum_{x' \leq x} \pi_{x'}^* \quad \text{and} \quad \sum_{x' < x} \pi_{x'}^* \leq \sum_{i'=1}^i \rho_{i'}(t) \right\};$$

where each value $x \in \mathcal{X}_i(t)$ is taken with probability

$$\Pr(X(t) = x | \theta = i, Y^{t-1} = y^{t-1}) = \frac{\min \left\{ \sum_{i'=1}^i \rho_{i'}(t), \sum_{x' \leq x} \pi_{x'}^* \right\} - \max \left\{ \sum_{i'=1}^{i-1} \rho_{i'}(t), \sum_{x' < x} \pi_{x'}^* \right\}}{\rho_i(t)}.$$

Proposition 1. *Under the above variable-length posterior matching encoding, and for each $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and all possible output sequences y^{t-1} ,*

$$EJS(\boldsymbol{\rho}(t), \Gamma^{\text{PM}}) \geq C.$$

The proof of Proposition 1 is given in Appendix III-A.

Remark 3. By Theorem 1 and Proposition 1, under the variable-length posterior matching encoding

$$\mathbb{E}_{\Gamma^{\text{PM}}}[\tilde{\tau}_\epsilon] \leq \frac{\log M + \log \frac{1}{\epsilon} + 1 + \log \log \frac{M}{\epsilon}}{C} + \frac{6(4C_2)^2}{C^2}. \quad (37)$$

C. Application II: MaxEJS Coding

We present a new coding scheme based on the greedy maximization of EJS divergence. At each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and given the posterior vector $\boldsymbol{\rho}(t)$, MaxEJS chooses the γ^* that maximizes the EJS divergence:

$$\gamma^* := \arg \max_{\gamma \in \mathcal{E}} EJS(\boldsymbol{\rho}(t), \gamma). \quad (38)$$

Proposition 2. *For every $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and all possible output sequences y^{t-1} , MaxEJS encoding satisfies*

$$EJS(\boldsymbol{\rho}(t), \gamma^*) \geq C, \quad (39a)$$

and furthermore,

$$EJS(\boldsymbol{\rho}(t), \gamma^*) \geq \tilde{\rho}C_1 \quad \text{if} \quad \max_{i \in \Omega} \rho_i(t) \geq \tilde{\rho}. \quad (39b)$$

The proof of Proposition 2 is given in Appendix III-B.

Remark 4. By Theorem 1 and Proposition 2,

$$\mathbb{E}_{\Gamma^{\text{MaxEJS}}}[\tilde{r}_\epsilon] \leq \frac{\log M + \log \log \frac{M}{\epsilon}}{C} + \frac{\log \frac{1}{\epsilon} + 1}{C_1} + \frac{6(4C_2)^2}{CC_1}, \quad (40)$$

and thus MaxEJS encoding together with the decoding and stopping rules described in (25) and (26) achieves Burnashev's optimal asymptotic performance in (22), see Corollary 1.

Remark 5. The presented deterministic one-phase sequential scheme differs from the previous schemes achieving Burnashev's optimal asymptotic performance, which are randomized and have two phases [1]–[4].

The computational complexity of the MaxEJS coding scheme could be prohibitive. In Section V-B, we propose simpler coding schemes for a class of binary-input channels that achieve Burnashev's optimal asymptotic performance in (22).

V. CODING FOR SYMMETRIC BINARY-INPUT CHANNELS

In this subsection, we focus on channels with binary inputs $\mathcal{X} = \{0, 1\}$ and with the following property

$$P(Y = y|X = 0) = P(Y = z - y|X = 1), \quad \forall y \in \mathcal{Y} \quad (41)$$

for some $z \in \mathbb{R}$.

The first attempt to address the problem of coding over a symmetric binary-input channel goes back to Horstein's coding scheme [6] over a binary symmetric channel (BSC) with a crossover probability $p \in (0, 1/2)$. Horstein considered the message to be a point in the interval $[0, 1]$ and suggested that to achieve the capacity of the channel, at any given time the transmitter selects the input of the channel such as to signal to the receiver whether the message is smaller than the median of the posterior or larger. Later, Burnashev and Zigangirov [7], presented a similar (randomized) coding scheme for discrete message sets as in (17) and proved that this scheme achieves capacity.

In Section V-A, we present and analyze a *deterministic* scheme for arbitrary symmetric binary-input channels satisfying (41), which resembles the Burnashev-Zigangirov scheme, when specialized to the BSC. In Section V-B, we then improve our scheme so that it achieves

Burnashev's optimal asymptotic performance in (22) over this class of symmetric binary-input channels.

A. Generalized Horstein-Burnashev-Zigangirov Scheme

Our generalization of the Horstein-Burnashev-Zigangirov scheme is deterministic. For each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and given the posterior vector $\boldsymbol{\rho}(t)$, we choose the encoding function:

$$\gamma^{\text{GHBZ}}(i) = \begin{cases} 0 & 1 \leq i \leq k^* \\ 1 & k^* < i \leq M \end{cases} \quad (42)$$

where

$$k^* := \arg \min_{k \in \Omega} \left| \sum_{i=1}^k \rho_i(t) - \frac{1}{2} \right|. \quad (43)$$

Proposition 3. Consider the deterministic scheme proposed above over a binary-input DMC that satisfies (41). For every $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and all possible output sequences y^{t-1} ,

$$EJS(\boldsymbol{\rho}(t), \gamma^{\text{GHBZ}}) \geq C. \quad (44)$$

The proof is given in Appendix III-C.

Remark 6. By Theorem 1 and Proposition 3, the described encoding satisfies

$$\mathbb{E}_{\gamma^{\text{GHBZ}}}[\tilde{\tau}_\epsilon] \leq \frac{\log M + \log \frac{1}{\epsilon} + 1 + \log \log \frac{M}{\epsilon}}{C} + \frac{6(4C_2)^2}{C^2}. \quad (45)$$

Notice that, when specialized to a binary-input channel, the variable-length posterior matching scheme of Section IV-B, at each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and given the posterior vector $\boldsymbol{\rho}(t)$, chooses encoding function γ^{GHBZ} with probability

$$\lambda_{\gamma^{\text{GHBZ}}} = \frac{\delta_2(t)}{\delta_1(t) + \delta_2(t)} \quad (46)$$

where

$$\delta_1(t) := \left| \sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} \right|, \quad \delta_2(t) := \left| \sum_{i=1}^{k_2^*} \rho_i(t) - \frac{1}{2} \right|, \quad (47)$$

and

$$k_2^* := k^* - \text{sign} \left(\sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} \right); \quad (48)$$

and it chooses the encoding function

$$\bar{\gamma}^{\text{GHBZ}}(i) = \begin{cases} 0 & 1 \leq i \leq k_2^* \\ 1 & k_2^* < i \leq M \end{cases} \quad (49)$$

with probability $\bar{\lambda}_{\gamma^{\text{GHBZ}}} = 1 - \lambda_{\gamma^{\text{GHBZ}}}$.

Combining Proposition 3 with Proposition 1, we have that there exists a class (a continuum) of randomized schemes that satisfy (44):

Corollary 2. *Every (randomized) encoding function Γ that selects γ^{GHBZ} with probability $\lambda \geq \lambda_{\gamma^{\text{GHBZ}}}$ in (46) and selects $\bar{\gamma}^{\text{GHBZ}}$ with probability $\bar{\lambda} = 1 - \lambda$, satisfies (33) with $R_{\min} = E_{\min} = C$.*

This corollary provides an alternative proof that Burnashev and Zigangirov's variable-length coding scheme [7] satisfies (45) over the BSC with crossover probability $p \in (0, 1/2)$. In fact, their scheme selects γ^{GHBZ} and $\bar{\gamma}^{\text{GHBZ}}$ with probabilities $\lambda = \frac{\nu(\delta_2(t))}{\nu(\delta_1(t)) + \nu(\delta_2(t))}$ and $\bar{\lambda} = 1 - \lambda$, respectively, where $\nu(x) = \log \frac{0.5 + (1-2p)x}{0.5 - (1-2p)x}$. We next prove that $\frac{\nu(\delta_2(t))}{\nu(\delta_1(t)) + \nu(\delta_2(t))} \geq \frac{\delta_2(t)}{\delta_1(t) + \delta_2(t)}$, which by Corollary 2 establishes that the Burnashev-Zigangirov scheme indeed satisfies (45).

Notice that $\nu(x) = \log \left(-1 + \frac{1}{0.5 - (1-2p)x} \right)$ is convex for all x because $p \in (0, 1/2)$. Since also $f: x \mapsto \frac{\nu(x)}{\nu(\delta_2(t))}$ is convex and since $f(0) = 0$ and $f(\delta_2(t)) = 1$, we conclude that $\frac{\nu(x)}{\nu(\delta_2(t))} \leq \frac{x}{\delta_2(t)}$, for all $x \in [0, \delta_2(t)]$. By (47) and (48), $0 \leq \delta_1(t) \leq \delta_2(t)$ and hence $\frac{\nu(\delta_1(t))}{\nu(\delta_2(t))} \leq \frac{\delta_1(t)}{\delta_2(t)}$. This immediately establishes the desired inequality $\frac{\nu(\delta_2(t))}{\nu(\delta_1(t)) + \nu(\delta_2(t))} \geq \frac{\delta_2(t)}{\delta_1(t) + \delta_2(t)}$.

B. Optimal Binary Variable-Length Codes

Motivated by the analysis above, we strive to simplify our deterministic one-phase MaxEJS scheme for the simpler symmetric binary-input channels. We propose the following encoding scheme. At each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and each sequence of observations $Y^{t-1} = y^{t-1}$, we choose the encoding function γ in a way that for all $i \in \{j \in \Omega: \gamma(j) = 0\}$,

$$0 \leq \sum_{j \in \Omega: \gamma(j)=0} \rho_j(t) - \sum_{j \in \Omega: \gamma(j)=1} \rho_j(t) < \rho_i(t). \quad (50)$$

By condition (50), at each time t , the probabilities of sending a 0 or a 1 are approximately $(1/2, 1/2)$ when all posteriors $\{\rho_i(t)\}_{i \in \Omega}$ are small, and they are $(\max_{i \in \Omega} \rho_i(t), 1 - \max_{i \in \Omega} \rho_i(t))$ when $\max_{i \in \Omega} \rho_i(t)$ is larger than $1/2$.

Proposition 4. *If for every $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and every sequence of observations $Y^{t-1} = y^{t-1}$ the encoding function γ satisfies (50), then*

$$EJS(\boldsymbol{\rho}(t), \gamma) \geq C, \quad (51a)$$

and

$$EJS(\boldsymbol{\rho}(t), \gamma) \geq \tilde{\rho}C_1 \quad \text{if } \max_{i \in \Omega} \rho_i(t) \geq \tilde{\rho}. \quad (51b)$$

Remark 7. By Theorem 1 and Proposition 2,

$$\mathbb{E}[\tilde{\tau}_\epsilon] \leq \frac{\log M + \log \log \frac{M}{\epsilon}}{C} + \frac{\log \frac{1}{\epsilon} + 1}{C_1} + \frac{6(4C_2)^2}{CC_1}, \quad (52)$$

and thus the encoding rule described above together with the decoding and stopping rules described in (25) and (26) achieves Burnashev's optimal asymptotic performance in (22), see Corollary 1.

In the following we present two algorithms that at each time $t = 0, 1, \dots, \tilde{\tau}_\epsilon - 1$ and for given posterior vector $\boldsymbol{\rho}(t)$ implement encoding functions γ satisfying (50).

Algorithm 1:

```

1  $\delta = 1.$ 
2 for  $n = 1, \dots, 2^M$  do
3    $v = \text{dec2bin}(n, M)$    % binary representation of  $n$  with  $M$  digits.
4    $z = (2v - 1) \times [\rho_1(t), \rho_2(t), \dots, \rho_M(t)]^\top.$ 
5   if  $z > 0$  &&  $z < \delta$  then
6      $\delta = z.$ 
7      $\hat{v} = v.$ 
8   end
9 end
10 for  $i = 1, \dots, M$  do
11    $\gamma(i) = \hat{v}_i$    %  $\hat{v}_i$  denotes  $i$ -th bit of  $\hat{v}.$ 
12 end

```

Algorithm 2:

```

1  $S_0 = \{1, 2, \dots, M\}$  and  $S_1 = \emptyset$ .
2  $r_0 = 1, r_1 = 0, \rho_{\min} = 0$ , and  $\delta = 1$ .
3 while  $\rho_{\min} < \delta$  do
4    $k = \arg \min_{i \in S_0} \rho_i(t)$ .
5    $S_0 = S_0 - \{k\}$  and  $S_1 = S_1 \cup \{k\}$ .
6    $r_0 = r_0 - \rho_k(t)$  and  $r_1 = r_1 + \rho_k(t)$ .
7   if  $r_0 < r_1$  then
8     Swap  $S_0$  and  $S_1$ .
9     Swap  $r_0$  and  $r_1$ .
10  end
11   $\delta = r_0 - r_1$ .
12   $\rho_{\min} = \min_{i \in S_0} \rho_i(t)$ .
13 end
14 for  $i = 1, \dots, M$  do
15    $\gamma(i) = \begin{cases} 0 & \text{if } i \in S_0 \\ 1 & \text{if } i \in S_1 \end{cases}$ .
16 end

```

Proposition 5. *Both Algorithms 1 and 2 satisfy condition (50). Algorithm 1 has computational complexity of order $O(2^M)$ for each encoding step while Algorithm 2 has complexity of order $O(M^2)$.*³

The proof is given in Appendix III-E.

Remark 8. In contrast to the previous one-phase sequential schemes in [6]–[8], the encoding

³The computational complexity of Algorithm 1 is of the same order as that of MaxEJS which in each step requires to find an encoding function (among 2^M choices) that maximizes the EJS divergence between the conditional output distributions. However, implementation of Algorithm 1 is simpler since it only requires linear operations instead of computing the EJS divergence (which can be computationally intensive, especially for channels with large output alphabet set).

processes described by Algorithms 1 and 2 here are completely deterministic. By insisting on a deterministic encoding, we can match our scheme's inputs only *approximately* to the capacity-achieving input distribution of $(1/2, 1/2)$. On the other hand, the proposed deterministic schemes are such that once a particular message's posterior passes a certain threshold, the transmitter assigns this message exclusively to one of the two inputs. This is critical to achieve the optimal $E_{\min} = C_1$.

Remark 9. The proofs of Propositions 4 and 5 continue to hold for those binary-input channels with uniform capacity-achieving input distribution $\pi_0^* = \pi_1^* = 1/2$ where for ease of notation we assume that $C_1 = D(P_0 \| P_1)$. This class of channels includes the class of channels for which (41) holds, for example the binary symmetric channel (BSC) with cross-over probability $p \in (0, 1/2)$, as well as the non-symmetric channel in Fig. 3 for $\eta \in (0, 1/2)$.

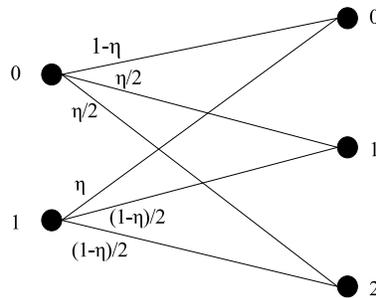


Fig. 3. Example of a non-symmetric (binary-input ternary-output) channel with capacity-achieving input distribution $\pi_0^* = \pi_1^* = 1/2$.

Remark 10. The results in Proposition 4 and Remark 7 above can also be extended to the case of K -ary symmetric channel with alphabet sets $\mathcal{X} = \mathcal{Y} = \{0, 1, \dots, K-1\}$ and transition probabilities of the form

$$P(Y = y | X = x) = \begin{cases} 1 - p & \text{if } x = y \\ \frac{p}{K-1} & \text{if } x \neq y \end{cases}$$

where $p \in (0, \frac{K-1}{K})$. Consider a coding scheme that at each time t prior to the stopping time

chooses the encoding function γ in a way that if for any $x, x' \in \mathcal{X}$,

$$\sum_{j \in \Omega: \gamma(j)=x} \rho_j(t) \geq \max \left\{ \frac{1}{K}, \sum_{j \in \Omega: \gamma(j)=x'} \rho_j(t) \right\},$$

then for all $i \in \{j \in \Omega: \gamma(j) = x\}$,

$$\sum_{j \in \Omega: \gamma(j)=x} \rho_j(t) - \sum_{j \in \Omega: \gamma(j)=x'} \rho_j(t) \leq \rho_i(t).$$

This coding scheme together with the decoding and stopping rules described in (25) and (26) achieves Burnashev's optimal asymptotic performance in (22) for the K -ary symmetric channel.

VI. RELIABILITY FUNCTION

Let a variable-length coding scheme \mathfrak{c} be given that for each positive integer ℓ can transmit one out of $M_{\mathfrak{c}_\ell}$ equiprobable messages at a probability $\text{Pe}_{\mathfrak{c}_\ell}$ and with an expected stopping time $\mathbb{E}_{\mathfrak{c}_\ell}[\tau]$. If for any small numbers $\delta > 0$, $0 \leq \epsilon < 1$ and all sufficiently large ℓ the following three conditions

$$\text{Pe}_{\mathfrak{c}_\ell} \leq \epsilon \tag{53a}$$

$$M_{\mathfrak{c}_\ell} \geq 2^{\ell(R-\delta)} \tag{53b}$$

$$\mathbb{E}_{\mathfrak{c}_\ell}[\tau] \leq \ell, \tag{53c}$$

hold for some positive real number R , then we say that the scheme \mathfrak{c} achieves (information) rate R .⁴

If \mathfrak{c} satisfies (53b) and (53c) but instead of (53a) it satisfies a stronger condition on exponential decay

$$\text{Pe}_{\mathfrak{c}_\ell} \leq 2^{-\ell(E-\delta)} \tag{54}$$

for some positive real number E , then we say that the scheme \mathfrak{c} achieves reliability E at rate R .

The capacity of a DMC is defined as the largest rate R that is achievable over this channel; it is equal to the Shannon capacity C as defined in (14) [15, p. 184]. For a given rate R below capacity, the reliability function $E(R)$ is defined as the maximum achievable error exponent at rate R . By Burnashev's lower bound in (21), we have the following lemma:

⁴It would be more precise to talk about *sequence of schemes* $\{\mathfrak{c}_\ell\}_{\ell \in \mathbb{Z}^+}$, where each \mathfrak{c}_ℓ is the general scheme \mathfrak{c} specialized to the message size $M_{\mathfrak{c}_\ell}$. However, this would make the notation overcomplicated.

Lemma 5. *No coding scheme can achieve diminishing error probability at rates higher than C . Furthermore,*

$$E(R) \leq C_1 \left(1 - \frac{R}{C}\right), \quad R \in (0, C). \quad (55)$$

Proof of Lemma 5: Let \mathfrak{c} be a coding scheme that for each $\ell \in \mathbb{Z}^+$ and for a message size $M_{\mathfrak{c}_\ell}$ satisfies (53) for a rate $R > 0$.

By (21) and (53), for each sufficiently large integer ℓ :

$$\begin{aligned} \ell \geq \mathbb{E}_{\mathfrak{c}_\ell}[\tau] &\geq \left(\frac{\log M_{\mathfrak{c}_\ell}}{C} + \frac{\log(1/\text{Pe}_{\mathfrak{c}_\ell})}{C_1}\right)(1 - o(1)) \\ &\geq \left(\frac{R\ell}{C} + \frac{\log(1/\text{Pe}_{\mathfrak{c}_\ell})}{C_1}\right)(1 - o(1)). \end{aligned} \quad (56)$$

In other words,

$$\begin{aligned} C &\geq \left(R + \frac{C}{C_1} \cdot \frac{\log(1/\text{Pe}_{\mathfrak{c}_\ell})}{\ell}\right)(1 - o(1)) \\ &\geq R(1 - o(1)) \end{aligned} \quad (57)$$

where the last inequality holds because $\log \frac{1}{\text{Pe}_{\mathfrak{c}_\ell}} \geq 0$. Since $o(1) \rightarrow 0$ as $\text{Pe}_{\mathfrak{c}_\ell} \rightarrow 0$, we obtain from (57) that $R \leq C$. This implies that no coding scheme can achieve diminishing error probability at rates higher than C .

Next we characterize an upper bound on the optimal reliability function $E(R)$. Let \mathfrak{c} be a coding scheme that for each $\ell \in \mathbb{Z}^+$ and for a message size $M_{\mathfrak{c}_\ell}$ satisfies (53b), (53c), and (54) for $E, R > 0$. By (21), (53b), and (54), for each sufficiently large integer ℓ :

$$\begin{aligned} \ell \geq \mathbb{E}_{\mathfrak{c}_\ell}[\tau] &\geq \left(\frac{\log M_{\mathfrak{c}_\ell}}{C} + \frac{\log(1/\text{Pe}_{\mathfrak{c}_\ell})}{C_1}\right)(1 - o(1)) \\ &\geq \left(\frac{R\ell}{C} + \frac{E\ell}{C_1}\right)(1 - o(1)). \end{aligned} \quad (58)$$

In other words,

$$1 \geq \left(\frac{R}{C} + \frac{E}{C_1}\right)(1 - o(1)). \quad (59)$$

Since $o(1) \rightarrow 0$ as $\ell \rightarrow \infty$, we obtain that $\frac{R}{C} + \frac{E}{C_1} \leq 1$. The desired inequality follows:

$$E \leq C_1 \left(1 - \frac{R}{C}\right). \quad (60)$$

■

On the other hand, we have the following achievable bound on rate–reliability function:

Lemma 6. *Suppose that we have a coding scheme \mathfrak{c} that for each message size $M > 0$ and each positive $\epsilon > 0$, satisfies $\text{Pe}_{\mathfrak{c}} \leq \epsilon$ with expected stopping time*

$$\mathbb{E}_{\mathfrak{c}}[\tau] \leq \left(\frac{\log M}{R_{\min}} + \frac{\log \frac{1}{\epsilon}}{E_{\min}} \right) (1 + o(1)) \quad (61)$$

for some positive integers E_{\min} and R_{\min} . Then, the scheme \mathfrak{c} can achieve any rate $R \in [0, R_{\min}]$ with reliability E , if

$$E \leq E_{\min} \left(1 - \frac{R}{R_{\min}} \right). \quad (62)$$

Thus, if a scheme \mathfrak{c} satisfies (61) for $R_{\min} = C$ and $E_{\min} = C_1$, then this scheme achieves Burnashev's optimal reliability function.

Proof of Lemma 6: Fix a small $\delta > 0$, a positive rate $R < R_{\min}$ and a positive error exponent E satisfying (62). Define for each $\ell \in \mathbb{Z}^+$, the small number $\epsilon_{\ell} \triangleq 2^{-\ell(E-\delta)}$ and the message size $M_{\ell} \triangleq 2^{\ell(R-\delta)}$. By assumption, for each $\ell \in \mathbb{Z}^+$, our coding scheme \mathfrak{c} attains a probability of error $\text{Pe}_{\mathfrak{c}_{\ell}} \leq \epsilon_{\ell}$ at an expected stopping time $\mathbb{E}_{\mathfrak{c}_{\ell}}[\tau_{\epsilon_{\ell}}]$ that is upper bounded as:

$$\begin{aligned} \mathbb{E}_{\mathfrak{c}_{\ell}}[\tau_{\epsilon_{\ell}}] &\leq \ell \left(\frac{R-\delta}{R_{\min}} + \frac{E-\delta}{E_{\min}} \right) (1 + o(1)) \\ &\leq \ell \left(1 - \frac{\delta}{R_{\min}} - \frac{\delta}{E_{\min}} \right) (1 + o(1)). \end{aligned} \quad (63)$$

Since $\delta > 0$ and since $o(1) \rightarrow 0$ as $\ell \rightarrow \infty$, we obtain that for sufficiently large ℓ ,

$$\mathbb{E}_{\mathfrak{c}_{\ell}}[\tau_{\epsilon_{\ell}}] \leq \ell. \quad (64)$$

Combined with our assumptions that $\text{Pe}_{\mathfrak{c}_{\ell}} \leq 2^{-\ell(E-\delta)}$ and $M_{\ell} \triangleq 2^{\ell(R-\delta)}$, this concludes the proof. ■

Corollary 1 combined with Lemma 6 provides the following:

Corollary 3 (Rate–Reliability Test). *Consider a DMC with $C > 0$ and $C_1 < \infty$ and a variable-length coding scheme \mathfrak{c} . If—irrespective of the size of the message set M —for any time t prior to the stopping time and for any posterior vector $\boldsymbol{\rho}(t)$ over the messages, the scheme selects (a possibly random) encoding function $\Gamma^{\mathfrak{c}}$ such that*

$$EJS(\boldsymbol{\rho}(t), \Gamma^{\mathfrak{c}}) \geq C, \quad (65a)$$

then it achieves the capacity C of the channel. Furthermore, if also,

$$EJS(\boldsymbol{\rho}(t), \Gamma^c) \geq \tilde{\rho}C_1 \quad \text{if } \max_{i \in \Omega} \rho_i(t) \geq \tilde{\rho}, \quad (65b)$$

then the scheme also achieves the optimal reliability function $E(R)$ of the channel.

The above corollary implies that all coding schemes described in Sections IV and V achieve the capacity C of the corresponding channels. Furthermore, the MaxEJS coding scheme and the simple coding scheme for the symmetric binary-input channel discussed in Section V-B achieve the reliability function $E(R)$.

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APPENDIX I

PROOF OF LEMMA 2

Property 1 is proved as follows:

$$\begin{aligned} JS(\boldsymbol{\rho}; P_1, \dots, P_M) &= \sum_{i=1}^M \rho_i D\left(P_i \parallel \sum_{j=1}^M \rho_j P_j\right) \\ &= \sum_{i=1}^M \rho_i D\left(P_i \parallel \rho_i P_i + (1 - \rho_i) \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) \\ &\stackrel{(a)}{\leq} \sum_{i=1}^M \left[\rho_i^2 D(P_i \parallel P_i) + \rho_i (1 - \rho_i) D\left(P_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) \right] \\ &= EJS(\boldsymbol{\rho}; P_1, \dots, P_M) - \sum_{i=1}^M \rho_i^2 D\left(P_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) \\ &\stackrel{(b)}{\leq} EJS(\boldsymbol{\rho}; P_1, \dots, P_M) \end{aligned}$$

where (a) and (b) follow respectively because KL divergence is convex (in both arguments) and nonnegative.

The proof of property 2 is provided next.

$$\begin{aligned}
EJS(\boldsymbol{\rho}; P_1, \dots, P_M) &= \sum_{i=1}^M \rho_i D\left(P_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) \\
&= \sum_{i=1}^M \rho_i \sum_{y \in \mathcal{Y}} P_i(y) \log \frac{P_i(y)}{\sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j(y)} \\
&= \sum_{i=1}^M \rho_i \log \frac{1 - \rho_i}{\rho_i} + \sum_{i=1}^M \sum_{y \in \mathcal{Y}} \rho_i P_i(y) \log \frac{\rho_i P_i(y)}{\sum_{j \neq i} \rho_j P_j(y)} \\
&= U(\boldsymbol{\rho}) + \sum_{y \in \mathcal{Y}} P_\rho(y) \sum_{i=1}^M \frac{\rho_i P_i(y)}{P_\rho(y)} \log \frac{\frac{\rho_i P_i(y)}{P_\rho(y)}}{1 - \frac{\rho_i P_i(y)}{P_\rho(y)}} \\
&= U(\boldsymbol{\rho}) - \sum_{y \in \mathcal{Y}} P_\rho(y) U\left(\left[\frac{\rho_1 P_1(y)}{P_\rho(y)}, \dots, \frac{\rho_M P_M(y)}{P_\rho(y)}\right]\right).
\end{aligned}$$

Property 3 is proved as follows.

Let P_1, P_2, \dots, P_M and Q_1, Q_2, \dots, Q_M be two set of distributions. For any $\lambda \in [0, 1]$ and $\bar{\lambda} = 1 - \lambda$,

$$\begin{aligned}
&EJS(\boldsymbol{\rho}; \lambda P_1 + \bar{\lambda} Q_1, \dots, \lambda P_M + \bar{\lambda} Q_M) \\
&= \sum_{i=1}^M \rho_i D\left(\lambda P_i + \bar{\lambda} Q_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} \lambda P_j + \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} \bar{\lambda} Q_j\right) \\
&\stackrel{(a)}{\leq} \sum_{i=1}^M \rho_i \left[\lambda D\left(P_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_j\right) + \bar{\lambda} D\left(Q_i \parallel \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} Q_j\right) \right] \\
&= \lambda EJS(\boldsymbol{\rho}; P_1, \dots, P_M) + \bar{\lambda} EJS(\boldsymbol{\rho}; Q_1, \dots, Q_M)
\end{aligned}$$

where (a) follows because KL divergence is convex in both arguments.

APPENDIX II

PROOF OF THEOREM 1

Let $\mathcal{F}(t)$ denote the history of the receiver's knowledge up to time t , i.e., $\mathcal{F}(t) = \sigma\{Y^{t-1}\}$. Moreover, for each time $t = 0, 1, \dots, \tau$, define

$$\tilde{U}(t) := \sum_{i=1}^M \rho_i(t) \log \frac{\rho_i(t)}{1 - \rho_i(t)} - \log \frac{\tilde{\rho}}{1 - \tilde{\rho}}$$

where recall that we defined $\tilde{\rho} = 1 - \frac{1}{1 + \max\{\log M, \log \frac{1}{\epsilon}\}}$. (For $M \geq 2$ and $\epsilon \leq 1$ which is the region of interest for these parameters, $\tilde{\rho} \geq \frac{1}{2}$.)

Notice that for all $i \in \Omega$ and given the observation $Y^{t-1} = y^{t-1}$, upon observing the new sample y_t , the belief state evolves as

$$\rho_i(t+1) = \frac{\rho_i(t)P(Y = y_t|X = \gamma_{y^{t-1}}(i))}{\sum_{j=1}^M \rho_j(t)P(Y = y_t|X = \gamma_{y^{t-1}}(j))}.$$

Furthermore,

$$\Pr(Y_t = y|Y^{t-1}) = \sum_{j=1}^M \rho_j(t)P(Y = y|X = \gamma_{y^{t-1}}(j)).$$

Under a (possibly randomized) coding scheme \mathfrak{c} ,

$$\begin{aligned} & \mathbb{E}_{\mathfrak{c}} \left[\sum_{i=1}^M \rho_i(t+1) \log \frac{\rho_i(t+1)}{1 - \rho_i(t+1)} \middle| \mathcal{F}(t) \right] \\ &= \sum_{\gamma \in \mathcal{E}} \Pr(\Gamma^{\mathfrak{c}} = \gamma | Y^{t-1} = y^{t-1}) \sum_{y \in \mathcal{Y}} \sum_{i=1}^M \rho_i(t) P_{\gamma(i)}(y) \log \frac{\rho_i(t) P_{\gamma(i)}(y)}{\sum_{j \neq i} \rho_j(t) P_{\gamma(j)}(y)} \\ &= \sum_{i=1}^M \rho_i(t) \log \frac{\rho_i(t)}{1 - \rho_i(t)} + \sum_{\gamma \in \mathcal{E}} \Pr(\Gamma^{\mathfrak{c}} = \gamma | Y^{t-1} = y^{t-1}) D \left(P_{\gamma(i)} \parallel \sum_{j \neq i} \frac{\rho_j(t)}{1 - \rho_i(t)} P_{\gamma(j)} \right) \\ &= \sum_{i=1}^M \rho_i(t) \log \frac{\rho_i(t)}{1 - \rho_i(t)} + EJS(\boldsymbol{\rho}(t), \Gamma^{\mathfrak{c}}). \end{aligned}$$

which implies that

$$\mathbb{E}_{\mathfrak{c}} \left[\tilde{U}(t+1) | \mathcal{F}(t) \right] = \tilde{U}(t) + EJS(\boldsymbol{\rho}(t), \Gamma^{\mathfrak{c}}). \quad (66)$$

From (66) and condition (33) of Theorem 1, the sequence $\{\tilde{U}(t)\}_{t=0}^{\tau}$ satisfies

$$\mathbb{E}_{\mathfrak{c}} \left[\tilde{U}(t+1) | \mathcal{F}(t) \right] \geq \begin{cases} \tilde{U}(t) + R_{\min} & \text{if } \tilde{U}(t) < 0 \\ \tilde{U}(t) + \tilde{\rho} E_{\min} & \text{if } \tilde{U}(t) \geq 0 \end{cases}. \quad (67)$$

The sequence $\{\tilde{U}(t)\}_{t=0}^{\tau}$ forms a submartingale with respect to the filtration $\{\mathcal{F}(t)\}$. Furthermore, from Lemma 7 below,

$$|\tilde{U}(t+1) - \tilde{U}(t)| \leq 4C_2 \text{ if } \max\{\tilde{U}(t), \tilde{U}(t+1)\} \geq 0. \quad (68)$$

Note that if $\rho_i(t) < 1 - \epsilon$ for all $i \in \Omega$, then

$$\tilde{U}(t) < \sum_{i=1}^M \rho_i(t) \log \frac{1 - \epsilon}{\epsilon} - \log \frac{\tilde{\rho}}{1 - \tilde{\rho}} \leq \log \frac{1 - \epsilon}{\epsilon}.$$

In other words, if $\tilde{U}(t) \geq \log \frac{1}{\epsilon}$, then there is an $i \in \Omega$ for which $\rho_i(t) \geq 1 - \epsilon$. Let $v := \min\{t : \tilde{U}(t) \geq \log \frac{1}{\epsilon}\}$. Note that by construction, $\tilde{\tau}_\epsilon \leq v$. Appealing to Lemma 10 at the end of this section, we obtain

$$\begin{aligned}
\mathbb{E}_\epsilon[\tilde{\tau}_\epsilon] &\leq \mathbb{E}_\epsilon[v] \\
&\leq \frac{\log \frac{1}{\epsilon} - \tilde{U}(0)}{\tilde{\rho}E_{\min}} + \tilde{U}(0)\mathbf{1}_{\{\tilde{U}(0) < 0\}} \left(\frac{1}{\tilde{\rho}E_{\min}} - \frac{1}{R_{\min}} \right) + \frac{3(4C_2)^2}{\tilde{\rho}R_{\min}E_{\min}} \\
&\leq \frac{\log \frac{1}{\epsilon}}{\tilde{\rho}E_{\min}} + \frac{-\tilde{U}(0)}{R_{\min}}\mathbf{1}_{\{\tilde{U}(0) < 0\}} + \frac{6(4C_2)^2}{R_{\min}E_{\min}} \\
&= \frac{\log \frac{1}{\epsilon}}{\tilde{\rho}E_{\min}} + \frac{\sum_{i=1}^M \rho_i(0) \log \frac{1-\rho_i(0)}{\rho_i(0)} + \log \frac{\tilde{\rho}}{1-\tilde{\rho}}}{R_{\min}}\mathbf{1}_{\{\tilde{U}(0) < 0\}} + \frac{6(4C_2)^2}{R_{\min}E_{\min}} \\
&\leq \frac{\log \frac{1}{\epsilon}}{\tilde{\rho}E_{\min}} + \frac{H(\boldsymbol{\rho}(0)) + \log \frac{\tilde{\rho}}{1-\tilde{\rho}}}{R_{\min}} + \frac{6(4C_2)^2}{R_{\min}E_{\min}} \\
&\leq \frac{\log \frac{1}{\epsilon}}{E_{\min}} \left(1 + \frac{1}{\max\{\log M, \log \frac{1}{\epsilon}\}} \right) + \frac{H(\boldsymbol{\rho}(0)) + \log \log \frac{M}{\epsilon}}{R_{\min}} + \frac{6(4C_2)^2}{R_{\min}E_{\min}} \\
&\leq \frac{H(\boldsymbol{\rho}(0)) + \log \log \frac{M}{\epsilon}}{R_{\min}} + \frac{\log \frac{1}{\epsilon} + 1}{E_{\min}} + \frac{6(4C_2)^2}{R_{\min}E_{\min}}. \tag{69}
\end{aligned}$$

Lemma 7. *If $\max\{\tilde{U}(t), \tilde{U}(t+1)\} \geq 0$, then*

$$|\tilde{U}(t+1) - \tilde{U}(t)| \leq 4C_2.$$

Proof: We first consider the case $\tilde{U}(t) \geq 0$. Note that if $\rho_i(t) < \tilde{\rho}$, $\forall i \in \Omega$, then $\tilde{U}(t) < 0$. Therefore, $\tilde{U}(t) \geq 0$ implies that $\exists i \in \Omega$ such that $\rho_i(t) \geq \tilde{\rho}$. Without loss of generality assume $\rho_1(t) \geq \tilde{\rho}$. We obtain,

$$\begin{aligned}
&|\tilde{U}(t+1) - \tilde{U}(t)| \\
&= \left| \sum_{i=1}^M \rho_i(t+1) \log \frac{\rho_i(t+1)}{1-\rho_i(t+1)} - \sum_{i=1}^M \rho_i(t) \log \frac{\rho_i(t)}{1-\rho_i(t)} \right| \\
&= \left| \sum_{i=1}^M \rho_i(t+1) \left(\log \frac{\rho_i(t+1)}{1-\rho_i(t+1)} - \log \frac{\rho_i(t)}{1-\rho_i(t)} \right) + \sum_{i=1}^M (\rho_i(t+1) - \rho_i(t)) \log \frac{\rho_i(t)}{1-\rho_i(t)} \right| \\
&\leq \max_{i \in \Omega} \left| \log \frac{\rho_i(t+1)}{1-\rho_i(t+1)} - \log \frac{\rho_i(t)}{1-\rho_i(t)} \right| + \left| \sum_{i=1}^M (\rho_i(t+1) - \rho_i(t)) \log \frac{\rho_i(t)}{1-\rho_i(t)} \right| \\
&\stackrel{(a)}{\leq} \log C_2 + \sum_{i=1}^M |\rho_i(t+1) - \rho_i(t)| \cdot \left| \log \frac{\rho_i(t)}{1-\rho_i(t)} \right|
\end{aligned}$$

$$\begin{aligned}
&\stackrel{(b)}{\leq} \log C_2 + C_2 \sum_{i=1}^M \rho_i(t)(1 - \rho_i(t)) \left| \log \frac{\rho_i(t)}{1 - \rho_i(t)} \right| \\
&\leq \log C_2 + C_2 \rho_1(t)(1 - \rho_1(t)) \log \frac{\rho_1(t)}{1 - \rho_1(t)} + C_2 \sum_{i \neq 1} \rho_i(t) \log \frac{1}{\rho_i(t)} \\
&\stackrel{(c)}{\leq} \log C_2 + C_2 + C_2 \left(\sum_{i \neq 1} \rho_i(t) \right) \log \frac{M-1}{\sum_{i \neq 1} \rho_i(t)} \\
&\leq \log C_2 + C_2 + C_2((1 - \tilde{\rho}) \log(M-1) + 1) \\
&= \log C_2 + C_2 + C_2 \left(\frac{\log(M-1)}{1 + \max\{\log M, \log \frac{1}{\tilde{c}}\}} + 1 \right) \\
&\leq \log C_2 + 3C_2 \\
&\leq 4C_2
\end{aligned}$$

where (a) and (b) follow respectively from Lemmas 8 and 9 below, and (c) follows from Jensen's inequality and the fact that

$$\left| x(1-x) \log \frac{x}{1-x} \right| \leq 1, \quad x \in [0, 1].$$

This completes the proof for the case $\tilde{U}(t) \geq 0$. The proof for the case $\tilde{U}(t+1) \geq 0$ is done by following the similar lines and interchanging time indices (t) and $(t+1)$. ■

Lemma 8. For any $i \in \Omega$,

$$\left| \log \frac{\rho_i(t+1)}{1 - \rho_i(t+1)} - \log \frac{\rho_i(t)}{1 - \rho_i(t)} \right| \leq \log C_2.$$

Proof:

$$\begin{aligned}
\left| \log \frac{\rho_i(t+1)}{1 - \rho_i(t+1)} - \log \frac{\rho_i(t)}{1 - \rho_i(t)} \right| &= \left| \log \frac{P(Y = y_t | X = \gamma_{y^{t-1}}(i))}{\sum_{j \neq i} \frac{\rho_j(t)}{1 - \rho_j(t)} P(Y = y_t | X = \gamma_{y^{t-1}}(j))} \right| \\
&\leq \max_{y \in \mathcal{Y}} \log \frac{\max_{x \in \mathcal{X}} P(Y = y | X = x)}{\min_{x \in \mathcal{X}} P(Y = y | X = x)} = \log C_2.
\end{aligned}$$

Lemma 9. For any $i \in \Omega$,

$$|\rho_i(t+1) - \rho_i(t)| \leq \min \{ \rho_i(t)(1 - \rho_i(t)), \rho_i(t+1)(1 - \rho_i(t+1)) \} C_2.$$

Proof:

$$\begin{aligned}
|\rho_i(t+1) - \rho_i(t)| &= \rho_i(t) \left| \frac{P(Y = y_t | X = \gamma_{y^{t-1}}(i))}{\sum_{j=1}^M \rho_j(t) P(Y = y_t | X = \gamma_{y^{t-1}}(j))} - 1 \right| \\
&\leq \rho_i(t) \left| \frac{(1 - \rho_i(t)) \max_{x \in \mathcal{X}} P(Y = y_t | X = x)}{\sum_{j=1}^M \rho_j(t) P(Y = y_t | X = \gamma_{y^{t-1}}(j))} \right| \\
&\leq \rho_i(t) (1 - \rho_i(t)) \max_{y \in \mathcal{Y}} \frac{\max_{x \in \mathcal{X}} P(Y = y | X = x)}{\min_{x \in \mathcal{X}} P(Y = y | X = x)} \\
&= \rho_i(t) (1 - \rho_i(t)) C_2.
\end{aligned} \tag{70}$$

Similarly we can show that

$$\begin{aligned}
|\rho_i(t+1) - \rho_i(t)| &= \rho_i(t+1) \left| 1 - \rho_i(t) - \frac{\sum_{j \neq i} \rho_j(t) P(Y = y_t | X = \gamma_{y^{t-1}}(j))}{P(Y = y_t | X = \gamma_{y^{t-1}}(i))} \right| \\
&= \rho_i(t+1) (1 - \rho_i(t+1)) \times \\
&\quad \left| \frac{1 - \rho_i(t)}{1 - \rho_i(t+1)} - \frac{\sum_{j=1}^M \rho_j(t) P(Y = y_t | X = \gamma_{y^{t-1}}(j))}{P(Y = y_t | X = \gamma_{y^{t-1}}(i))} \right| \\
&\leq \rho_i(t+1) (1 - \rho_i(t+1)) \max_{y \in \mathcal{Y}} \frac{\max_{x \in \mathcal{X}} P(Y = y | X = x)}{\min_{x \in \mathcal{X}} P(Y = y | X = x)} \\
&= \rho_i(t+1) (1 - \rho_i(t+1)) C_2.
\end{aligned} \tag{71}$$

Combining (70) and (71), we have the assertion of the lemma. ■

Lemma 10. *Assume that the sequence $\{\xi(t)\}$, $t = 0, 1, 2, \dots$ forms a submartingale with respect to a filtration $\{\mathcal{F}(t)\}$. Furthermore, assume there exist positive constants K_1 , K_2 , and K_3 such that*

$$\mathbb{E}[\xi(t+1) | \mathcal{F}(t)] \geq \xi(t) + K_1 \text{ if } \xi(t) < 0,$$

$$\mathbb{E}[\xi(t+1) | \mathcal{F}(t)] \geq \xi(t) + K_2 \text{ if } \xi(t) \geq 0,$$

$$|\xi(t+1) - \xi(t)| \leq K_3 \text{ if } \max\{\xi(t+1), \xi(t)\} \geq 0.$$

Consider the stopping time $v = \min\{t : \xi(t) \geq B\}$, $B > 0$. Then we have the inequality

$$\mathbb{E}[v] \leq \frac{B - \xi(0)}{K_2} + \xi(0)\mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right) + \frac{3K_3^2}{K_1K_2}.$$

Proof: This lemma is a generalization of Lemma 1 in [25]. The proof is provided below.

Consider the sequence $\{\eta(t)\}$ defined as follows

$$\eta(t) = \begin{cases} -A + \frac{\xi(t)}{K_1} - t & \text{if } \xi(t) < 0 \\ -Ae^{-\alpha\xi(t)} + \frac{\xi(t)}{K_2} - t & \text{if } \xi(t) \geq 0 \end{cases}$$

where $A = \left[\frac{3K_3^2}{K_2} \left(\frac{1}{K_1} - \frac{1}{K_2} \right) \right]^+$ and $\alpha = \frac{0.5K_2}{K_3^2}$.

Claim 1. The sequence $\{\eta(t)\}$ forms a submartingale with respect to the filtration $\{\mathcal{F}(t)\}$.

By Doob's Stopping Theorem,

$$\begin{aligned} \eta(0) &\leq \mathbb{E}[\eta(v)] \\ &\leq \mathbb{E} \left[\frac{\xi(v)}{K_2} - v \right] \\ &= \frac{\mathbb{E}[\xi(v-1)] + \mathbb{E}[\xi(v) - \xi(v-1)]}{K_2} - \mathbb{E}[v] \\ &\leq \frac{B + K_3}{K_2} - \mathbb{E}[v]. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \eta(0) &= \left(-A + \frac{\xi(0)}{K_1} \right) \mathbf{1}_{\{\xi(0) < 0\}} + \left(-Ae^{-\alpha\xi(0)} + \frac{\xi(0)}{K_2} \right) \mathbf{1}_{\{\xi(0) \geq 0\}} \\ &\geq -A + \frac{\xi(0)}{K_2} - \xi(0)\mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right). \end{aligned}$$

Combining the above inequalities, we obtain

$$\begin{aligned} \mathbb{E}[v] &\leq \frac{B + K_3}{K_2} - \eta(0) \\ &\leq \frac{B + K_3}{K_2} + A - \frac{\xi(0)}{K_2} + \xi(0)\mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right) \\ &= \frac{B - \xi(0)}{K_2} + \xi(0)\mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right) + \left[\frac{3K_3^2}{K_2} \left(\frac{1}{K_1} - \frac{1}{K_2} \right) \right]^+ + \frac{K_3}{K_2} \\ &\stackrel{(a)}{\leq} \frac{B - \xi(0)}{K_2} + \xi(0)\mathbf{1}_{\{\xi(0) < 0\}} \left(\frac{1}{K_2} - \frac{1}{K_1} \right) + \frac{3K_3^2}{K_1K_2} \end{aligned} \tag{72}$$

where (a) holds since by definition $K_1, K_2 \leq K_3$ and hence, $\frac{K_3}{K_2} \leq \min \left\{ \frac{3K_3^2}{K_1K_2}, \frac{3K_3^2}{K_2^2} \right\}$. ■

Proof of Claim 1: We will show that $\mathbb{E}[\eta(t+1)|\mathcal{F}(t)] \geq \eta(t)$. There are two cases:

Case I. $\xi(t) < 0$:

If $\xi(t+1) < 0$, then

$$\eta(t+1) = -A + \frac{\xi(t+1)}{K_1} - (t+1). \quad (73)$$

On the other hand, if $\xi(t+1) \geq 0$, then by the assumption of Lemma 10, $\xi(t+1) \leq K_3$, and we have

$$\begin{aligned} \eta(t+1) &= -Ae^{-\alpha\xi(t+1)} + \frac{\xi(t+1)}{K_2} - (t+1) \\ &\stackrel{(a)}{\geq} -A + \frac{\xi(t+1)}{K_1} - (t+1) \end{aligned} \quad (74)$$

where (a) follows from the fact that 1) if $K_1 \geq K_2$, then by definition $A = 0$, and $\frac{x}{K_2} \geq \frac{x}{K_1}$ for $x \geq 0$; and 2) if $K_1 < K_2$, then $-Ae^{-\alpha x} + \frac{x}{K_2}$ is concave in x , $-Ae^{-\alpha x} + \frac{x}{K_2} = -A + \frac{x}{K_1}$ for $x = 0$, and for $x = K_3$

$$\begin{aligned} -Ae^{-\alpha K_3} + \frac{K_3}{K_2} &\geq -A(1 - \alpha K_3 + \frac{1}{2}(\alpha K_3)^2) + \frac{K_3}{K_2} \\ &= -A + A\alpha K_3(1 - \frac{1}{4}\frac{K_2}{K_3}) + \frac{K_3}{K_2} \\ &\geq -A + \frac{9}{8}K_3 \left(\frac{1}{K_1} - \frac{1}{K_2} \right) + \frac{K_3}{K_2} \\ &\geq -A + \frac{K_3}{K_1}. \end{aligned}$$

Combining (73) and (74), we obtain

$$\begin{aligned} \mathbb{E}[\eta(t+1)|\mathcal{F}(t)] &\geq \mathbb{E}\left[-A + \frac{\xi(t+1)}{K_1} - (t+1)|\mathcal{F}(t)\right] \\ &\geq -A + \frac{\xi(t) + K_1}{K_1} - (t+1) \\ &= -A + \frac{\xi(t)}{K_1} - t = \eta(t). \end{aligned} \quad (75)$$

Case II. $\xi(t) \geq 0$:

If $\xi(t+1) \geq 0$, then

$$\eta(t+1) = -Ae^{-\alpha\xi(t+1)} + \frac{\xi(t+1)}{K_2} - (t+1). \quad (76)$$

On the other hand, if $\xi(t+1) < 0$, then we have

$$\begin{aligned}\eta(t+1) &= -A + \frac{\xi(t+1)}{K_1} - (t+1) \\ &\stackrel{(a)}{\geq} -Ae^{-\alpha\xi(t+1)} + \frac{\xi(t+1)}{K_2} - (t+1)\end{aligned}\quad (77)$$

where (a) follows from the fact that 1) if $K_1 \geq K_2$, then by definition $A = 0$, and $\frac{x}{K_1} \geq \frac{x}{K_2}$ for $x < 0$; and 2) if $K_1 < K_2$, then $-Ae^{-\alpha x} + \frac{x}{K_2}$ is concave in x , $-Ae^{-\alpha x} + \frac{x}{K_2} = -A + \frac{x}{K_1}$ for $x = 0$, and $-Ae^{-\alpha K_3} + \frac{K_3}{K_2} \geq -A + \frac{K_3}{K_1}$.

Combining (76) and (77), we obtain

$$\begin{aligned}\mathbb{E}[\eta(t+1)|\mathcal{F}(t)] &\geq \mathbb{E}\left[-Ae^{-\alpha\xi(t+1)} + \frac{\xi(t+1)}{K_2} - (t+1)|\mathcal{F}(t)\right] \\ &\geq \mathbb{E}\left[-Ae^{-\alpha\xi(t+1)}|\mathcal{F}(t)\right] + \frac{\xi(t) + K_2}{K_2} - (t+1) \\ &= \mathbb{E}\left[-Ae^{-\alpha\xi(t+1)}|\mathcal{F}(t)\right] + Ae^{-\alpha\xi(t)} + \eta(t) \\ &= \eta(t) - Ae^{-\alpha\xi(t)}\mathbb{E}\left[e^{-\alpha(\xi(t+1)-\xi(t))} - 1|\mathcal{F}(t)\right] \\ &\stackrel{(a)}{\geq} \eta(t) - Ae^{-\alpha\xi(t)}\mathbb{E}\left[-\alpha(\xi(t+1) - \xi(t)) + \frac{1}{2}\alpha^2(\xi(t+1) - \xi(t))^2 e^{\alpha K_3}|\mathcal{F}(t)\right] \\ &\geq \eta(t) + A\alpha e^{-\alpha\xi(t)}\left[K_2 - \frac{1}{2}\alpha K_3^2 e^{\alpha K_3}\right] \\ &\stackrel{(b)}{\geq} \eta(t)\end{aligned}\quad (78)$$

where (a) follows from the fact that for $|x| \leq K$,

$$\begin{aligned}e^x &= 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \\ &\leq 1 + x + \frac{x^2}{2} \left(1 + \frac{K}{3} + \frac{K^2}{12} + \dots\right) \\ &\leq 1 + x + \frac{x^2}{2} e^K;\end{aligned}$$

and (b) holds since

$$\frac{1}{2}\alpha K_3^2 e^{\alpha K_3} = \frac{1}{4}K_2 e^{\frac{0.5K_2}{K_3}} \leq \frac{e^{0.5}}{4}K_2 \leq K_2.$$

■

APPENDIX III

PROOF OF THE PROPOSITIONS

A. Proof of Proposition 1

Fix a time instant t and assume that $Y^{t-1} = y^{t-1}$. For ease of notation, in the following we drop the time index t for $\rho_i(t)$ and simply write ρ_i .

Let

$$\lambda_\gamma := \Pr(\Gamma^{\text{PM}} = \gamma | Y^{t-1} = y^{t-1}).$$

Define for each $i \in \Omega$ and $x \in \mathcal{X}$:

$$\Lambda_{i,x} := \sum_{\gamma: \gamma(i)=x} \lambda_\gamma = \Pr(X = x | \theta = i, Y^{t-1} = y^{t-1}) \quad (79)$$

and

$$\hat{\rho}_{i,x} := \rho_i \Lambda_{i,x} = \Pr(X = x, \theta = i | Y^{t-1} = y^{t-1}). \quad (80)$$

Notice that for each $i, j \in \Omega$, $x, x' \in \mathcal{X}$, and for a fixed posterior distribution, the various messages are mapped into inputs of the channel independently of each other and hence,

$$\sum_{\gamma: \substack{\gamma(i)=x \\ \gamma(j)=x'}} \lambda_\gamma = \Lambda_{i,x} \Lambda_{j,x'}. \quad (81)$$

Rearranging terms and using Jensen's inequality, we obtain

$$\begin{aligned} EJS(\boldsymbol{\rho}(t), \Gamma^{\text{PM}}) &= \sum_{\gamma \in \mathcal{E}} \lambda_\gamma \sum_{i=1}^M \rho_i D \left(P_{\gamma(i)} \left\| \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_{\gamma(j)} \right. \right) \\ &= \sum_{i=1}^M \rho_i \sum_{x \in \mathcal{X}} \sum_{\gamma: \gamma(i)=x} \lambda_\gamma D \left(P_x \left\| \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} P_{\gamma(j)} \right. \right) \\ &\geq \sum_{i=1}^M \sum_{x \in \mathcal{X}} \rho_i \Lambda_{i,x} D \left(P_x \left\| \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} \sum_{\gamma: \gamma(i)=x} \frac{\lambda_\gamma}{\Lambda_{i,x}} P_{\gamma(j)} \right. \right) \\ &= \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} D \left(P_x \left\| \sum_{j \neq i} \frac{\rho_j}{1 - \rho_i} \sum_{x' \in \mathcal{X}} \sum_{\substack{\gamma: \gamma(i)=x \\ \gamma(j)=x'}} \frac{\lambda_\gamma}{\Lambda_{i,x}} P_{x'} \right. \right) \\ &\stackrel{(a)}{=} \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} D \left(P_x \left\| \frac{\sum_{j \neq i} \sum_{x' \in \mathcal{X}} \rho_j \Lambda_{j,x'} P_{x'}}{1 - \rho_i} \right. \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} D \left(P_x \left\| \frac{\sum_{x' \in \mathcal{X}} (\pi_{x'}^* P_{x'} - \hat{\rho}_{i,x'} P_{x'})}{1 - \rho_i} \right\| \right), \\
&= \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} D \left(P_x \left\| \frac{\sum_{x' \in \mathcal{X}} (\pi_{x'}^* P_{x'} - \hat{\rho}_{i,x'} P_{x'})}{1 - \rho_i} \right\| \right) \\
&\quad + \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} \frac{\rho_i}{1 - \rho_i} D \left(P_x \left\| \frac{\sum_{x'} \hat{\rho}_{i,x'} P_{x'}}{\rho_i} \right\| \right) \\
&\quad - \sum_{i=1}^M \sum_{x \in \mathcal{X}} \hat{\rho}_{i,x} \frac{\rho_i}{1 - \rho_i} D \left(P_x \left\| \frac{\sum_{x'} \hat{\rho}_{i,x'} P_{x'}}{\rho_i} \right\| \right) \\
&\geq \sum_{i=1}^M \sum_{x \in \mathcal{X}} \frac{\hat{\rho}_{i,x}}{1 - \rho_i} D \left(P_x \left\| \sum_{x' \in \mathcal{X}} \pi_{x'}^* P_{x'} \right\| \right) \\
&\quad - \sum_{i=1}^M \frac{\rho_i^2}{1 - \rho_i} \sum_{x \in \mathcal{X}} \Lambda_{i,x} D \left(P_x \left\| \sum_{x' \in \mathcal{X}} \Lambda_{i,x'} P_{x'} \right\| \right) \\
&\stackrel{(b)}{\geq} \sum_{i=1}^M \sum_{x \in \mathcal{X}} \frac{\hat{\rho}_{i,x}}{1 - \rho_i} C - \sum_{i=1}^M \frac{\rho_i^2}{1 - \rho_i} C \\
&= \sum_{i=1}^M \frac{\rho_i}{1 - \rho_i} C - \sum_{i=1}^M \frac{\rho_i^2}{1 - \rho_i} C \\
&= C
\end{aligned} \tag{82}$$

where (a) follows from (81); and inequality (b) follows from Fact 1 and that

$$\sum_{x \in \mathcal{X}} \Lambda_{i,x} D \left(P_x \left\| \sum_{x' \in \mathcal{X}} \Lambda_{i,x'} P_{x'} \right\| \right)$$

is the mutual information $I(X; Y)$ between an input X with probability mass function $\{\Lambda_{i,x}\}_{x \in \mathcal{X}}$ and the output produced by the channel (see property (7) of the JS divergence), and thus is smaller than the capacity C .

B. Proof of Proposition 2

Fix a time t and assume that $Y^{t-1} = y^{t-1}$. Recall that Γ^{PM} denotes the random encoding function of the variable-length posterior matching scheme in Section IV-B. By definition (38) and by Proposition 1,

$$EJS(\boldsymbol{\rho}(t), \gamma^*) \geq EJS(\boldsymbol{\rho}(t), \Gamma^{\text{PM}}) \geq C.$$

Now, assume that $\max_{i \in \Omega} \rho_i(t) \geq \tilde{\rho}$ and define

$$\hat{i} := \arg \max_{i \in \Omega} \rho_i(t). \quad (83)$$

Then,

$$\rho_{\hat{i}}(t) \geq \tilde{\rho}. \quad (84)$$

Let $x, x' \in \mathcal{X}$ be two inputs of the channel satisfying $D(P_x \| P_{x'}) = C_1$. Also, define the encoding function

$$\hat{\gamma}(i) := \begin{cases} x & \text{if } i = \hat{i} \\ x' & \text{otherwise.} \end{cases} \quad (85)$$

By definition (38), from (84), and by the selection of x, x' :

$$EJS(\boldsymbol{\rho}(t), \gamma^*) \geq EJS(\boldsymbol{\rho}(t), \hat{\gamma}) \geq \rho_{\hat{i}}(t) D(P_x \| P_{x'}) \geq \tilde{\rho} C_1. \quad (86)$$

C. Proof of Proposition 3

Let

$$\pi_x(t) := \sum_{i \in \Omega: \gamma^{\text{GHBZ}}(i)=x} \rho_i(t), \quad x \in \{0, 1\}. \quad (87)$$

Let

$$k_2^* := k^* - \text{sign} \left(\sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} \right),$$

and define

$$\delta_1(t) := \left| \sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} \right|, \quad \delta_2(t) := \left| \sum_{i=1}^{k_2^*} \rho_i(t) - \frac{1}{2} \right|.$$

Suppose $\sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} < 0$ which implies that $k_2^* = k^* + 1$. Note that by definition, $\pi_0(t) = \frac{1}{2} - \delta_1(t)$, $\rho_{k_2^*}(t) = \delta_1(t) + \delta_2(t)$, and $\pi_1(t) = \frac{1}{2} + \delta_1(t)$. In this case, the EJS divergence is bounded as

$$\begin{aligned} EJS(\boldsymbol{\rho}(t), \gamma^{\text{GHBZ}}) &= \sum_{i=1}^{k^*} \rho_i(t) D \left(P_0 \left\| \frac{\pi_0(t) - \rho_i(t)}{1 - \rho_i(t)} P_0 + \frac{\pi_1(t)}{1 - \rho_i(t)} P_1 \right. \right) \\ &\quad + \rho_{k_2^*}(t) D \left(P_1 \left\| \frac{\pi_0(t)}{1 - \rho_{k_2^*}(t)} P_0 + \frac{\pi_1(t) - \rho_{k_2^*}(t)}{1 - \rho_{k_2^*}(t)} P_1 \right. \right) \\ &\quad + \sum_{i=k_2^*+1}^M \rho_i(t) D \left(P_1 \left\| \frac{\pi_0(t)}{1 - \rho_i(t)} P_0 + \frac{\pi_1(t) - \rho_i(t)}{1 - \rho_i(t)} P_1 \right. \right) \end{aligned}$$

$$\begin{aligned}
&\stackrel{(a)}{\geq} \pi_0(t)D\left(P_0\left\|\pi_0(t)P_0 + \pi_1(t)P_1\right.\right) \\
&\quad + \rho_{k_2^*}(t)D\left(P_1\left\|\frac{1}{2}P_0 + \frac{1}{2}P_1\right.\right) \\
&\quad + (\pi_1(t) - \rho_{k_2^*}(t))D\left(P_1\left\|\pi_0(t)P_0 + \pi_1(t)P_1\right.\right) \\
&\stackrel{(b)}{=} \pi_0(t)D\left(P_0\left\|\pi_0(t)P_0 + \pi_1(t)P_1\right.\right) \\
&\quad + \rho_{k_2^*}(t)D\left(P_0\left\|\frac{1}{2}P_0 + \frac{1}{2}P_1\right.\right) \\
&\quad + (\pi_1(t) - \rho_{k_2^*}(t))D\left(P_0\left\|\pi_1(t)P_0 + \pi_0(t)P_1\right.\right) \\
&\stackrel{(c)}{\geq} D\left(P_0\left\|\frac{1}{2}P_0 + \frac{1}{2}P_1\right.\right) \\
&= C
\end{aligned}$$

where (a) follows from the facts that $\frac{\pi_0(t) - \rho_i(t)}{1 - \rho_i(t)} \leq \pi_0(t)$, $\frac{\pi_1(t) - \rho_{k_2^*}(t)}{1 - \rho_{k_2^*}(t)} \leq \frac{1}{2}$, $\frac{\pi_1(t) - \rho_i(t)}{1 - \rho_i(t)} \leq \pi_1(t)$, and by Lemma 1; (b) holds because of condition (41); and (c) follows from the facts that KL divergence is convex, $(\pi_0(t))^2 + \frac{1}{2}\rho_{k_2^*}(t) + (\pi_1(t) - \rho_{k_2^*}(t))\pi_1(t) = \frac{1}{2} + \delta_1(t)(\delta_1(t) - \delta_2(t)) \leq \frac{1}{2}$, and by Lemma 1.

The proof for the case $\sum_{i=1}^{k^*} \rho_i(t) - \frac{1}{2} \geq 0$ follows similarly.

D. Proof of Proposition 4

Suppose γ is an encoding function that satisfies (50). Let

$$\pi_x(t) = \sum_{i \in \Omega: \gamma(i)=x} \rho_i(t) \quad \text{for } x \in \mathcal{X} = \{0, 1\},$$

and define $\delta(t) = \pi_0(t) - \pi_1(t)$. From (50),

$$0 \leq \delta(t) \leq \rho_i(t), \quad \forall i \in \{j \in \Omega: \gamma(j) = 0\}. \quad (88)$$

We have

$$\begin{aligned}
EJS(\boldsymbol{\rho}(t), \gamma) &= \sum_{i=1}^M \rho_i(t)D\left(P_{\gamma(i)}\left\|\sum_{j \neq i} \frac{\rho_j(t)}{1 - \rho_i(t)} P_{\gamma(j)}\right.\right) \\
&= \sum_{i \in \Omega: \gamma(i)=0} \rho_i(t)D\left(P_0\left\|\frac{\pi_0(t) - \rho_i(t)}{1 - \rho_i(t)} P_0 + \frac{\pi_1(t)}{1 - \rho_i(t)} P_1\right.\right)
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{i \in \Omega: \gamma(i)=1} \rho_i(t) D\left(P_1 \left\| \frac{\pi_0(t)}{1-\rho_i(t)} P_0 + \frac{\pi_1(t) - \rho_i(t)}{1-\rho_i(t)} P_1\right.\right) \\
 & \stackrel{(a)}{\geq} \sum_{i \in \Omega: \gamma(i)=0} \rho_i(t) D\left(P_0 \left\| \frac{1}{2} P_0 + \frac{1}{2} P_1\right.\right) \\
 & + \sum_{i \in \Omega: \gamma(i)=1} \rho_i(t) D\left(P_1 \left\| \frac{1}{2} P_0 + \frac{1}{2} P_1\right.\right) \\
 & \stackrel{(b)}{=} C
 \end{aligned}$$

where (a) follows from the facts that $\pi_0(t) - \rho_i(t) \leq \pi_1(t)$ for any i with $\gamma(i) = 0$, $\pi_1(t) \leq \pi_0(t)$, and since for two distributions P and Q and $\alpha \in [0, 1]$, $D(P \|\alpha P + (1-\alpha)Q)$ is decreasing in α (see Lemma 1); and (b) follows from Fact 1 and since the capacity of the channel is achieved by the uniform input distribution.

On the other hand, if $\rho_i(t) \geq \frac{1}{2}$, then condition (50) is only satisfied by the encoding function $\hat{\gamma}$ under which $\hat{\gamma}(\hat{i}) = 0$ and $\hat{\gamma}(j) = 1$ for all $j \neq \hat{i}$. Therefore, if $\rho_i(t) \geq \tilde{\rho}$ we obtain

$$EJS(\boldsymbol{\rho}(t), \hat{\gamma}) \geq \rho_i(t) D(P_0 \| P_1) \geq \tilde{\rho} C_1.$$

E. Proof of Proposition 5

For any encoding function $\gamma \in \mathcal{E}$, let

$$\delta_\gamma(t) = \sum_{i \in \Omega: \gamma(i)=0} \rho_i(t) - \sum_{i \in \Omega: \gamma(i)=1} \rho_i(t). \quad (89)$$

Algorithm 1 computes $\delta_\gamma(t)$ for all 2^M encoding functions $\gamma \in \mathcal{E}$ and selects γ^{Alg1} such that

$$\gamma^{\text{Alg1}} := \arg \min_{\gamma \in \mathcal{E}: \delta_\gamma(t) \geq 0} \delta_\gamma(t). \quad (90)$$

Next we prove by contradiction that γ^{Alg1} satisfies (50), i.e.,

$$\delta_{\gamma^{\text{Alg1}}}(t) \leq \rho_i(t), \quad \forall i \in \{j \in \Omega: \gamma^{\text{Alg1}}(j) = 0\}. \quad (91)$$

Suppose there exists $k \in \Omega$ such that $\gamma^{\text{Alg1}}(k) = 0$ and $\rho_k(t) < \delta_{\gamma^{\text{Alg1}}}(t)$. We consider two cases:

Case I. $0 < \rho_k(t) \leq \frac{1}{2} \delta_{\gamma^{\text{Alg1}}}(t)$:

Define the encoding function $\hat{\gamma}_1$ as follows

$$\hat{\gamma}_1(i) = \begin{cases} 1 & \text{if } i = k \\ \gamma^{\text{Alg1}}(i) & \text{otherwise} \end{cases}. \quad (92)$$

We have

$$0 \leq \delta_{\hat{\gamma}_1}(t) = \delta_{\gamma^{\text{Alg1}}}(t) - 2\rho_k(t) < \delta_{\gamma^{\text{Alg1}}}(t),$$

which contradicts (90).

Case II. $\frac{1}{2}\delta_{\gamma^{\text{Alg1}}}(t) < \rho_k(t) < \delta_{\gamma^{\text{Alg1}}}(t)$:

Define the encoding function $\hat{\gamma}_2$ as follows

$$\hat{\gamma}_2(i) = 1 - \hat{\gamma}_1(i), \quad \forall i \in \Omega. \quad (93)$$

We have

$$0 < \delta_{\hat{\gamma}_2}(t) = 2\rho_k(t) - \delta_{\gamma^{\text{Alg1}}}(t) < \delta_{\gamma^{\text{Alg1}}}(t),$$

which again contradicts (90).

Algorithm 2 constructs an encoding function that satisfies (50). Algorithm 2 terminates in at most $M(M-1)/2$ rounds of operations, where in each round the main computational burden is to find an element of S_0 with the lowest belief. Note that we do not have to search for the element with the lowest belief in each round if we sort all the beliefs once in the beginning, which has complexity order $O(M \log M)$.

APPENDIX IV

PROOF OF LEMMAS 3 AND 4

A. Proof of Lemma 3

From the described optimal decoding rule of (25), the constraint on the probability of error is satisfied by any coding scheme with the stopping rule (26):

$$\text{Pe} = \mathbb{E}[1 - \max_{i \in \Omega} \rho_i(\tilde{\tau}_\epsilon)] \leq \epsilon,$$

hence, by construction,

$$\mathbb{E}[\tau_\epsilon^*] \leq \mathbb{E}[\tilde{\tau}_\epsilon^*]. \quad (94)$$

On the other hand, let us consider $\mathbb{E}[\tilde{\tau}_\epsilon^*]$ for any $\iota > \epsilon$. Under any coding scheme,

$$\begin{aligned} \mathbb{E}[\tau_\epsilon] &\geq \mathbb{E}[\tau_\epsilon | \max_{j \in \Omega} \rho_j(\tau_\epsilon) \geq 1 - \iota] P(\max_{j \in \Omega} \rho_j(\tau_\epsilon) \geq 1 - \iota) \\ &\stackrel{(a)}{\geq} \mathbb{E}[\tau_\epsilon | \max_{j \in \Omega} \rho_j(\tau_\epsilon) \geq 1 - \iota] (1 - \iota^{-1} \mathbb{E}[1 - \max_{j \in \Omega} \rho_j(\tau_\epsilon)]) \end{aligned}$$

$$\begin{aligned}
 &\stackrel{(b)}{\geq} \mathbb{E}[\tau_\epsilon | \max_{j \in \Omega} \rho_j(\tau_\epsilon) \geq 1 - \iota] (1 - \frac{\epsilon}{\iota}) \\
 &\geq \mathbb{E}[\tilde{\tau}_\iota^*] (1 - \frac{\epsilon}{\iota})
 \end{aligned} \tag{95}$$

where (a) follows from Markov inequality and (b) follows from the definition of τ_ϵ which implies that $\text{Pe} = \mathbb{E}[1 - \max_{j \in \Omega} \rho_j(\tau_\epsilon)] \leq \epsilon$. From (95),

$$\mathbb{E}[\tilde{\tau}_\iota^*] (1 - \frac{\epsilon}{\iota}) \leq \mathbb{E}[\tau_\epsilon^*]. \tag{96}$$

B. Proof of Lemma 4

This proof is based on the dynamic programming (DP) characterization of $\mathbb{E}[\tilde{\tau}_\iota^*]$.

Let $\mathbb{P}(\Omega) := \{\boldsymbol{\rho} \in [0, 1]^M : \sum_{i=1}^M \rho_i = 1\}$. Let $V_\iota^* : \mathbb{P}(\Omega) \rightarrow \mathbb{R}_+$, referred to as the *optimal value function*, be the minimal solution to the following fixed point equation:

$$V(\boldsymbol{\rho}) = \begin{cases} 0 & \text{if } \min_{j \in \Omega} \{1 - \rho_j\} \leq \iota \\ 1 + \min_{\gamma \in \mathcal{E}} \sum_y P_\rho^\gamma(y) V(\Phi^\gamma(\boldsymbol{\rho}, y)) & \text{otherwise} \end{cases} \tag{97}$$

where $P_\rho^\gamma(y) := \sum_{i=1}^M \rho_i P_{\gamma(i)}(y)$ is the channel output density under encoding rule γ and

$$\Phi^\gamma(\boldsymbol{\rho}, y) := \left[\frac{\rho_1 P_{\gamma(1)}(y)}{P_\rho^\gamma(y)}, \dots, \frac{\rho_M P_{\gamma(M)}(y)}{P_\rho^\gamma(y)} \right] \tag{98}$$

represents the evolution of the belief vector in one transmission step and under encoding γ according to the Bayes' rule.

Fact 3 (Proposition 9.8 in [26]). *For the uniform initial belief $\boldsymbol{\rho}(0) = [\frac{1}{M} \dots \frac{1}{M}]$, $V_\iota^*(\boldsymbol{\rho}(0)) = \mathbb{E}[\tilde{\tau}_\iota^*]$. Furthermore, given the (suboptimal) stopping rule $\tilde{\tau}_\iota$, an optimum encoding rule at any time t prior to the stopping and any belief $\boldsymbol{\rho}(t)$ is the mapping*

$$\tilde{\gamma}^* = \arg \min_{\gamma \in \mathcal{E}} \sum_{y \in \mathcal{Y}} P_\rho^\gamma(y) V_\iota^*(\Phi^\gamma(\boldsymbol{\rho}, y)).$$

In lieu of full characterization of V_ι^* , the following fact, specialized for

$$\underline{V}_\iota(\boldsymbol{\rho}) = \left[\frac{H(\boldsymbol{\rho}) - F_M(\delta) - F_M(\iota)}{C} + \frac{\log \frac{1-\iota}{\iota} - \log \frac{1-\delta}{\delta} - \log C_2 - 1}{C_1} \mathbf{1}_{\{\max_{i \in \Omega} \rho_i \leq 1-\delta\}} \right]^+$$

and in combination with Fact 3, provides the assertion of the lemma.

Fact 4 (Lemma 1 in [21]). Let $\underline{V}_\iota : \mathbb{P}(\Omega) \rightarrow \mathbb{R}_+$ satisfy the following:

$$\underline{V}_\iota(\boldsymbol{\rho}) \leq \begin{cases} 0 & \text{if } \min_{j \in \Omega} \{1 - \rho_j\} \leq \iota \\ 1 + \min_{\gamma \in \mathcal{E}} \sum_{y \in \mathcal{Y}} P_\rho^\gamma(y) \underline{V}_\iota(\Phi^\gamma(\boldsymbol{\rho}, y)) & \text{otherwise} \end{cases}.$$

Then \underline{V}_ι is a uniform lower bound for the optimal value function V_ι^* .

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