The pre-log of Gaussian broadcast with feedback can be two

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***Abstract*— A generic intuition says that the pre-log, or multiplexing gain, cannot be larger than the minimum of the number of transmit and receive dimensions. This suggests that for the scalar broadcast channel, the pre-log cannot exceed one. By contrast, in this note, we show that when the noises are anti-correlated and feedback is present, then a pre-log of two can be attained. In other words, in this special case, in the limit of high SNR, the scalar Gaussian broadcast channel turns into two parallel AWGN channels. Achievability is established via a coding strategy due to Schalkwijk, Kailath, and Ozarow.**

I. Introduction

The significance of feedback in a capacity sense has been thoroughly studied for point-to-point and several network scenarios. Many results point to the lack of such a significance, starting with Shannon's proof that the capacity of a memoryless channel is unchanged by feedback. For networks, even for memoryless ones, feedback can increase capacity, as first shown by Gaarder and Wolf [1]. However, in most cases, the increase in capacity due to feedback remains modest, as expressed for example in a general conjecture in [2].

The exact feedback capacity remains unknown for most networks, with the notable exception of the two-user Gaussian multiple-access channel (MAC), whose capacity was found by Ozarow [3]. Some recent progress concerns the M-user Gaussian MAC [4]. Again, these results emphasize the lack of significance of feedback in a capacity sense.

By contrast, the result presented in this short note shows that feedback *can* have a rather significant impact on capacity in a certain *broadcast* setting. More specifically, we consider the problem of two-user broadcast subject to additive white Gaussian noise. This scenario has been studied previously by Ozarow [5], Ozarow and Leung [6], as well as Willems and van der Meulen [7].

The main result of this paper is that for the special case where the two noises are (fully) *anti-correlated*, in the limit as P becomes large, the trade-off between the two broadcast clients vanishes, and each client attains a rate as if the other did not exist. Formally, the result is presented as a "pre-log," or "multiplexing gain."

To our knowledge the considered setting is the first example of a channel where the "pre-log" can indeed be larger than the number of transmit antennas.

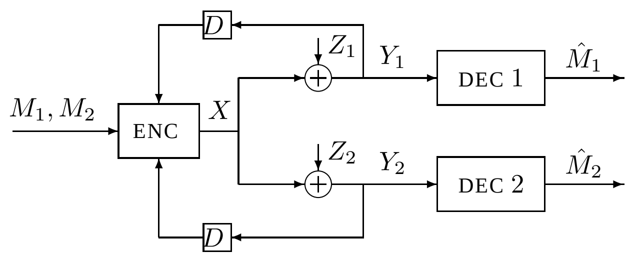

Fig. 1. The two-user AWGN broadcast channel with full causal output feedback.

This behavior is interesting in view of the result by Telatar [8] who showed that for *uncorrelated* noise sequences, the "pre-log" is upper bounded by the number of transmit antennas and by the number of receive antennas even if the two receivers are allowed cooperate. Therefore, in the setting at hand when the noise sequences are uncorrelated the "pre-log" cannot be larger than 1.

One motivation for the study of anti-correlated noises is that the signals Z_1 and Z_2 in Figure 1 are due to one and the same outside interferer, but appear with different (more precisely, opposite) phase shifts at the two receivers.

II. The Model

The communication system studied in this note is illustrated in Figure 1. For a given time-t channel input x_t the channel outputs observed at receivers 1 and 2 are

$$Y_{1,t} = x_t + Z_{1,t} \tag{1}$$

$$Y_{2,t} = x_t + Z_{2,t} \tag{2}$$

where the sequence of pairs of random variables $\{(Z_{1,t}, Z_{2,t})\}$ is drawn in an independent and identically distributed (iid) fashion for a normal distribution with zero mean and covariance matrix

$$\mathsf{K} = \begin{pmatrix} \sigma_1^2 & \rho_z\sigma_1\sigma_2 \\ \rho_z\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \tag{3}$$

for $\sigma_1, \sigma_2 > 0$ and $-1 \leq \rho_z \leq 1$.

The goal of the transmission is to convey message M_1 to Receiver 1 and an independent message M_2 to Receiver 2, where M_1 is uniformly distributed over the set

$\{1,\ldots,\lfloor 2^{nR_1}\rfloor\}$ and M_2 is uniformly distributed over the set $\{1,\ldots,\lfloor 2^{nR_2}\rfloor\}$, n being the block-length and R_1 and R_2 the respective rates of transmission.

Having access to perfect feedback the encoder can produce its time-t channel inputs not only as a function of the messages M_1 and M_2 but also based on the previous channel outputs. Thus a block-length n encoding scheme consists of n functions $f_t^{(n)}$, for $t = 1,\ldots,n$, such that

$$X_t = f_t^{(n)}\left(M_1, M_2, \mathbf{Y}_1^{t-1}, \mathbf{Y}_2^{t-1}\right)$$

where $\mathbf{Y}_1^{t-1} \triangleq (Y_{1,1},\ldots,Y_{1,t-1})$ and $\mathbf{Y}_2^{t-1} \triangleq (Y_{2,1},\ldots,Y_{2,t-1})$. We impose an average block-power constraint $P > 0$ on the sequence of channel inputs:

$$\frac{1}{n}\mathsf{E}\left[\sum_{t=1}^{n} X_t^2\right] \le P.$$

Of particular interest to this note is the *sum-rate capacity* $C(P,\sigma_1^2,\sigma_2^2,\rho_z)$, namely, the largest value of $R_1 + R_2$ for which reliable communication is feasible.

III. The Main Result

The main result of this note concerns the so-called "pre-log", defined as follows.

Definition 1: Letting the sum-rate capacity be given by $C(P,\sigma_1^2,\sigma_2^2,\rho_z)$, its corresponding pre-log is defined as

$$\kappa = \lim_{P\to\infty} \frac{C(P,\sigma_1^2,\sigma_2^2,\rho_z)}{\frac{1}{2}\log_2(1+P)}. \tag{4}$$

In the context of fading communication channels, the pre-log is often referred to as the *multiplexing gain.*

We start by noting that a pre-log of one is trivially attainable. Moreover, from the fact that for a broadcast channel without feedback, the capacity region only depends on the conditional marginals (see e.g. [9, p.599]), we have:

Lemma 1: For the two-user AWGN broadcast channel without feedback, the pre-log is 1 irrespective of the noise correlation ρ_z.

Also, by merely merging the two decoders into a single decoder, thus turning the problem into a point-to-point communication system, we find:

Lemma 2: For the two-user AWGN broadcast channel with full (causal) feedback, if the noise correlation satisfies $|\rho_z| < 1$, then the pre-log is 1.

The main result of this note is the following:

Theorem 1: For the two-user AWGN broadcast channel with full (causal) feedback, if the noise correlation is $\rho_z = -1$, then the pre-log is two.

The converse follows trivially by observing that with or without feedback, the following simple "single-user" upper bounds hold:

$$R_1 \le \frac{1}{2}\log_2\left(1+\frac{P}{\sigma_1^2}\right) \tag{5}$$

$$R_2 \le \frac{1}{2}\log_2\left(1+\frac{P}{\sigma_2^2}\right). \tag{6}$$

Thus, the pre-log cannot exceed two.

The somewhat more interesting part of the theorem concerns the achievability. This is proved via a strategy developed by Ozarow [5]. Details will be presented in a longer version of this note.

IV. Some Extensions

A. Limited Feedback

It can be shown that even if only one of the two channel outputs are fed back, a pre-log of two is attainable for the case of fully anti-correlated noises.

B. Interference Channel

An extension of our result concerns the two-user Gaussian *interference* channel. Recently, the pre-log of Gaussian interference networks attained a lot of attention as for networks with more than 2 transmitter/receiver pairs the pre-log is in general still unknown. However, in the case of only two transmitter/receiver pairs an analogous result to Lemma 1 is well known, i.e., that irrespective of the noise correlation the pre-log is 1. Our main result in this section is that when restricting attention to the case of unit-gains on all links, then it is possible to also extend the results in Lemma 2 and Theorem 1 to two-user Gaussian interference channels. Thus, for the two-user Gaussian interference channel with all unit-gains and full feedback, if the noise correlation satisfies $|\rho_z| < 1$, then the pre-log equals 1, and if the noise correlation is $\rho_z = -1$, then the pre-log equals 2. These extensions can also be obtained in the case of only limited feedback, i.e., when each transmitter observes feedback only from the corresponding receiver.

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