Side-Information with a Grain of Salt

Pliny's Naturalis Historia, 77 A.D.:
Take two dried walnuts, two figs, and twenty leaves of rue; pound them all together, with the addition of a grain of salt*; if a person takes this mixture fasting, he will be proof against all poisons for that day.*

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Based on collaborations with
Shraga Bross, Michael Gastpar, Gerhard Kramer, Amos Lapidoth, Shlomo Shamai

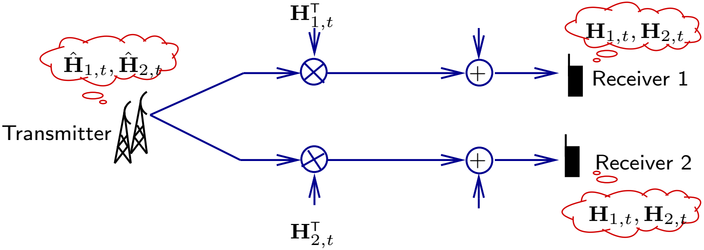
Robustness w.r.t. Precision of Side-Information

- Performance with imprecise side-information
- Robust schemes

Four Communication Scenarios with Side-Information

A on channel states in MIMO broadcast channels

Fading MIMO BC with Imprecise State-Information



Lapidoth/Shamai/Wigger'05

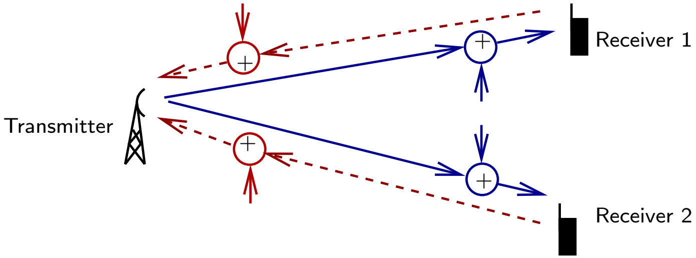
Gain (i.e., 2 efficient links thanks to two antennas) collapses with imprecise transmitter state-information!

Four Communication Scenarios with Side-Information

A on channel states in MIMO broadcast channels

B via feedback in Gaussian broadcast channels

Scalar Gaussian Broadcast Channel with Noisy Feedback



Gastpar/Lapidoth/Wigger'09

Gain collapses completely when feedback is noisy!
I.e. # efficient links collapses from 2 to 1!

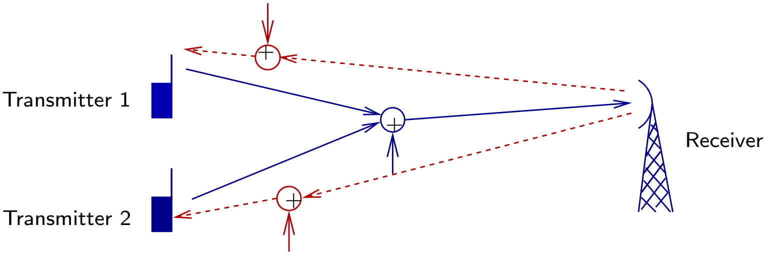
Four Communication Scenarios with Side-Information

A on channel states in MIMO broadcast channels

B via feedback in Gaussian broadcast channels

C via feedback in Gaussian multiple-access channels

Gaussian MAC with Noisy Feedback



Lapidoth&Wigger'06

- Noisy feedback is almost as good as perfect feedback!
- Even if noisy, feedback is *always* beneficial!

Four Communication Scenarios with Side-Information

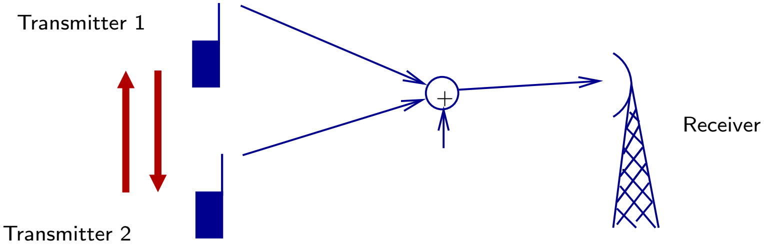
A on channel states in MIMO broadcast channels

B via feedback in Gaussian broadcast channels

C via feedback in Gaussian multiple-access channels

D on other transmitter's message in Gaussian multiple-access channels

Gaussian MAC with Conferencing Encoders à la Willems'83



Bross/Lapidoth/Wigger'08

- Capacity region
- Best conferencing: optimally describe message to other transmitter

Four Communication Scenarios with Side-Information

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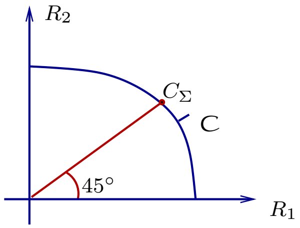
D on other transmitter's message in Gaussian multiple-access channels

Capacity Region

- Rates of communication R_1 and R_2
- Capacity region $\mathbb{C}$: Set of (R_1, R_2) s.t. p(error) arbitrarily small
- Sum-rate capacity C_Σ: maximum throughput s.t. p(error) arbitrarily small



Degrees of Freedom η

$$C_\Sigma(P) \approx \eta \cdot \frac{1}{2}\log\left(1+\frac{P}{N}\right), \qquad\qquad \frac{P}{N} \gg 1.$$

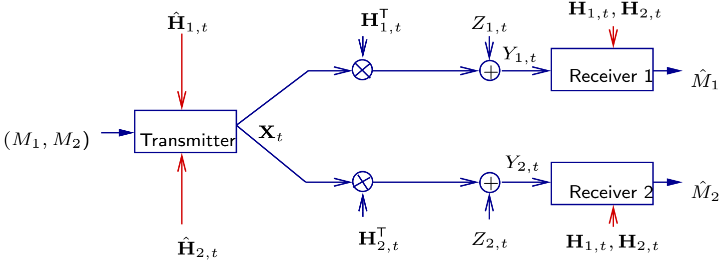
Engineering intuition:

- $\#$ interference-free Gaussian channels in a system
- $\eta \leq \min\{\#\text{ tx-antennas}, \#\text{rx-antennas}\}$

Part A

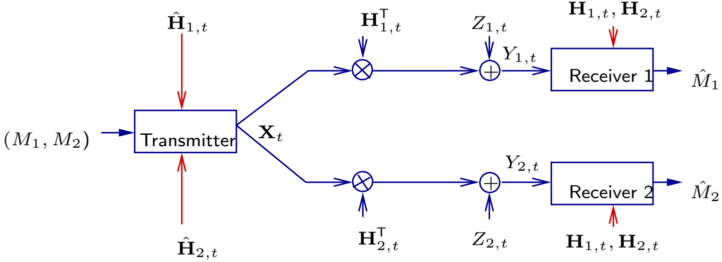
Fading MIMO Gaussian Broadcast Channel with Imprecise State-Information

Gaussian Fading MIMO Broadcast Channel



- Transmitter has 2 antennas, receivers 1 antenna
- $Y_{\nu,t} = \mathbf{H}^{\mathsf{T}}_{\nu,t}\mathbf{X}_t + Z_{\nu,t}$; $\{Z_{\nu,t}\}$ IID $\sim \mathcal{N}(0, N)$
- Power constraint: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[\|\mathbf{X}_t\|^2\right] \leq P$

Gaussian Fading MIMO Broadcast Channel



- $\mathbf{H}_{\nu,t} = \hat{\mathbf{H}}_{\nu,t} + \tilde{\mathbf{H}}_{\nu,t}$
- Tx knows a-causally *realizations* of $\{\hat{\mathbf{H}}_{1,t}\}, \{\hat{\mathbf{H}}_{2,t}\}$
- Rxs know *realizations* of $\{\tilde{\mathbf{H}}_{1,t}\}, \{\tilde{\mathbf{H}}_{2,t}\}, \{\hat{\mathbf{H}}_{1,t}\}, \{\hat{\mathbf{H}}_{2,t}\}$ (optimistic)

Perfect Channel State Information (CSI)

- 2 transmit antennas [Caire & Shamai'03]: $\eta_{\text{PerfectCSI}} = 2$
- 1 transmit antenna: $\eta_{\text{PerfectCSI}} = 1$
- $\Longrightarrow$ 2 antennas double throughput at high powers!
- Beamforming: Transmission in two orthogonal directions

Approximate Channel State Information

Theorem 1: (Lapidoth/Shamai/Wigger'05)

Degrees of freedom collapse from 2 to at most

$$\eta_{\text{Approx.CSI}} \leq \frac{4}{3}$$

- Conjecture: 1 degree of freedom! $\Rightarrow$ "No gain" from 2 tx-antennas!
- Here: precision of transmitter CSI fixed, not improved as $P \to \infty$

Subsequent Work

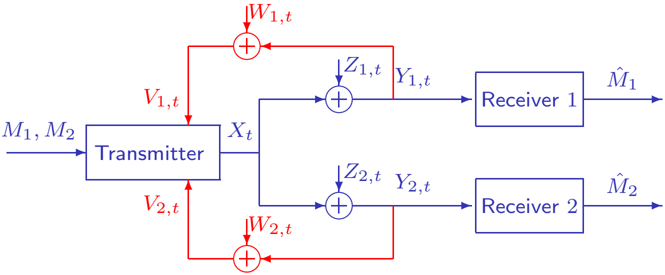
- Precision becomes exact as $P \to \infty$

- [Jindal'06], [Marzetta'06], [Caire/Jindal/Kobayashi/Ravindran'07], [Shamai/Caire/Jindal'07]

Part B

Scalar Gaussian Broadcast Channel with Noisy Feedback

Scalar Gaussian Broadcast Channel with Noisy Feedback

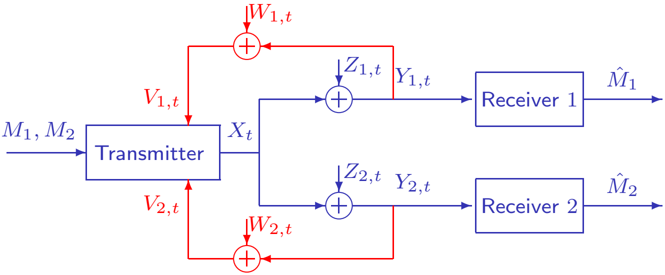


- Transmitter & receivers: each 1 antenna!

- $Y_{\nu,t} = X_t + Z_{\nu,t},$ $\qquad \begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} N & \rho_z N \\ \rho_z N & N \end{pmatrix}\right)$

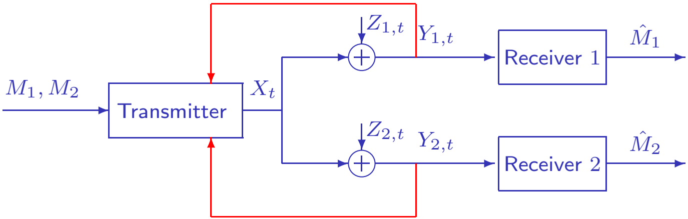
- Noise correlation caused by interference

Scalar Gaussian Broadcast Channel with Noisy Feedback



- $V_{\nu,t} = Y_{\nu,t} + W_{\nu,t}$, $\qquad \left\{ \begin{pmatrix} W_{1,t} \\ W_{2,t} \end{pmatrix} \right\}$ IID $\sim \mathcal{N}\left(\mathbf{0}, \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\right)$
- $\{W_{1,t}, W_{2,t}\}$ independent of $\{Z_{1,t}, Z_{2,t}\}$
- $X_t = f_t(M_1, M_2 \mathbf{V}_{1,1}^{t-1}, \mathbf{V}_{2,1}^{t-1})$
- Power constraint: $\frac{1}{n} \sum_{t=1}^{n} \mathsf{E}\left[X_t^2\right] \leq P$

Perfect Feedback



Theorem 2: (Gastpar&Wigger'08)

$$\eta_{\text{PerfectFB}} = \begin{cases} 1, & -1 < \rho_z \leq 1, \\ 2, & \rho_z = -1. \end{cases}$$

For $\rho_z = -1$:

- 2 degrees of freedom with 1 transmit antenna!
- Perfect feedback doubles degrees of freedom

Nosiy Feedback: Degrees of Freedom Collapse

Theorem 3: (Gastpar/Lapidoth/Wigger'09, in preparation)

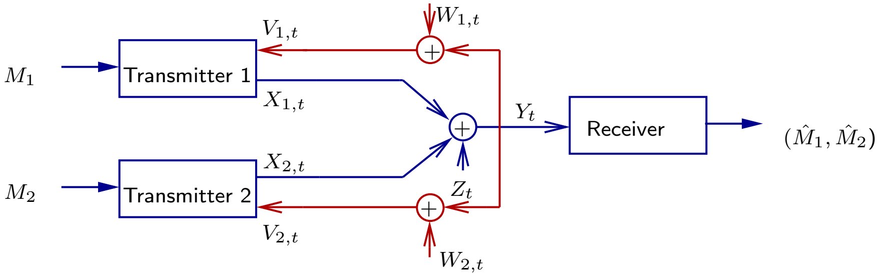
$$\eta_{\text{NoisyFB}} = 1, \qquad \text{for all } \rho_z \in [-1, 1].$$

- Feedback noise variances $\sigma_1^2, \sigma_2^2 > 0$ fixed
- Gain of [Gastpar/Wigger'08] collapses!
- Engineering intuition "$\eta \leq \min\{\#\text{tx-antennas}, \#\text{rx-antennas}\}$" OK!

Part C

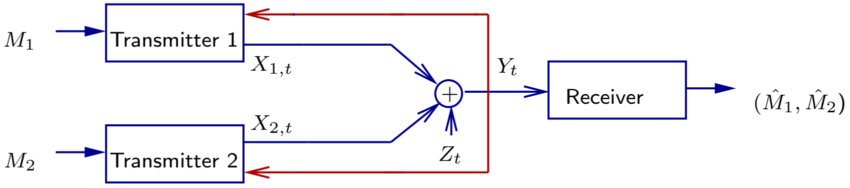
Gaussian MAC with Noisy Feedback

Gaussian MAC with Noisy Feedback



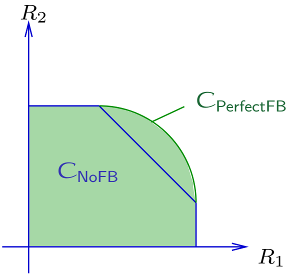
- $Y_t = X_{1,t} + X_{2,t} + Z_t$, $\quad \{Z_t\}$ IID $\sim \mathcal{N}(0, N)$
- $V_{\nu,t} = Y_t + W_{\nu,t}$, $\quad \{(W_{1,t}, W_{2,t})^\mathsf{T}\} \sim$ IID $\mathcal{N}(\mathbf{0}, \mathsf{K}_{W_1 W_2})$
- Power constraints: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[X_{\nu,t}^2\right] \le P_\nu$, $\quad \nu \in \{1, 2\}$

Perfect Feedback



- Ozarow'84:

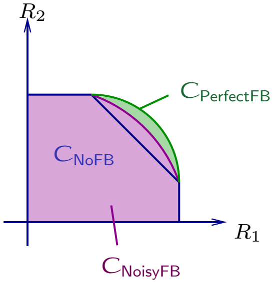
$$C_{\text{PerfectFB}} = \bigcup_{\rho \in [0,1]} \left\{ \begin{array}{lcl} (R_1, R_2) : & & \\ R_1 & \leq & \frac{1}{2}\log\left(1 + \frac{P_1(1-\rho^2)}{N}\right) \\ R_2 & \leq & \frac{1}{2}\log\left(1 + \frac{P_2(1-\rho^2)}{N}\right) \\ R_1 + R_2 & \leq & \frac{1}{2}\log\left(1 + \frac{P_1+P_2+2\sqrt{P_1P_2}\rho}{N}\right) \end{array} \right\}$$



- Capacity increased!

Imperfect Feedback: Noisy Feedback

- C_{NoisyFB} open
- Ozarow's strategy doesn't work!



- a) Carleial'82, b) Willems/van der Meulen/Schalkwijk'83, c) Gastpar'05
  - Collapse to C_{NoFB} if feedback too noisy!
  - a)&b) do not approach $C_{\text{PerfectFB}}$ when feedback becomes noise-free

Shortcomings of schemes or inherent in problem?

Results for Noisy Feedback

- We robustify Ozarow's scheme to noisy feedback

Theorem 4: (Lapidoth&Wigger'06)

Noisy-feedback capacity converges to perfect-feedback capacity

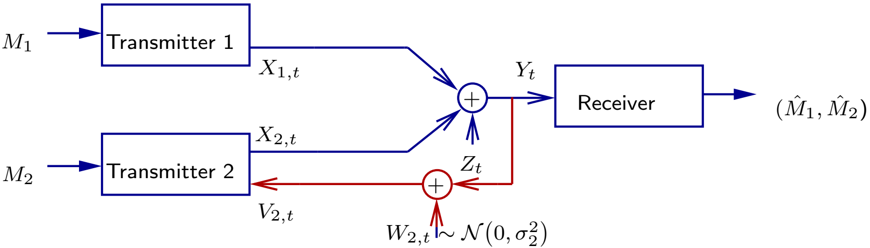
$$\mathsf{cl}\Big(\bigcup_{\sigma^2 \geq 0} \bigcap_{\mathsf{K}:\ \mathrm{tr}(\mathsf{K}) \leq \sigma^2} \mathcal{C}_{\mathsf{NoisyFB}}(P_1, P_2, N, \mathsf{K}) \Big) = \mathcal{C}_{\mathsf{PerfectFB}}(P_1, P_2, N)$$

Theorem 5a: (Lapidoth&Wigger'06)

Noisy feedback is always beneficial!

$$\mathcal{C}_{\mathsf{NoFB}}(P_1, P_2, N) \subsetneq \mathcal{C}_{\mathsf{NoisyFB}}(P_1, P_2, N, \mathsf{K}_{W_1 W_2}) \qquad \forall\ \mathsf{K}_{W_1 W_2} \succeq 0.$$

Imperfect Feedback: Noisy Partial Feedback



- Situation even worse! $\longrightarrow$ Feedback beneficial?
- If $\sigma_2^2 = 0$, Cover-Leung region achievable

Van der Meulen'87:
Is Cover-Leung region capacity when $\sigma_2^2 = 0$?

Results for Partial Feedback

- Our robust scheme still works!

Theorem 5b: (Lapidoth&Wigger'06)

Noisy partial feedback is always beneficial!

$$C_{\mathsf{NoFB}}(P_1, P_2, N) \subsetneq C_{\mathsf{NoisyPartialFB}}(P_1, P_2, N, \sigma_2^2), \qquad \forall\ \sigma_2^2 \geq 0.$$

Theorem 6: Answer to van der Meulen'87 (Lapidoth&Wigger'06)

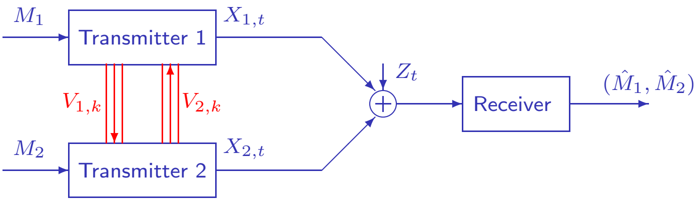
Perfect partial-feedback capacity $\neq$ *Cover-Leung region!*

$$\mathcal{R}_{\mathsf{CL}}(P_1, P_2, N) \subsetneq C_{\mathsf{PerfectPartialFB}}(P_1, P_2, N), \qquad \text{for some } P_1, P_2, N > 0.$$

Part D

Gaussian MAC with Conferencing Encoders

Gaussian MAC with Conferencing Encoders

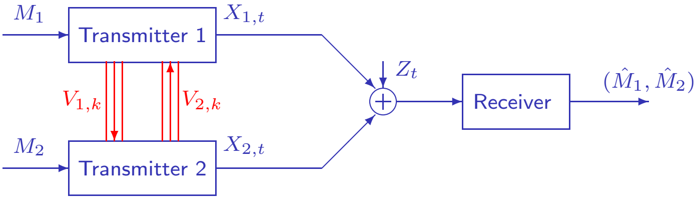


1. phase: Conference (Willems'83)

- κ sequential uses of perfect bit-pipes
- $V_{1,k} = \varphi_{1,k}\left(M_1, V_{2,1}^{k-1}\right)$; $\qquad V_{2,k} = \varphi_{2,k}\left(M_2, V_{1,1}^{k-1}\right)$
- Rate-limitations:

$$\sum_{k=1}^{\kappa} \log|\mathcal{V}_{1,k}| \leq nC_{12} \qquad \text{and} \qquad \sum_{k=1}^{\kappa} \log|\mathcal{V}_{2,k}| \leq nC_{21}$$

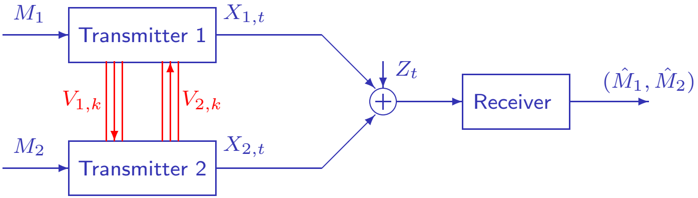
Gaussian MAC with Conferencing Encoders



2. phase: Transmission over channel:

- $Y_t = X_{1,t} + X_{2,t} + Z_t; \qquad \{Z_t\}$ IID $\sim \mathcal{N}(0, N)$
- $X_{1,t} = f_{1,t}\left(M_1, V_{2,1}^{\kappa}\right); \qquad X_{2,t} = f_{2,t}\left(M_2, V_{1,1}^{\kappa}\right)$
- Power constraints: $\frac{1}{n}\sum_{t=1}^{n} \mathsf{E}\left[X_{\nu,t}^2\right] \le P_\nu, \qquad \nu \in \{1, 2\}$

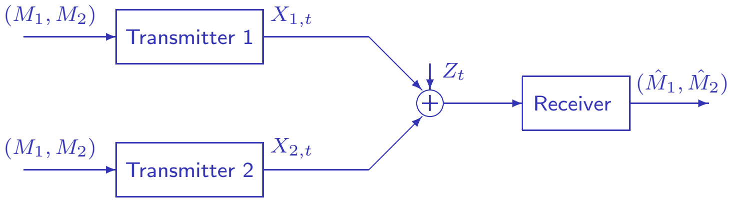
Gaussian MAC with Conferencing Encoders



Special Cases:

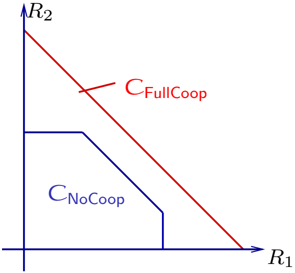
- $C_{12} = C_{21} = \infty$: full cooperation (both txs know (M_1, M_2))
- $C_{12} = 0$, $C_{21} = \infty$: Tx 1 knows (M_1, M_2), Tx 2 only M_2
- $C_{12} = C_{21} = 0$: no conferencing

Full Cooperation



- Transmitters 1 and 2 know (M_1, M_2):

$$C_{\text{FullCoop}} = \Bigg\{(R_1, R_2) : \\ R_1 + R_2 \leq \frac{1}{2}\log\left(1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N}\right)\Bigg\}$$



Partial Cooperation: Conferencing Encoders

Theorem 7: (Bross/Lapidoth/Wigger'08)

$$C_{\mathsf{Conf}} = \bigcup_{\substack{\rho_1,\rho_2 \\ \in[0,1]}} \left\{ (R_1,R_2): \begin{array}{rcl} R_1 & \le & \frac{1}{2}\log\left(1+\frac{P_1\left(1-\rho_1^2\right)}{N}\right)+C_{12} \\ R_2 & \le & \frac{1}{2}\log\left(1+\frac{P_2\left(1-\rho_2^2\right)}{N}\right)+C_{21} \\ R_1+R_2 & \le & \frac{1}{2}\log\left(1+\frac{P_1\left(1-\rho_1^2\right)+P_2\left(1-\rho_2^2\right)}{N}\right)+C_{12}+C_{21} \\ R_1+R_2 & \le & \frac{1}{2}\log\left(1+\frac{P_1+P_2+2\rho_1\rho_2\sqrt{P_1P_2}}{N}\right) \end{array} \right\}$$

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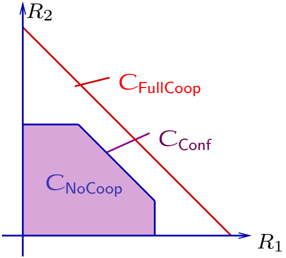
$C_{12} = C_{21} = 0$



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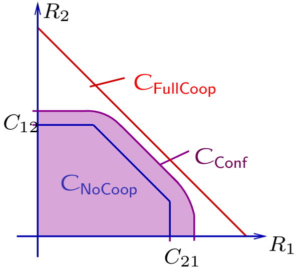
$C_{12}, C_{21} \neq 0$,
but "small"



If $C_{12} > 0$ or $C_{21} > 0$

$$\mathcal{C}_{\text{NoCoop}} \subsetneq \mathcal{C}_{\text{Conf}}$$

Partial Cooperation: Conferencing Encoders

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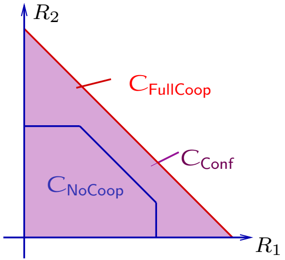
$$C_{12}, C_{21} \ge \frac{1}{2}\log\left(1+\frac{P_1+P_2+2\sqrt{P_1P_2}}{N}\right)$$



Multi-Antenna Extension

- Transmitters/receiver have multiple antennas
- $\mathbf{Y}_t = \mathsf{H}_1 \mathbf{X}_{1,t} + \mathsf{H}_2 \mathbf{X}_{2,t} + \mathbf{Z}_t$; $\{\mathbf{Z}_t\}$ IID $\sim \mathcal{N}(\mathbf{0}, \mathsf{I})$

Theorem 8: (Wigger&Kramer'09)

Capacity region for MIMO extension:

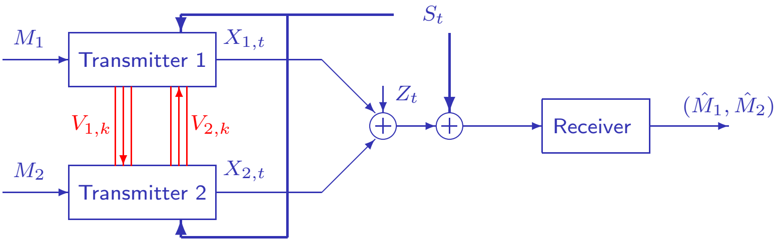
$$\bigcup_{\substack{\mathsf{A}_1,\mathsf{A}_2,\mathsf{B}_1,\mathsf{B}_2: \\ \mathrm{tr}(\mathsf{A}_1\mathsf{A}_1^\mathsf{T}+\mathsf{B}_1\mathsf{B}_1^\mathsf{T})\le P_1 \\ \mathrm{tr}(\mathsf{A}_2\mathsf{A}_2^\mathsf{T}+\mathsf{B}_2\mathsf{B}_2^\mathsf{T})\le P_2}} \left\{ \begin{array}{lcl} (R_1, R_2): & & \\ R_1 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}\right)\right) + C_{12} \\ R_2 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}\right)\right) + C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log\left(\det\left(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}+\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}\right)\right) \\ & & +C_{12}+C_{21} \\ R_1 + R_2 & \le & \frac{1}{2}\log\Big(\det\Big(\mathsf{I}+\mathsf{H}_1\mathsf{A}_1\mathsf{A}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}+\mathsf{H}_1\mathsf{B}_1\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T} \\ & & \quad +\mathsf{H}_2\mathsf{A}_2\mathsf{A}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}+\mathsf{H}_2\mathsf{B}_2\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T} \\ & & \quad +\mathsf{H}_1\mathsf{B}_1\mathsf{B}_2^\mathsf{T}\mathsf{H}_2^\mathsf{T}+\mathsf{H}_2\mathsf{B}_2\mathsf{B}_1^\mathsf{T}\mathsf{H}_1^\mathsf{T}\Big)\Big) \end{array} \right\}$$

Capacity Achieving Scheme (inspired by Willems'83)

- Transmitters split messages: $M_1 = (M_{1,c}, M_{1,p})$ and $M_2 = (M_{2,c}, M_{2,p})$
- Conference $M_{1,c}$ and $M_{2,c} \Rightarrow$ Common Message $(M_{1,c}, M_{2,c})$
- Rate of $M_{1,c} < C_{12}$ and rate of $M_{2,c} < C_{21}$
- Superposition $M_{1,p}$ or $M_{2,p}$ on top of $(M_{1,c}, M_{2,c})$

- MIMO: conferenced bits describe common beamforming direction

Dirty-Paper Extension



- 2 Settings: Transmitters can learn S^n before or after the conference

Theorem 9: (Bross/Lapidoth/Wigger'08)

$$C_{\text{Int,before}} = C_{\text{Int,after}} = C_{\text{Conf}}$$

Part D2: A Proof

Converse (Outer Bound on Capacity) for the Gaussian MAC with Conferencing Encoders

Converse for Original Setup (Outer Bound)

- Step 1: Willems's outer bound *with power constraints*:

$$C_{\mathsf{Conf}} \subseteq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \leq P_1,\, \mathsf{E}[X_2^2] \leq P_2}} \mathcal{R}_{X_1,U,X_2}, \quad (1)$$

  where

$$\mathcal{R}_{X_1,U,X_2} \triangleq \left\{ (R_1,R_2) : \begin{array}{rcl} R_1 & \leq & I(X_1;Y|X_2U) + C_{12}, \\ R_2 & \leq & I(X_2;Y|X_1U) + C_{21}, \\ R_1 + R_2 & \leq & I(X_1X_2;Y|U) + C_{12} + C_{21}, \\ R_1 + R_2 & \leq & I(X_1X_2;Y) \end{array} \right\}$$

- Step 2: In (1) suffices to take *Gaussian* Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

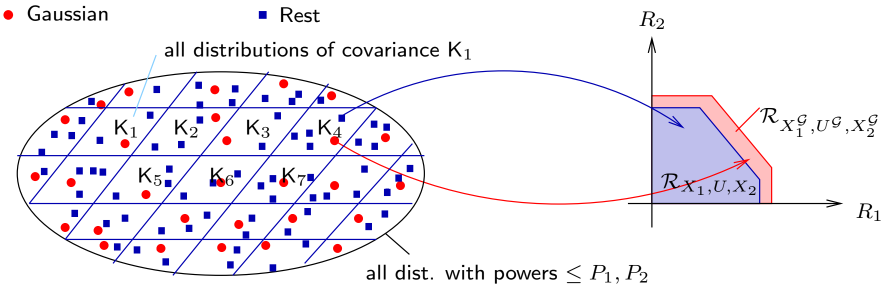
- Step 3: Evaluate $\mathcal{R}_{X_1,U,X_2}$ $\forall$ Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$

Optimization subject to Markov Constraints

Step 2: Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ are optimal for $\mathcal{R}_{X_1,U,X_2}$:

$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1,\, \mathsf{E}[X_2^2] \le P_2}} \mathcal{R}_{X_1,U,X_2} = \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2] \le P_1,\, \mathsf{E}[(X_2^{\mathcal{G}})^2] \le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

If there were no Markov condition:

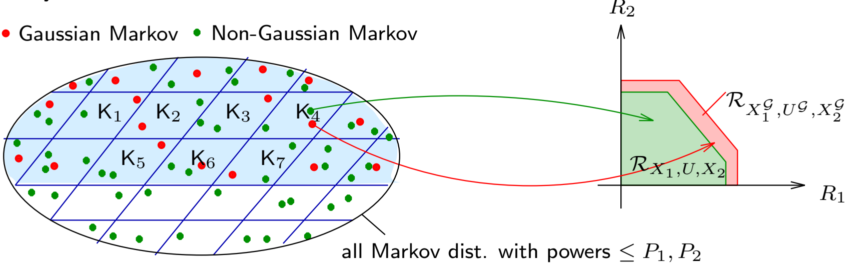


Optimization subject to Markov Constraints

Step 2: Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ are optimal for $\mathcal{R}_{X_1,U,X_2}$:

$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2]\le P_1,\ \mathsf{E}[X_2^2]\le P_2}} \mathcal{R}_{X_1,U,X_2} = \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2]\le P_1,\ \mathsf{E}[(X_2^{\mathcal{G}})^2]\le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

First try with Markov condition $\rightarrow$ same as without Markov condition



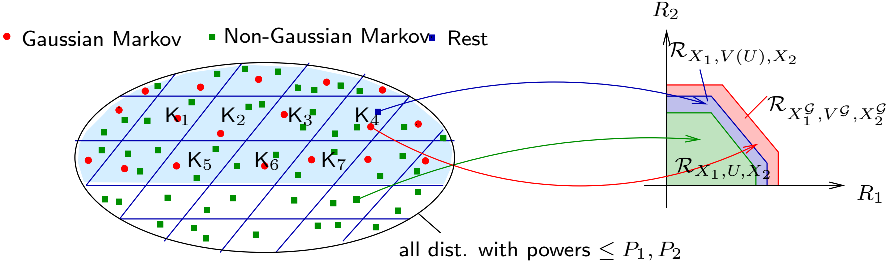
Problem: $\forall \mathsf{K} \succeq 0$ there is a Markov triple but not necess. a Gaussian Markov!

Optimization subject to Markov Constraints

Step 2: Gaussians $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$ are optimal for $\mathcal{R}_{X_1,U,X_2}$:

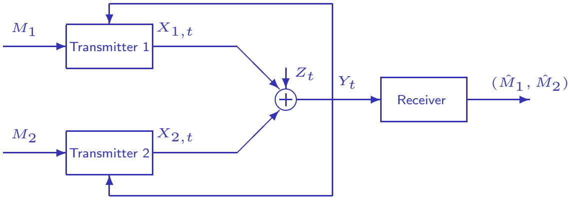
$$\bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1,\ \mathsf{E}[X_2^2] \le P_2}} \mathcal{R}_{X_1,U,X_2} = \bigcup_{\substack{X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}} \\ \mathsf{E}[(X_1^{\mathcal{G}})^2] \le P_1,\ \mathsf{E}[(X_2^{\mathcal{G}})^2] \le P_2}} \mathcal{R}_{X_1^{\mathcal{G}},U^{\mathcal{G}},X_2^{\mathcal{G}}}$$

Trick: Consider X_1, V, X_2, where $V = \mathsf{E}[X_1|U] - \mathsf{E}[X_1]$



Because for covariance of X_1, V, X_2 there is a Gaussian Markov triple $\square$

New Tool also Applies for Cover-Leung Region



Achievable region for Gaussian MAC with perfect partial feedback

$$C_{\text{PerfectFB}} \supseteq C_{\text{PerfectPartialFB}} \supseteq \mathcal{R}_{\text{CL}}$$

$$\mathcal{R}_{\text{CL}} \triangleq \bigcup_{\substack{X_1 - U - X_2 \\ \mathsf{E}[X_1^2] \le P_1, \\ \mathsf{E}[X_2^2] \le P_2}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \le & I(X_1; Y | X_2 U) \\ R_2 & \le & I(X_2; Y | X_1 U) \\ R_1 + R_2 & \le & I(X_1 X_2; Y) \end{array} \right\}$$

Suffices to consider Gaussian Markov triples $X_1^{\mathcal{G}} - U^{\mathcal{G}} - X_2^{\mathcal{G}}$!

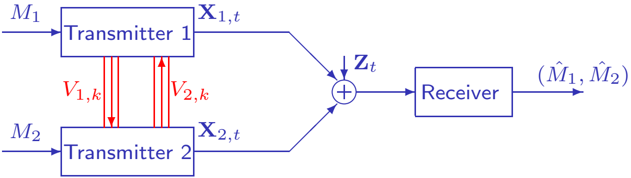
New Tool Applies for even More Settings

In expressions for capacity regions of :

- Two-users MAC with a common and two private messages (Slepian&Wolf'73)
- Interference channels with partial transmitter cooperation (Maric/Yates/Kramer'07)
- Compound MAC with conferencing encoders (Maric/Yates/Kramer'08)

it suffices to consider Gaussian Markov triples!

New Tool Extends to Vector-Case



Capacity of Gaussian MIMO MAC with Conferencing Encoders

$$C_{\text{Conf}} = \mathcal{R}_{\text{Conf}}$$

$$\mathcal{R}_{\text{Conf}} \triangleq \bigcup_{\substack{\mathbf{X}_1 - \mathbf{U} - \mathbf{X}_2 \\ \text{tr}(\mathsf{K}_{\mathbf{X}_1}) \leq P_1, \\ \text{tr}(\mathsf{K}_{\mathbf{X}_2}) \leq P_2}} \left\{ (R_1, R_2) : \begin{array}{rcl} R_1 & \leq & I(\mathbf{X}_1; \mathbf{Y} | \mathbf{X}_2 \mathbf{U}) + C_{12}, \\ R_2 & \leq & I(\mathbf{X}_2; \mathbf{Y} | \mathbf{X}_1 \mathbf{U}) + C_{21}, \\ R_1 + R_2 & \leq & I(\mathbf{X}_1 \mathbf{X}_2; \mathbf{Y} | \mathbf{U}) + C_{12} + C_{21}, \\ R_1 + R_2 & \leq & I(\mathbf{X}_1 \mathbf{X}_2; \mathbf{Y}) \end{array} \right\}$$

Suffices to consider Gaussian Markov triples $\mathbf{X}_1^{\mathcal{G}} - \mathbf{U}^{\mathcal{G}} - \mathbf{X}_2^{\mathcal{G}}$!

New Tool Extends to Multiple Markov Chains (Wigger&Kramer'09)

Capacity of 3 Users AWGN MAC with Common Msgs (Slepian&Wolf'73)

$$C_{\text{3Users,CommonMsgs}} = \mathcal{R}_{3,\text{SW}}$$

$$\mathcal{R}_{3,\text{SW}} \triangleq \bigcup_{\substack{U_0,U_{12},U_{13},U_{23} \text{ indep.} \\ X_1-(U_0,U_{12},U_{13})-(X_2,X_3,U_{23}) \\ X_2-(U_0,U_{12},U_{23})-(X_1,X_3,U_{13}) \\ X_3-(U_0,U_{13},U_{23})-(X_1,X_2,U_{12}) \\ \mathsf{E}[X_\nu^2]\leq P_\nu,\ \nu\in\{1,2,3\}}} \Big\{ (R_0, R_1, R_2, R_3, R_{12}, R_{13}, R_{23}) :$$

$$\begin{aligned}
R_1 &\leq I(X_1; Y | X_2, X_3, U_0, U_{12}, U_{13}) \\
R_2 &\leq I(X_2; Y | X_1, X_3, U_0, U_{12}, U_{23}) \\
R_3 &\leq I(X_3; Y | X_1, X_2, U_0, U_{13}, U_{23}) \\
R_1 + R_2 &\leq I(X_1, X_2; Y | X_3, U_0, U_{12}, U_{13}, U_{23}) \\
R_1 + R_3 &\leq I(X_1, X_3; Y | X_2, U_0, U_{12}, U_{13}, U_{23}) \\
\dots &\quad \dots \\
R_0 + R_{12} + R_{13} + R_{23} + R_1 + R_2 + R_3 & \\
&\leq I(X_1, X_2, X_3; Y) \Big\}
\end{aligned}$$

Suffices to consider Gaussians satisfying independence and Markov conditions!

Summary of Talk

Fading MIMO BC with Channel State Information @ Tx/Rxs

- Imprecisions in CSI @ tx $\Rightarrow$ degrees of freedom collapse from 2 to $\leq \frac{4}{3}$

BC with Correlated Noises and Feedback

- 2 degrees of freedom with 1 tx-antenna and perfect fb for $\rho_z = -1$
- Noisy feedback $\Rightarrow$ degrees of freedom collapse to 1

MAC with Feedback

- Almost noise-free feedback $\approx$ noise-free feedback
- Even noisy feedback is always beneficial
- Answer van der Meulen's question

MAC with Conferencing Encoders

- Capacity region (also for MIMO and Costa extensions)
- Solved Optimization problem subject to Markovity conditions

Other Research Topics

- Cognitive Interference Networks, Wyner's Linear Network

  "Equivalence cognition at txs and joint processing at rxs"

  [Lapidoth/Shamai/Wigger ISIT'07 & ITW'07; Lapidoth/Levy/Shamai/Wigger ISIT'09]

- Relay Channels with Feedback

  "With feedback, amplify&forward at relay $\gg$ block-Markov schemes"

  [Bross/Wigger, ISIT'07; Bross/Wigger, IT Jan. 2009]

- Free-Space Optical Intensity Channels

  "High and low SNR asymptotics under nonnegativity, peak, and average power constraints"

  [Moser/Lapidoth/Wigger, ISIT'08 and submitted to IT-Trans.]

See also: http://people.ethz.ch/∼wiggerm