Asymptotic High-SNR Capacity of MISO Optical Intensity Channels

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***Abstract*—This paper derives the asymptotic capacity for the multiple-input single-output free-space optical intensity channel in the regime of high signal-to-noise ratio (SNR). The asymptotic result is proven via upper and lower bounds on capacity at finite SNR.**

I. Introduction

Optical wireless communication is a form of communication in which visible, infrared, or ultraviolet light is transmitted in free space (air or vacuum) to carry a message to its destination. Recent works suggest that it is a promising solution to replacing some of the existing radio-frequency (RF) wireless communication systems in order to prevent future rate bottlenecks [1]–[3]. Particularly attractive are simple *intensity-modulation–direct-detection (IM-DD)* systems. In such a system, the transmitter modulates the intensity of optical signals coming from light emitting diodes (LEDs) or laser diodes (LDs), and the receiver measures incoming optical intensities by means of photodetectors. The electrical output signals of the photodetectors are essentially proportional to the incoming optical intensities, but are corrupted by thermal noise of the photodetectors, relative-intensity noise of random intensity fluctuations inherent to low-cost LEDs and LDs, and shot noise caused by ambient light. In a first approximation, noise coming from these sources is usually modeled as being additive Gaussian and independent of the transmitted light signal; see [1], [2].

The free-space optical intensity channel has been extensively studied in the literature. In the *single-input single-output (SISO)* scenario, where the transmitter employs a single transmit LED or LD, and the receiver a single photodetector, the work [4] established upper and lower bounds on the capacity of this channel that are asymptotically tight in both high-signal-to-noise ratio (SNR) and low-SNR limits. Improved bounds at finite SNR have subsequently been presented in [5]–[8]. For the *multiple-input and multiple-output (MIMO)* optical intensity channel, where the transmitter is equipped with multiple LEDs or LDs, and the receiver with multiple photodetectors, the recent work [9] has derived several asymptotic capacity results in the high-SNR limit; see details below. Previous to [9], various code constructions for this setup have been proposed in [10]–[13]. When there is no crosstalk so the MIMO channel can be modeled through a diagonal channel matrix, bounds on capacity were presented in [8], [14].

The main contributions of [9] are the asymptotic high-SNR capacity for the MIMO channel in the following cases:

- The channel matrix is of full column rank, i.e., its rank equals the number of transmit LEDs, and the inputs are subject to any peak- and average-power constraints;
- The channel is multiple-input and single-output (MISO), and the inputs are subject to only a peak-power constraint, or only an average-power constraint, or both constraints but with the average-power constraint being sufficiently loose.

In the current work, we consider the MISO channel in the regime where there are both average- and peak-power constraints, and where the average-power constraint is not "sufficiently loose." Together with the MISO results in [9], this completely characterizes the asymptotic high-SNR capacity of any MISO channel for all parameter ranges.

The basic tools for proving our new result are similar to those used in [4], [8], [9]: our capacity lower bound is derived using the Entropy Power Inequality (EPI), and our upper bound is based on duality [15]. However, the proofs are much more involved. On one hand, the optimal input distribution for the lower bound turns out to be more complex than those in [4], [9]: it involves LED cooperation (compared to independent signaling in the MIMO full-column-rank case [9]), and, with certain probabilities, assigns to each LED a truncated exponential distribution (compared to either exponential or uniform distribution in the special MISO cases in [9]), whose parameters must be optimized. On the other hand, the output distribution chosen in the duality-based upper bound depends on the capacity-achieving input distribution, and its analysis is based on a new lemma that bounds the capacity-achieving probability measure on a given interval.

II. Channel Model and Main Result

A. Channel Model

Consider a communication link where the transmitter is equipped with n_{T} LEDs (or LDs) and the receiver with a single photodetector. The photodetector receives a superposition of the signals sent by the LEDs, and we assume that the crosstalk between LEDs is constant. Hence, the channel output is given by

$$Y = \mathbf{h}^{\mathsf{T}}\mathbf{x} + Z, \tag{1}$$

where the n_T-vector $\mathbf{x}$ denotes the channel input, whose entries are proportional to the optical intensities of the corresponding LEDs, and are therefore nonnegative:

$$X_i \in \mathbb{R}_0^+, \quad i = 1, \ldots, n_\mathrm{T}; \tag{2}$$

where the length-n_T row vector $\mathbf{h}^\mathsf{T}$ is the constant channel state vector with nonnegative entries, which, without loss of generality, we assume to be ordered:

$$h_1 \geq h_2 \geq \cdots \geq h_{n_\mathrm{T}} > 0; \tag{3}$$

and where $Z \sim \mathcal{N}(0, \sigma^2)$ is additive Gaussian noise. Note that, in contrast to the input $\mathbf{x}$, the output Y can be negative.

Inputs are subject to a peak-power (peak-intensity) and an average-power (average-intensity) constraint:

$$\Pr[X_i > \mathsf{A}] = 0, \quad \forall i \in \{1, \ldots, n_\mathrm{T}\}, \tag{4}$$

$$\sum_{i=1}^{n_\mathrm{T}} \mathsf{E}[X_i] \leq \mathsf{E}, \tag{5}$$

for some fixed parameters $\mathsf{A}, \mathsf{E} > 0$. Note that the average-power constraint is on the expectation of the channel input and not on its square. Also note that A describes the maximum power of each single LED, while E describes the allowed average total power of all LEDs together.

We denote the ratio between the allowed average power and the allowed peak power by α:

$$\alpha \triangleq \frac{\mathsf{E}}{\mathsf{A}}, \tag{6}$$

where $0 < \alpha \leq n_\mathrm{T}$. For $\alpha = n_\mathrm{T}$ the average-power constraint is inactive in the sense that it is automatically satisfied whenever the peak-power constraint is satisfied. Thus, $\alpha = n_\mathrm{T}$ corresponds to the case with only a peak-power constraint. On the other hand, $\alpha \ll 1$ corresponds to having a dominant average-power constraint and only a very weak peak-power constraint.

We denote the capacity of the channel (1) with allowed peak power A and allowed average power E by $\mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{A}, \mathsf{E})$. The capacity is given by [16]

$$\mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{A}, \mathsf{E}) = \sup_Q \mathrm{I}(Q, W) \tag{7}$$

where the supremum is over all laws Q on $\mathbf{X}$ satisfying (2), (4), and (5). When only an average-power constraint is imposed, capacity is denoted by $\mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{E})$. It is given as in (7) except that the supremum is taken over all laws Q on $\mathbf{X}$ satisfying (2) and (5).

B. Previous Results

Denote

$$\bar{X} \triangleq \mathbf{h}^\mathsf{T}\mathbf{X}. \tag{8}$$

Because $\mathbf{X} \multimap\!\!- \bar{X} \multimap\!\!- Y$ form a Markov chain, and because $\bar{X}$ is a function of $\mathbf{X}$, we have [9]

$$\mathrm{I}(\mathbf{X}; Y) = \mathrm{I}(\bar{X}; Y). \tag{9}$$

Hence the MISO channel (1) is equivalent to the SISO channel whose input is $\bar{X}$, while the constraints (4) and (5) on $\mathbf{X}$ are transformed to a set of admissible distributions for $\bar{X}$.

Define the following quantities:

$$h_\mathrm{sum} \triangleq \sum_{i=1}^{n_\mathrm{T}} h_i \tag{10}$$

$$\alpha_\mathrm{th} \triangleq \frac{1}{2} + \frac{1}{h_\mathrm{sum}} \sum_{k=1}^{n_\mathrm{T}} h_k (k-1). \tag{11}$$

In the case where there is no peak-power constraint, or where $\alpha \geq \alpha_\mathrm{th}$, the asymptotic high-SNR capacity of the above MISO channel is given by [9, Cor. 7 & Thm. 8]:

Theorem 1 ([9]): When $\alpha \geq \alpha_\mathrm{th}$,

$$\lim_{\mathsf{A} \to \infty} \left\{ \mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{A}, \alpha\mathsf{A}) - \log \mathsf{A} \right\} = \frac{1}{2} \log \frac{h_\mathrm{sum}^2}{2\pi e \sigma^2}. \tag{12}$$

Without a peak-power constraint,

$$\lim_{\mathsf{E} \to \infty} \left\{ \mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{E}) - \log \mathsf{E} \right\} = \frac{1}{2} \log \frac{e h_1^2}{2\pi \sigma^2}. \tag{13}$$

C. Main Result

The main result of this paper characterizes the asymptotic high-SNR capacity of the MISO channel in the regime that is not covered by Theorem 1.

Theorem 2: If $\alpha < \alpha_\mathrm{th}$, then

$$\begin{aligned} &\lim_{\mathsf{A} \to \infty} \left\{ \mathsf{C}_{\mathbf{h}^\mathsf{T}, \sigma^2}(\mathsf{A}, \alpha\mathsf{A}) - \log \mathsf{A} \right\} \\ &= \frac{1}{2} \log \frac{h_\mathrm{sum}^2}{2\pi e \sigma^2} - \inf_{\lambda \in \left(0, \min\{\frac{1}{2}, \alpha\}\right)} \Bigg\{ \log \frac{\mu(\lambda)}{1 - e^{-\mu(\lambda)}} - 1 \\ &\quad + \frac{\mu(\lambda)\, e^{-\mu(\lambda)}}{1 - e^{-\mu(\lambda)}} + \inf_{\mathbf{p}:\ \sum_{k=1}^{n_\mathrm{T}} p_k (k-1) = \alpha - \lambda} \mathscr{D}\left(\mathbf{p} \middle\| \frac{\mathbf{h}}{h_\mathrm{sum}}\right) \Bigg\}, \end{aligned} \tag{14}$$

where $\mathbf{p}$ denotes a probability vector, i.e., $p_1, \ldots, p_{n_\mathrm{T}}$ are nonnegative and sum to one, where $\mathscr{D}(\cdot\|\cdot)$ denotes relative entropy, and where $\mu(\lambda)$ is the unique positive solution for μ to

$$\frac{1}{\mu} - \frac{e^{-\mu}}{1 - e^{-\mu}} = \lambda. \tag{15}$$

Theorem 2 is proven via lower and upper bounds on the capacity. The proof will be outlined in Section III. Before doing so, we make some remarks about this result.

Optimization in (14). For a fixed λ, the optimal choice for $\mathbf{p} = (p_1, \ldots, p_k)$ is (see, e.g., [17, Problem 12.2])

$$p_k^* = \frac{\frac{h_k}{h_\mathrm{sum}} a^k}{\sum_{k'=1}^{n_\mathrm{T}} \frac{h_{k'}}{h_\mathrm{sum}} a^{k'}}, \quad k = 1, \ldots, n_\mathrm{T}, \tag{16}$$

where $a \geq 0$ is obtained by solving the following equation:

$$\frac{\sum_{k=1}^{n_\mathrm{T}} \frac{h_k}{h_\mathrm{sum}} a^k k}{\sum_{k=1}^{n_\mathrm{T}} \frac{h_k}{h_\mathrm{sum}} a^k} = \alpha - \lambda + 1. \tag{17}$$

The optimization over λ can easily be performed numerically.

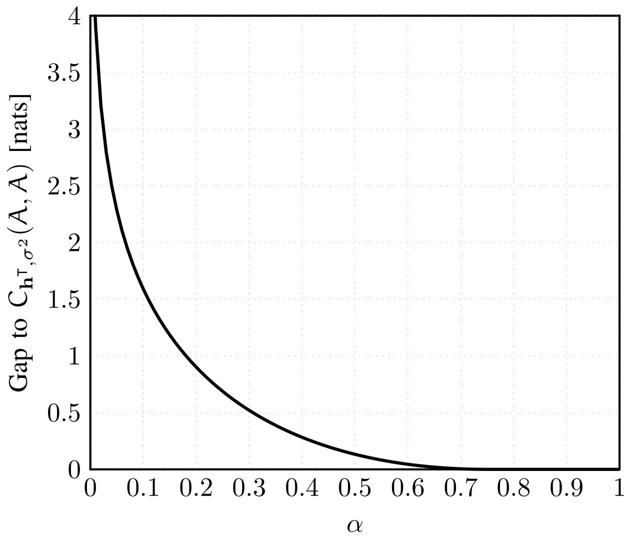

Fig. 1. The outer infimum in (14) as a function of α, for MISO channel with two transmit LEDs, $h_1 = 3$ and $h_2 = 1$. This infimum is the asymptotic capacity gap to the case with no average-power constraint $\mathsf{C}_{\mathbf{h}^\mathsf{T},\sigma^2}(\mathsf{A},\mathsf{A})$.

Asymptotically Optimal Input Distributions. As our capacity lower bound will show, in the high-SNR limit, the optimal strategy in terms of capacity is to maximize $\mathsf{h}(\bar{X})$. One can show that the optimal distribution for $\mathbf{X}$ in this sense is given by the following: with probability p_k, $k \in \{1,\dots,n_\mathrm{T}\}$,

$$X_j = \mathsf{A} \quad \text{w.p. } 1, \qquad j \in \{1,\dots,k-1\}, \tag{18a}$$
$$X_k \sim \text{truncated exponential of parameter } \mu, \tag{18b}$$
$$X_j = 0 \quad \text{w.p. } 1, \qquad j \in \{k+1,\dots,n_\mathrm{T}\}, \tag{18c}$$

where $\mathbf{p}$ and μ are the optimal choices of $\mathbf{p}$ and $\mu(\lambda)$, respectively, in (14). Note that, when $\alpha \geq \alpha_\mathrm{th}$, [9] shows that the truncated exponential distribution in (18b) should be replaced by a uniform distribution, and the optimal $\mathbf{p}$ is given by $p_k = h_k/h_\mathrm{sum}$, $k \in \{1,\dots,n_\mathrm{T}\}$. These choices yield a uniform distribution on $[0, h_\mathrm{sum}\mathsf{A}]$ for $\bar{X}$. When $\alpha < \alpha_\mathrm{th}$, it is no longer possible to achieve a uniform distribution for $\bar{X}$ without violating the average-power constraint (5). Instead, the optimal distribution for $\bar{X}$ is a concatenation of n_T truncated exponentials.

An Example. Consider a MISO channel with two transmit LEDs and with channel parameters $h_1 = 3$ and $h_2 = 1$. We plot the outer infimum in (14) against α in Fig. 1. Note that this infimum characterizes the capacity gap to the case with no average-power constraint, in the high-SNR limit. As expected, the gap becomes zero when α reaches $\alpha_\mathrm{th} = 0.75$, and approaches infinity when α tends to zero.

III. Proof Outline

A. Achievability: Lower Bound

We prove the following lower bound on capacity, from which the achievability part of Theorem 2 follows by optimization over the parameters.

Proposition 3: For any $\lambda \in \left(0, \min\{\frac{1}{2}, \alpha\}\right)$ and $\mathbf{p}$ satisfying $\sum_{k=1}^{n_\mathrm{T}} p_k(k-1) = \alpha - \lambda$, we have

$$\mathsf{C}_{\mathbf{h}^\mathsf{T},\sigma^2}(\mathsf{A},\alpha\mathsf{A}) \geq \log \mathsf{A} + \frac{1}{2}\log\frac{h_\mathrm{sum}^2}{2\pi e\sigma^2} - \log\frac{\mu(\lambda)}{1-e^{-\mu(\lambda)}} - 1 + \frac{\mu(\lambda)\,e^{-\mu(\lambda)}}{1-e^{-\mu(\lambda)}} + \mathscr{D}\left(\mathbf{p}\middle\|\frac{\mathbf{h}}{h_\mathrm{sum}}\right). \tag{19}$$

Proof: For a particular input distribution, using the EPI [17, Thm. 17.7.3] we obtain

$$\begin{aligned}
&\mathsf{C}_{\mathbf{h}^\mathsf{T},\sigma^2}(\mathsf{A},\alpha\mathsf{A}) \\
&\quad\geq \mathrm{I}(\bar{X};\bar{X}+Z) && (20)\\
&\quad= \mathsf{h}(\bar{X}+Z) - \mathsf{h}(Z) && (21)\\
&\quad\geq \frac{1}{2}\log\left(e^{2\,\mathsf{h}(\bar{X})} + e^{2\,\mathsf{h}(Z)}\right) - \mathsf{h}(Z) && (22)\\
&\quad= \mathsf{h}(\bar{X}) - \frac{1}{2}\log 2\pi e\sigma^2 + \frac{1}{2}\log\left(1 + \frac{2\pi e\sigma^2}{e^{2\,\mathsf{h}(\bar{X})}}\right). && (23)
\end{aligned}$$

Now fix probabilities $p_1,\dots,p_{n_\mathrm{T}}$ summing to one and satisfying

$$\sum_{k=1}^{n_\mathrm{T}} p_k(k-1) = \alpha - \lambda. \tag{24}$$

Define a random variable $U \in \{1,\dots,n_\mathrm{T}\}$ with $\Pr[U=k] = p_k$. Conditional on $U = k$, we choose $\mathbf{X}$ according to (18) where, more precisely, for (18b) we choose X_k to have the probability density function

$$f(x) = \frac{1}{\mathsf{A}}\frac{\mu(\lambda)}{1-e^{-\mu(\lambda)}}\,e^{-\frac{\mu(\lambda)x}{\mathsf{A}}}\,\mathbf{I}\{0 < x \leq \mathsf{A}\}, \tag{25}$$

where $\mu(\lambda)$ is the unique nonzero solution to (15). Note that

$$\begin{aligned}
\sum_{j=1}^{n_\mathrm{T}} \mathsf{E}[X_j] &= \sum_{k=1}^{n_\mathrm{T}} p_k \sum_{j=1}^{n_\mathrm{T}} \mathsf{E}[X_j \,|\, U=k] && (26)\\
&= \sum_{k=1}^{n_\mathrm{T}} p_k\left((k-1)\mathsf{A} + \left(\frac{1}{\mu(\lambda)} - \frac{e^{-\mu(\lambda)}}{1-e^{-\mu(\lambda)}}\right)\mathsf{A}\right) && (27)\\
&= \sum_{k=1}^{n_\mathrm{T}} p_k\left((k-1)\mathsf{A} + \lambda\mathsf{A}\right) && (28)\\
&= \alpha\mathsf{A}, && (29)
\end{aligned}$$

where the second last equality holds because of (15), and the last equality because of (24). Thus, this choice of input satisfies both (4) and (5). Also note that, conditional on U, only one LED has a random input, while the others are deterministic. Therefore it is possible to deduce from $\bar{X}$ the value of U:

$$U = k \iff \bar{X} \in \left(\sum_{j=1}^{k-1} h_j, \sum_{j=1}^{k} h_j\right]. \tag{30}$$

Thus,

$$\begin{aligned}
\mathsf{h}(\bar{X}) &= \mathsf{h}(\bar{X}) - \mathsf{h}(\bar{X}|U) + \mathsf{h}(\bar{X}|U) && (31)\\
&= \mathrm{I}(U;\bar{X}) + \mathsf{h}(\bar{X}|U) && (32)\\
&= \mathsf{H}(U) - \underbrace{\mathsf{H}(U|\bar{X})}_{=0} + \mathsf{h}(\bar{X}|U) && (33)
\end{aligned}$$

$$= \mathsf{H}(\mathbf{p}) + \sum_{k=1}^{n_{\mathrm{T}}} p_k \, \mathsf{h}(\bar{X}|U=k) \tag{34}$$

$$= \mathsf{H}(\mathbf{p}) + \sum_{k=1}^{n_{\mathrm{T}}} p_k \log h_k + \log \mathsf{A} - \log \frac{\mu(\lambda)}{1-e^{-\mu(\lambda)}} + 1 - \frac{\mu(\lambda)\, e^{-\mu(\lambda)}}{1-e^{-\mu(\lambda)}}. \tag{35}$$

Combining (35) with (23) and maximizing over $\mathbf{p}$ and λ prove the bound. ■

B. Converse: Upper Bound

We first derive the following upper bound on capacity.

Proposition 4: For every $k \in \{0, \ldots, n_{\mathrm{T}}\}$, define

$$s_k \triangleq \sum_{j=1}^{k} h_j. \tag{36}$$

Let p_k, $k \in \{2, \ldots, n_{\mathrm{T}}\}$, denote the probability that $\bar{X} \in (s_{k-1}\mathsf{A}, s_k\mathsf{A}]$, and p_1 that $\bar{X} \in [0, s_1\mathsf{A}]$, under the capacity-achieving input distribution. For any choice of parameters $\mu, \delta, \eta \in (0,\infty)$, $\beta \in (0, h_{n_{\mathrm{T}}})$, and n_{T}-dimensional probability vector $\mathbf{q}$, we have

$$\begin{aligned}
&\mathsf{C}_{\mathbf{h}^{\mathsf{T}},\sigma^2}(\mathsf{A}, \alpha\mathsf{A}) \\
&\le \max\bigl\{1, \log 2\pi e\sigma^2\bigr\} \cdot \mathcal{Q}\biggl(\frac{\delta\mathsf{A}}{\sigma}\biggr) + \frac{\delta\mathsf{A}}{\sqrt{2\pi}\sigma}\, e^{-\frac{\delta^2\mathsf{A}^2}{2\sigma^2}} \\
&\quad - \log\biggl(1 - 2\,\mathcal{Q}\biggl(\frac{\delta\mathsf{A}}{\sigma}\biggr) - 2(n_{\mathrm{T}}-1)\frac{\delta\mathsf{A}}{\eta}\biggr) \\
&\quad + \log\eta \sum_{k=1}^{n_{\mathrm{T}}} \Biggl(\mathcal{Q}\biggl(\frac{(h_k-\delta)\mathsf{A}}{\sigma}\biggr) - \mathcal{Q}\biggl(\frac{(h_k+\delta)\mathsf{A}}{\sigma}\biggr) \\
&\qquad\qquad + \mathcal{Q}\biggl(\frac{(h_{k+1}-\delta)\mathsf{A}}{\sigma}\biggr) - \mathcal{Q}\biggl(\frac{(h_{k+1}+\delta)\mathsf{A}}{\sigma}\biggr)\Biggr) \\
&\quad + 2n_{\mathrm{T}}\log\eta \cdot \frac{\log 2 + \frac{1}{e} + \frac{1}{2}\log\Bigl(1+\frac{\beta^2\mathsf{A}^2}{4\sigma^2}\Bigr)}{\frac{1}{2}\log\Bigl(1+\frac{\mathsf{A}^2}{2\pi e\sigma^2}\cdot\Upsilon(\alpha,\delta)\Bigr)} \\
&\quad + 2\log\eta\cdot\biggl(\mathcal{Q}\biggl(\frac{(\beta-\delta)\mathsf{A}}{\sigma}\biggr) - \mathcal{Q}\biggl(\frac{(\beta+\delta)\mathsf{A}}{\sigma}\biggr)\biggr) \\
&\quad + \mathcal{Q}\biggl(\frac{\delta\mathsf{A}}{\sigma}\biggr) \\
&\qquad \cdot \sum_{k=1}^{n_{\mathrm{T}}} \max\Biggl\{0, \log\Biggl(\frac{\mathsf{A}\bigl(h_k - 2\delta\bigr)}{1-2\,\mathcal{Q}\bigl(\frac{\delta\mathsf{A}}{\sigma}\bigr) - 2(n_{\mathrm{T}}-1)\frac{\delta\mathsf{A}}{\eta}}\Biggr)\Biggr\} \\
&\quad + \log\mathsf{A} - \frac{1}{2}\log 2\pi e\sigma^2 + \mathcal{Q}\biggl(\frac{\delta\mathsf{A}}{\sigma}\biggr)\sum_{k=1}^{n_{\mathrm{T}}}\biggl(\frac{\mu s_{n_{\mathrm{T}}}}{h_k} - \log q_k\biggr) \\
&\quad + \frac{\mu\sigma}{\mathsf{A}\sqrt{2\pi}}\sum_{k=1}^{n_{\mathrm{T}}}\frac{1}{h_k} + \sum_{k=1}^{n_{\mathrm{T}}} p_k\log\frac{h_k}{q_k} - \log\mu \\
&\quad + \log\bigl(1-e^{-\mu}\bigr) + \mu\biggl(\alpha - \sum_{k=1}^{n_{\mathrm{T}}} p_k(k-1)\biggr),
\end{aligned} \tag{37}$$

where

$$\Upsilon(\alpha,\delta) \triangleq \begin{cases} e^{2\alpha\delta}\Bigl(\frac{1-e^{-\delta}}{\delta}\Bigr)^2 & \text{if } \alpha \in \bigl(0, \frac{1}{2}\bigr), \\ 1 & \text{if } \alpha \ge \frac{1}{2}. \end{cases} \tag{38}$$

Proof Outline: We apply the duality-based bounding technique [15]

$$\begin{aligned}
&\mathsf{C}_{\mathbf{h}^{\mathsf{T}},\sigma^2}(\mathsf{A}, \alpha\mathsf{A}) \le \\
&\quad \mathsf{E}_{Q^*}\biggl[-\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{(y-\bar{X})^2}{2\sigma^2}}\log f(y)\,\mathrm{d}y\biggr] - \frac{1}{2}\log 2\pi e\sigma^2,
\end{aligned} \tag{39}$$

where Q^* is the capacity-achieving input distribution, and f is any probability density function on the real line. Here we define the parameters

$$\overline{s}_k \triangleq s_k + \delta, \qquad k = 1, \ldots, n_{\mathrm{T}} - 1; \tag{40}$$

$$\underline{s}_k \triangleq s_k - \delta, \qquad k = 1, \ldots, n_{\mathrm{T}} - 1; \tag{41}$$

$$\overline{s}_0 \triangleq s_0 - \delta; \tag{42}$$

$$\underline{s}_{n_{\mathrm{T}}} \triangleq s_{n_{\mathrm{T}}} + \delta; \tag{43}$$

$$\mu_k \triangleq \frac{\mu}{h_k}, \qquad k = 1, \ldots, n_{\mathrm{T}}, \tag{44}$$

and choose

$$f(y) = \begin{cases}
\frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{y^2}{2\sigma^2}} & \text{if } y \le \overline{s}_0\mathsf{A}, \\
q_k \frac{\mu_k\Bigl(1-2\,\mathcal{Q}\bigl(\frac{\delta\mathsf{A}}{\sigma}\bigr) - 2(n_{\mathrm{T}}-1)\frac{\delta\mathsf{A}}{\eta}\Bigr)}{\mathsf{A}\bigl(e^{-\mu_k\overline{s}_{k-1}} - e^{-\mu_k\underline{s}_k}\bigr)}\,e^{-\frac{\mu_k}{\mathsf{A}}y} & \text{if } \overline{s}_{k-1}\mathsf{A} < y \le \underline{s}_k\mathsf{A},\ k \in \{1,\ldots,n_{\mathrm{T}}\}, \\
\frac{1}{\eta} & \text{if } \underline{s}_k\mathsf{A} < y \le \overline{s}_k\mathsf{A},\ k \in \{1,\ldots,n_{\mathrm{T}}-1\}, \\
\frac{1}{\sqrt{2\pi}\sigma}\,e^{-\frac{(y-s_{n_{\mathrm{T}}})^2}{2\sigma^2}} & \text{if } y > \underline{s}_{n_{\mathrm{T}}}\mathsf{A},
\end{cases} \tag{45}$$

Note that, on the interval $[0, h_{\mathrm{sum}}\mathsf{A}]$, our choice of f resembles the asymptotically optimal distribution for $\bar{X}$, the latter being a concatenation of truncated exponential distributions. But, for technical reasons, we add "buffers" between neighboring truncated exponentials, as well as Gaussian tails on both sides of the interval.

Calculations are omitted due to space limitations. ■

To use Proposition 4 to derive the desired asymptotic upper bound, we choose

$$q_k = \frac{\mathsf{A}p_k + 1}{\mathsf{A} + n_{\mathrm{T}}}, \qquad k \in \{1, \ldots, n_{\mathrm{T}}\}. \tag{46}$$

Also, denote $\lambda(\mathbf{p}) \triangleq \alpha - \sum_{k=1}^{n_{\mathrm{T}}} p_k(k-1)$, fix some $\zeta \in (0,1)$, and choose

$$\mu = \begin{cases} \mu(\mathbf{p}) & \text{if } \mathsf{A}^{-(1-\zeta)} < \lambda(\mathbf{p}) < \frac{1}{2}, \\ \mathsf{A}^{1-\zeta} & \text{if } \lambda(\mathbf{p}) \le \mathsf{A}^{-(1-\zeta)}, \\ \frac{1}{\mathsf{A}} & \text{if } \lambda(\mathbf{p}) \ge \frac{1}{2}, \end{cases} \tag{47}$$

where $\mu(\mathbf{p})$ is the unique positive solution for μ to

$$\frac{1}{\mu} - \frac{e^{-\mu}}{1-e^{-\mu}} = \lambda(\mathbf{p}). \tag{48}$$

We then choose

$$\delta = \frac{\log(1 + \mathsf{A})}{\mathsf{A}}, \tag{49}$$

$$\beta = \frac{\log^2(1 + \mathsf{A})}{\mathsf{A}}, \tag{50}$$

$$\eta = \log^2(1 + \mathsf{A}), \tag{51}$$

and let $\mathsf{A} \to \infty$. After some further calculations, which are again omitted, we obtain that the right-hand side of (14) is an upper bound on its left-hand side.

IV. Concluding Remarks

We have shown that, in the high-SNR limit, the capacity of the peak- and average-intensity limited optical intensity channel is given by the maximum value of $\mathsf{h}(\mathbf{h}^\mathsf{T}\mathbf{X})$ minus the differential entropy of the Gaussian noise. There may be alternative ways to show this asymptotic result. For example, if one can show that, at high SNR, the optimal input distribution has a bounded density, then one may be able to come to the above conclusion via convergence properties of the differential entropy; see, e.g., [18]. On the other hand, such an approach, if successful, would not provide finite-SNR bounds on capacity as our Propositions 3 and 4.

We have also identified the input distribution that maximizes $\mathsf{h}(\mathbf{h}^\mathsf{T}\mathbf{X})$. This input distribution has the following structure: whenever a weaker transmit LED is switched on, all the stronger transmit LEDs must be transmitting at maximum intensity with probability one. Furthermore, conditional on the transmit signal of a specific LED being "random," this signal has a truncated exponential distribution, with a parameter that is the same for all transmit LEDs.

Recall that [9] has established the high-SNR asymptotic capacity of the MIMO optical intensity channel when the channel matrix has full column rank. A close look at the results in [9] confirms that, in the full-column-rank case, the asymptotic capacity is also given by the maximum differential entropy of $\mathsf{H}\mathbf{X}$, H being the channel matrix, minus that of the noise vector.

With the current work and [9], the only MIMO optical intensity channels whose asymptotic high-SNR capacities are not yet known are those with more than one receive antennas (photodetectors), and with channel matrices that do not have full column rank. It is natural to conjecture that, for those channels, the asymptotic high-SNR capacity is again given by the maximum of $\mathsf{h}(\mathsf{H}\mathbf{X})$ minus the differential entropy of the noise.

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