On the Sum Capacity of the Gaussian Multiple Access Channel with Feedback

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***Abstract*—This paper studies the sum capacity $C(P)$ of the N-sender additive white Gaussian noise (AWGN) multiple access channel (MAC), under equal power constraint P, when noiseless output feedback is available to all the N senders. The multi-letter characterization of the sum capacity, in terms of directed information, is considered as an optimization problem. The main result of this paper is to solve this problem when it is restricted to *Gaussian* causally conditional input distributions. Also, a dependence balance bound in terms of directed information is introduced, which for the case of memoryless channels is the same as the bound introduced by Kramer and Gastpar. This bound is used to capture the causality, however, since it is in general "non-convex" makes the problem technically hard. A general upper bound is obtained by forming the Lagrange dual problem and it is then shown that this upper bound coincides with the sum-rate achieved by Kramer's Fourier-MEC scheme. This result generalizes earlier work by Kramer and Gastpar on the achievable sum rate under a "per-symbol" power constraint to the one under the standard "block" power constraint.**

I. Introduction

Consider the communication problem between N senders and a receiver over a multiple access channel (MAC) with additive white Gaussian noise (AWGN) when channel output is noiselessly fed back to all the senders. Each sender $j \in \mathcal{S} := \{1, \ldots, N\}$ wishes to reliably transmit a message $M_j \in \mathcal{M}_j := \{1, \ldots, 2^{nR_j}\}$ to the receiver. At each time i, the output of the channel is

$$Y_i = \sum_{j=1}^{N} X_{ji} + Z_i, \tag{1}$$

where $\{Z_i\}$ is a discrete-time zero-mean white Gaussian noise process with unit average power $\big(\mathrm{E}(Z_i^2) = 1\big)$, and is independent of $M_1, \ldots, M_N$. The transmitted symbol X_{ji} for sender j at time i depends on the message M_j and the previous channel output sequence $Y^{i-1} := \{Y_1, Y_2, \ldots, Y^{i-1}\}$, and must satisfy the expected block power constraint

$$\sum_{i=1}^{n} \mathrm{E}(X_{ji}^2(m_j, Y^{i-1})) \leq nP_j, \quad m_j \in \mathcal{M}_j.$$

We define a $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ code with power constraints $P_1, \ldots, P_N$ as

1) N message sets $\mathcal{M}_1, \ldots, \mathcal{M}_N$, where $\mathcal{M}_j = \{1, 2, \ldots, 2^{nR_j}\}$,
2) a set of N encoders, where encoder j, at each time i, (stochastically) maps the pair (m_j, y^{i-1}) to a symbol x_{ji} such that X_{ji} satisfies
$$\sum_{i=1}^{n} \mathrm{E}(X_{ji}^2(m_j, Y^{i-1})) \leq nP_j, \quad m_j \in \mathcal{M}_j,$$
and
3) a decoder map which assigns indices $\hat{m}_j \in \mathcal{M}_j$, $j \in \{1, \ldots, N\}$, to each received sequence y^n.

Let $X(\mathcal{A}) := \{X_j : j \in \mathcal{A}\}$, $\mathcal{A} \subseteq \mathcal{S} = \{1, \ldots, N\}$, be an ordered subset of random variables $X_1, \ldots, X_N$. We assume throughout that $M(\mathcal{S}) := (M_1, \ldots, M_N)$ is a random variable uniformly distributed over $\mathcal{M}_1 \times \cdots \times \mathcal{M}_N$. The probability of error is defined as

$$P_e^{(n)} := \mathrm{P}\{\hat{M}(\mathcal{S}) \neq M(\mathcal{S})\}.$$

A rate tuple $(R_1, \ldots, R_N)$ is called achievable if there exists a sequence of $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ codes such that $P_e^{(n)} \to 0$ as $n \to \infty$. The capacity region $\mathscr{C}$ is defined as the closure of the set of achievable rates and the sum capacity C is defined as

$$C := \sup \left\{ R_{\text{sum}} = \sum_{k=1}^{N} R_j : \ (R_1, \ldots, R_N) \in \mathcal{C} \right\}.$$

The capacity region for the N-sender AWGN-MAC with feedback is not known except for the case $N = 2$. In this case the capacity region was found by Ozarow [8]. Ozarow's capacity achieving scheme is an extension of a scheme by Schalkwijk and Kailath [9] for the single-user AWGN channel. Kramer [4] further generalized Ozarow's scheme to $N \geq 2$ senders. However, it is not known whether Kramer's Fourier-MEC scheme is optimal in general.

In this paper we consider the sum capacity under equal block power constraint. From [5] we know that the sum-rate capacity is given by

$$C = \lim_{n \to \infty} C_n,$$

where for each nonnegative integer n:

$$C_n := \lim_{n \to \infty} \sup_{p(x_1^n, \ldots, x_N^n || y^{n-1})} \frac{1}{n} I(X_1^n, \ldots, X_N^n \to Y^n), \tag{2}$$

where $I(X_1^n,\ldots,X_N^n \to Y^n)$ denotes Massey's directed information [7], i.e.,

$$I(X_1^n,\ldots,X_N^n \to Y^n) := \sum_{i=1}^{n} I(X_1^i,\ldots,X_N^i;Y_i|Y^{i-1}), \tag{3}$$

and where the supremum in (2) is over all causally conditional distributions of the form

$$\begin{aligned} p(x_1^n,\ldots,x_N^n||y^{n-1}) \\ &:= p(x_{1,i},\ldots,x_{N,i}|x_1^{i-1},\ldots,x_N^{i-1},y^{i-1}) \\ &= \prod_{j=1}^{N} p(x_{j,i}|x_j^{i-1},y^{i-1}). \end{aligned} \tag{4}$$

For discrete memoryless channels, the directed information in (3) reduces to

$$I(X_1^n,\ldots,X_N^n \to Y^n)\frac{1}{n}\sum_{i=1}^{n} I(X_{1i},\ldots,X_{Ni};Y_i|Y^{i-1}).$$

Thus, the sum-rate capacity of the N-user AWGN MAC with feedback and equal power constraints P is defined as follows:

$$C = \lim_{n\to\infty} \sup \frac{1}{n}\sum_{i=1}^{n} I(X_{1i},\ldots,X_{Ni};Y_i|Y^{i-1}), \tag{5}$$

where the supremum is over distributions of the form given in (4) such that for every nonnegative integer n:

$$\sum_{i=1}^{n} \mathrm{E}(X_{ji}^2) \le nP, \quad j\in\{1,\ldots,N\}. \tag{6}$$

We evaluate this multi-letter sum capacity expression for causally conditional distribution of the form

$$X_{ji} = V_{ji} + L_{ji}(Y^{i-1}), \tag{7}$$

where V_{ji} are zero-mean Gaussian random variables such that V_j^n is independent of Z^n and $V_{j'}^n$ for all $j \neq j'$, and where L_{ji}'s are linear functions. This class of distributions is the multi-user analogous of the distributions considered by Cover and Pombra [3]. We show that when taking the supremum in (5) only over causally conditional distributions of the form (7), the result meets with the sum rate achievable by Kramer's Fourier-MEC scheme. This generalizes the recent work by Kramer and Gastpar [6] who considered the more restrictive symmetric "per-symbol" power constraint:

$$\mathrm{E}(X_{ji}^2) \le P, \quad \forall i \in \{1,\ldots,n\},\ j\in\{1,\ldots,N\}.$$

In the following theorem, we state our main result.

Theorem 1. *Let a nonnegative power P be given, and let*

$$C^G(P) := \lim_{n\to\infty} \sup \frac{1}{n} I(X_{1i},\ldots,X_{Ni};Y_i|Y^{i-1})$$

where

$$X_{ji} = V_{ji} + L_{ji}(Y^{i-1}),$$
$$\sum_{i=1}^{n} \mathrm{E}(X_{ji}^2) \le nP, \forall n, j,$$

and V_{ji} are zero-mean Gaussian random variables such that V_j^n is independent of Z^n and $V_{j'}^n$ for all $j \neq j'$, and L_{ji}'s are linear functions. Then

$$C^G(P) = C_1(P,\phi(P)) = C_2(P,\phi(P)),$$

where for each real number $\phi \in [1,N]$:

$$C_1(P,\phi) := \frac{1}{2}\log(1+NP\phi)$$
$$C_2(P,\phi) := \frac{N}{2(N-1)}\log(1+(N-\phi)P\phi),$$

and where $\phi(P)\in[1,N]$ is defined as the unique solution of $C_1(P,\phi) = C_2(P,\phi)$.

Proof: The lower bound, $C^G(P) \ge C_1(P,\phi(P))$ follows from the distribution corresponding to Kramer's Fourier-MEC scheme [4] with proper initialization. The upper bound, $C^G(P) \le C_1(P,\phi(P))$, is proved in Section II.

II. Proof of the Upper Bound of Theorem 1

In this section we prove the upper bound $C^G(P) \le C_1(P,\phi(P))$. To this end, we show that for every nonnegative integer n:

$$C_n^G(P) \le C_1(P,\phi(P))$$

where

$$C_n^G(P) := \sup \frac{1}{n} I(X_{1i},\ldots,X_{Ni};Y_i|Y^{i-1}),$$

with the supremum taken over all distributions satisfying (4), (6), (7).

First, we prove a dependence balance bound in terms of directed informations for any distribution of the form (4).

Lemma 1. *For every causally conditional distribution on the input sequences $\{X_{1,i}\}_{i=1}^n,\ldots,\{X_{N,i}\}_{i=1}^n$ satisfying (4) we have*

$$I(X^n(\mathcal{S}) \to Y^n) \le \frac{1}{N-1}\sum_{j=1}^{N} I(X^n(\mathcal{S}\backslash\{j\} \to Y^n||X_j^n). \tag{8}$$

Proof: See Section III.

Moreover, since the channel is memoryless:

$$I(X^n(\mathcal{S}) \to Y^n) = \sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) \tag{9}$$

and

$$\begin{aligned} I(X^n(\mathcal{S}\backslash\{j\}) \to Y^n||X_j^n) \\ \le \sum_{i=1^n} I(X_i(\mathcal{S}\backslash\{j\});Y_i|Y^{i-1},X_{ji}), \end{aligned} \tag{10}$$

where the last inequality comes from the fact that conditioning reduces entropy. Combining (8), (9), (10), we obtain the dependence balance bound as in [6]:

$$\begin{aligned} \sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) \\ \le \frac{1}{N-1}\sum_{j=1}^{N}\sum_{i=1}^{n} I(X_i(\mathcal{S}\backslash\{j\});Y_i|Y^{i-1},X_{ji}), \end{aligned} \tag{11}$$

which we refer to as the dependence balance bound (DBB) in the proof.

Since, for any distribution of the form (4) the bound given in (11) holds, C_n is equal to the solution of the following optimization problem

$$\begin{aligned} \text{maximize} \quad & \frac{1}{n}\sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) \qquad (12) \\ \text{subject to} \quad & X_{ji} = V_{ji} + L_{ji}(Y^{i-1}) \\ & \sum_{i=1}^{n} \mathbb{E}(X_{ji}^2) \le nP_j,\ j\in\{1,\dots,N\} \\ & \quad \text{DBB}(11) \end{aligned}$$

Since $V^n(\mathcal{S})$ are Gaussian and L_{ji}'s are linear, $(X^n(\mathcal{S}),Y^n)$ are jointly Gaussian and we can replace the mutual information terms by functions of the covariance matrix $K_i := K_{X_i(\mathcal{S})}$ as follows.

$$\begin{aligned} C_1(K_i) &:= \frac{1}{2}\log\Big(1+\sum_{j,j'}K_i(j,j')\Big) \\ &= I(X_i(\mathcal{S});Y_i|Y^{i-1}) \end{aligned} \qquad (13)$$

and

$$\begin{aligned} &C_2(K_i) := \\ &\frac{1}{2(N-1)}\sum_{j=1}^{N}\log\left[1+\sum_{j',j''}K_i(j',j'')-\frac{\Big(\sum_{j'}K_i(j,j')\Big)^2}{K_i(j,j)}\right] \\ &= \frac{1}{N-1}\sum_{j=1}^{N} I(X_i(\mathcal{S}\backslash\{j\});Y_i|Y^{i-1},X_{j,i}). \end{aligned} \qquad (14)$$

Substituting (13) and (14) in (12) and removing the functional relationship $X_{ji} = V_{ji} + L_{ji}(Y^{i-1})$ from the optimization problem we have

$$C_n \le p^*,$$

where p^* is the solution to the following problem.

$$\begin{aligned} P:\ \text{maximize} \quad & \tfrac{1}{n}\textstyle\sum_{i=1}^{n} C_1(K_i) \\ \text{subject to} \quad & K_i \succeq 0, \quad i\in\{1,\dots,n\}, \\ & \textstyle\sum_{i=1}^{n} K_i(j,j) \le nP, \quad j\in\{1,\dots,k\}, \\ & \textstyle\sum_{i=1}^{n} C_1(K_i) - C_2(K_i) \le 0. \end{aligned}$$

We prove the following lemma for problem P.

Lemma 2. *The solution p^* to the optimization problem P satisfies*

$$p^* \le C_1(P,\phi(P)) = C_2(P,\phi(P)),$$

where $\phi(P)\in[1,N]$ is the unique solution to

$$C_1(P,\phi(P)) = C_2(P,\phi(P)).$$

Proof:

We form the dual problem using equal Lagrange multipliers $\lambda_j = \lambda \ge 0,\ j\in\{1,\dots,N\}$ for the N power constraints and $\gamma \ge 0$ for the DBB. Then, by weak duality, we have $p^* \le U(\lambda,\gamma)$, where

$$U(\lambda,\gamma) = \max_{K\succeq 0}(1-\gamma)C_1(K) + \gamma C_2(K) + \lambda\sum_{j=1}^{N}(P-K_{jj}). \qquad (15)$$

Next, we show that there exists an optimal matrix K of the following form.

$$K = x\cdot\begin{pmatrix} 1 & \rho & \rho & \dots & \rho \\ \rho & 1 & \rho & \dots & \rho \\ \rho & \rho & 1 & \dots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \dots & 1 \end{pmatrix}. \qquad (16)$$

This form of K was also considered in [10], [4]. However, in those cases the objective function was concave. Here, we show that although (15) is not in general concave in K it is still sufficient to look at matrices of the form (16).

To show the sufficiency of matrices of the form in (16), we fix an arbitrary matrix K' (not necessarily as in (16)). Then, we construct a matrix $\bar{K}$ of the form in (16) and such that

$$\begin{aligned} &(1-\gamma)C_1(\bar{K}) + \gamma C_2(\bar{K}) + \lambda\sum_{j=1}^{N}(P-\bar{K}_{jj}) \\ &\quad \ge (1-\gamma)C_1(K') + \gamma C_2(K') + \lambda\sum_{j=1}^{N}(P-K'_{jj}). \end{aligned} \qquad (17)$$

To this end, we consider for each permutation π on the indices $\{1,\dots,N\}$ the matrix that is obtained by permuting the rows and the columns of K' according to the permutation π, and denote this matrix by $\pi(K')$. By symmetry, it is easily seen that for each permutation π:

$$\begin{aligned} &(1-\gamma)C_1(\pi(K')) + \gamma C_2(\pi(K')) + \lambda\sum_{j=1}^{N}(P-\pi(K')_{jj}) \\ &= (1-\gamma)C_1(K') + \gamma C_2(K') + \lambda\sum_{j=1}^{N}(P-K'_{jj}). \end{aligned}$$

Now consider all $N!$ possible permutations on the set $\{1,\dots,N\}$ and denote them by $\pi_1,\dots,\pi_{N!}$. Then, consider the arithmetic average over the corresponding permuted matrices:

$$\bar{K} := \frac{1}{N!}\sum_{\ell=1}^{N!}\pi_\ell(K').$$

Since $C_1(K)$ and $\sum_{j=1}^{N}K_{jj}$ are functions only of the sum of the entries of K and since $C_2(K)$ is concave in K (see Lemma 5), it follows that this arithmetic average satisfies (17). Since the matrix $\bar{K}$ is also of the form in (16) this establishes the desired sufficiency of the matrices in (16).

Thus, we can continue our analysis with matrices of the form in (16), and define

$$C_1(K) = C_1(x,\phi) := \frac{1}{2}\log(1+Nx\phi)$$
$$C_2(K) = C_2(x,\phi) := \frac{N}{2(N-1)}\log(1+(N-\phi)x\phi),$$

where $\phi := 1+(N-1)\rho$. We know that $\rho > -1/(N-1)$, so $\phi \geq 0$ and we have

$$U(\lambda,\gamma) = \max_x \max_{0\leq\phi\leq N} g(\gamma,x,\phi) + \lambda N(P-x),$$

where

$$g(\gamma,x,\phi) := (1-\gamma)C_1(x,\phi) + \gamma C_2(x,\phi).$$

By Lemma 6, for fixed $x,\gamma \geq 0$, $g(\gamma,x,\phi)$ is concave in ϕ and the maximum happens at $\phi^*(\gamma,x) > 0$ such that

$$\frac{(1-\gamma)(N-1)}{1+\ Nx\phi^*} = \frac{\gamma(2\phi^*-N)}{1+x\phi^*(N-\phi^*)},$$

which is equivalent to the condition that the first derivative is zero. Hence,

$$U(\lambda,\gamma) = \max_x g(x,\phi^*(\gamma,x)) + \lambda N(P-x),$$

Recall that $U(\lambda,\gamma)$ is an upper bound for R_{sum}, for any $\gamma,\lambda \geq 0$. Hence, for any $\gamma \geq 0$ we have

$$\begin{aligned} R_{\text{sum}} &\leq \min_{\lambda\geq 0} U(\lambda,\gamma) \\ &= \min_{\lambda\geq 0}\max_x g(x,\phi^*(\gamma,x)) + \lambda N(P-x). \end{aligned}$$

To evaluate the last expression we use the following lemma.

Lemma 3. *Let $\gamma,x \geq 0$ and $\phi^*(\gamma,x) > 0$ be the solution to*

$$\frac{(1-\gamma)(N-1)}{1+\ Nx\phi} = \frac{\gamma(2\phi-N)}{1+x\phi(N-\phi)}. \tag{18}$$

Then,

$$g(x,\phi^*(\gamma,x)) = (1-\gamma)C_1(x,\phi^*(\gamma,x)) + \gamma C_2(x,\phi^*(\gamma,x)),$$

is concave in x.

Proof: See Section III.

Since $g(x,\phi^*(\gamma,x))$ is concave in x and is unbounded as $x\to\infty$ we have

$$\min_{\lambda\geq 0}\max_x g(x,\phi^*(\gamma,x)) + \lambda N(P-x) = g(P,\phi^*(\gamma,P)).$$

This is true since $\min_{\lambda\geq 0}\max_x g(x,\phi^*(\gamma,x)) + \lambda N(P-x)$ is the dual problem of

$$\begin{aligned} &\text{maximize} \quad g(x,\phi^*(\gamma,x)) \\ &\text{subject to} \quad x \leq P, \end{aligned}$$

which is a convex optimization (concave maximization) [1] for which Slater's condition is satisfied, hence, strong duality holds. Also, since the objective is unbounded for large x the optimum of the dual problem can not happen at $\lambda = 0$ and by complementary slackness condition the optimum happens at $x = P$. Therefore, $\min_{\lambda\geq 0} U(\lambda,\gamma) = g(P,\phi^*(\gamma,P))$ and

$$R_{\text{sum}} \leq g(P,\phi^*(\gamma,P)), \quad \forall\gamma \geq 0.$$

It remains to find the proper γ. We use the following lemma.

Lemma 4. *Let*

$$C_1(x,\phi) = \frac{1}{2}\log(\ 1+\ \ Nx\phi\ )$$
$$C_2(x,\phi) = \frac{N}{2(N-1)}\log(\ 1+\ (N-\phi)x\phi\ ),$$

Then for every $x > 0$, there exists a unique $\phi(x)$ satisfying $C_1(x,\phi) = C_2(x,\phi)$. Moreover,

$$1 + \frac{(2\phi(x)-N)(1+Nx\phi(x))}{(N-1)(1+x\phi(x)(N-\phi(x)))} > 0. \tag{19}$$

Proof: See Section III.

Let $1 \leq \phi(P) \leq N$ be the unique solution for $C_1(P,\phi) = C_2(P,\phi)$. Then, by (19) it is always possible to pick $\gamma^* \geq 0$ such that $\gamma^*, P, \phi(P)$ satisfy (18) and for $\gamma = \gamma^*$ we have $\phi^*(\gamma^*,P) = \phi(P)$. Then, we have

$$\begin{aligned} R_{\text{sum}} &\leq g(P,\phi^*(\gamma^*,P)) \\ &= (1-\gamma^*)C_1(P,\phi^*(\gamma^*,P)) + \gamma^* C_2(P,\phi^*(\gamma^*,P)) \\ &= (1-\gamma^*)C_1(P,\phi(P)) + \gamma^* C_2(P,\phi(P)) \quad (20) \\ &= C_1(P,\phi(P)), \quad (21) \end{aligned}$$

where (20) and (21) follow from $\phi^*(\gamma^*,P) = \phi(P)$ and $C_1(P,\phi) = C_2(P,\phi)$, respectively. This completes the proof of the upper bound.

III. Proof of the Lemmas

Lemma 1. *For every causally conditional distribution on the input sequences $\{X_{1,i}\}_{i=1}^n,\ldots,\{X_{N,i}\}_{i=1}^n$ satisfying*

$$p(x_1^n,\ldots,x_N^n||y^{n-1}) = \prod_{i=1}^n\prod_{j=1}^N p(x_{ji}|x_j^{i-1},y^{i-1}) \tag{22}$$

we have

$$I(X^n(\mathcal{S})\to Y^n) \leq \frac{1}{N-1}\sum_{j=1}^N I(X^n(\mathcal{S}\backslash\{j\})\to Y^n|X_j^n) \tag{23}$$

Proof: Consider

$$I(X^n(\mathcal{S}) \to Y^n) - \sum_{j=1}^{N} I(X_j^n \to Y^n)$$
$$= \Big[\sum_{i=1}^{n} h(X_i(\mathcal{S})|X^{i-1}(\mathcal{S}), Y^{i-1}) + h(X^{i-1}(\mathcal{S})|Y^{i-1}) - h(X^i(\mathcal{S})|Y^i)\Big] - \Big[\sum_{j=1}^{N}\sum_{i=1}^{n} h(X_{ji}|X_j^{i-1}, Y^{i-1}) + h(X_j^{i-1}|Y^{i-1}) - h(X_j^i|Y^i)\Big]$$
$$= \left[\sum_{i=1}^{n} h(X_i(\mathcal{S})|X^{i-1}(\mathcal{S}), Y^{i-1}) - \sum_{j=1}^{N} h(X_{ji}|X_j^{i-1}, Y^{i-1})\right] + \sum_{j=1}^{N} h(X_j^n|Y^n) - h(X^n(\mathcal{S})|Y^n) \tag{24}$$
$$= \sum_{j=1}^{N} h(X_j^n|Y^n) - h(X^n(\mathcal{S})|Y^n), \tag{25}$$
$$\geq 0 \tag{26}$$

where the last inequality holds since conditioning reduces entropy and (24) follows form the fact that for any distribution of the form (22) the bracket term in (24) is zero. Hence, we have

$$I(X^n(\mathcal{S}) \to Y^n) - \sum_{j=1}^{N} I(X_j^n \to Y^n) \geq 0$$

Adding $(N-1)I(X(\mathcal{S}) \to Y^n)$ to both sides and rearranging terms using the chain rule of directed information we have

$$I(X^n(\mathcal{S}) \to Y^n) \leq \frac{1}{N-1} I(X^n(\mathcal{S}\backslash\{j\}) \to Y^n || X_j^n). \tag{27}$$

■

Lemma 3. *Let $\gamma, x \geq 0$ and $\phi^*(x) > 0$ be the solution to*

$$\frac{(1-\gamma)(N-1)}{1+\ Nx\phi} = \frac{\gamma(2\phi - N)}{1 + x\phi(N-\phi)}. \tag{28}$$

Then,

$$g(x, \phi^*(x)) = (1-\gamma)C_1(x, \phi^*(x)) + \gamma C_2(x, \phi^*(x)),$$

is concave in x.

Proof: If $0 \leq \gamma \leq 1$ the concavity is immediate since we know $C_1(K)$ and $C_2(K)$ are concave in K. For $\gamma > 1$ let

$$g(x, \phi) := (1-\gamma)C_1(x, \phi) + \gamma C_2(x, \phi).$$

where

$$C_1(x, \phi) = \log(1 + Nx\phi)$$
$$C_2(x, \phi) = \frac{N}{N-1}\log(1 + x\phi(N-\phi)).$$

Consider

$$\max_x \max_\phi g(x, \phi)$$

For a fixed x the objective is concave in ϕ. Setting the first derivative with respect to ϕ we have

$$\frac{(1-\gamma)(N-1)}{1+\ Nx\phi} = \frac{\gamma(2\phi - N)}{1 + x\phi(N-\phi)}. \tag{29}$$

or

$$a\phi^2 + b\phi + c = 0, \tag{30}$$

where

$$a = (N + \gamma - 1 + \gamma N)x$$
$$b = -N(N + \gamma - 1)x + 2\gamma$$
$$c = -(N + \gamma - 1)$$

Since $ac < 0$, there is only one positive solution

$$0 < \phi^*(x) = \frac{-b + \sqrt{b^2 - 4ac}}{2a} < \frac{N}{2}, \tag{31}$$

where the last inequality follows form (29) and the fact that $\gamma > 1$.

Monotonicity: For $\gamma > 1$ and $x \geq 0$, $\phi^*(x)$ is increasing in x and

$$\phi^*(x) \geq \frac{N + \gamma - 1}{2\gamma}.$$

From (29) we have $\phi^*(0) = \frac{N+\gamma-1}{2\gamma}$. Taking the derivative of (30) with respect to x we have

$$\frac{d\phi^*}{dx} = \frac{-\phi^*(a'\phi^* + b')}{2a\phi^* + b}, \tag{32}$$

where $a' = N + \gamma - 1 + \gamma N,\ b' = -N(N + \gamma - 1)$ are derivatives of a, b with respect to x. From (31) we know $2a\phi^* + b > 0$. We need to show $a'\phi^* + b' \leq 0$. Note that

$$\frac{d\phi^*}{dx} = 0 \iff \phi^* = -b'/a',$$

and $\phi^*(\infty) = -\frac{b'}{a'}$. For $\gamma > 1$, we can show $\phi^*(0) < -b'/a'$. Therefore,

$$\frac{d\phi^*}{dx}\Big|_{x=0} > 0.$$

Hence, we can conclude $\phi^*(x) \leq -\frac{b'}{a'}$ for all $x \geq 0$. The reason is that if we start with $\phi^* < -b'/a'$ and ϕ^* wants to become larger than $-b'/a'$, the first derivative has to be positive at $\phi^* = -b'/a'$, which can not happen.

Concavity: Let $f(x) := g(x, \phi^*(x))$. For $x \geq 0$, $f(x)$ is concave and

$$\frac{df(x)}{dx} = \frac{N(\gamma - 1)(\phi^*(x))^2}{(1 + Nx\phi^*(x))(N - 2\phi^*(x))} \geq 0.$$

We have

$$\frac{df(x)}{dx} = \frac{\partial g(x,\phi)}{\partial x} + \frac{\partial g(x,\phi)}{\partial \phi}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)}$$

From the definition of $\phi^*(x)$ we know

$$\frac{\partial g(x,\phi)}{\partial \phi}\Big|_{x,\phi^*(x)} = 0.$$

Hence,

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{\partial g(x,\phi)}{\partial x}\Big|_{x,\phi^*(x)}\\ &= (1-\gamma)\frac{N\phi}{1+Nx\phi}+\gamma\cdot\frac{N}{N-1}\cdot\frac{\phi(N-\phi)}{1+x\phi(N-\phi)}\Big|_{x,\phi^*(x)}\\ &= \frac{N\phi}{1+Nx\phi}\left(1-\gamma+\gamma\cdot\frac{N-\phi}{N-1}\cdot\frac{1+Nx\phi}{1+x\phi(N-\phi)}\right)\Big|_{x,\phi^*(x)}\\ &= \frac{N(\gamma-1)(\phi^*(x))^2}{(1+Nx\phi^*(x))(N-2\phi^*(x))} \qquad (33)\\ &\geq 0,\end{aligned}$$

where (33) follows from the fact that $\phi^*(x)$ satisfies (29) and the last inequality holds since $\phi^*(x) < \frac{N}{2}$.

To prove the concavity of $f(x)$ we show that

$$\frac{d^2 f(x)}{dx^2} < 0.$$

From (33) we have

$$\frac{df(x)}{dx} = N(\gamma-1)\tilde{f}(x),$$

where

$$\tilde{f}(x) := h(x,\phi^*(x))$$
$$h(x,\phi) := \frac{\phi^2}{(1+Nx\phi)(N-2\phi)}.$$

Therefore it is enough to show that

$$\frac{\tilde{f}(x)}{dx} < 0.$$

Consider

$$\begin{aligned}\frac{d\tilde{f}(x)}{dx} &= \frac{\partial h(x,\phi)}{\partial x}+\frac{\partial h(x,\phi)}{\partial \phi}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)}\\ &= \frac{-N\phi^3}{(1+Nx\phi)^2(N-2\phi)}+\frac{\phi(N^2x\phi+2(N-\phi))}{(1+Nx\phi)^2(N-2\phi)^2}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)}\\ &= \phi\cdot\frac{\frac{d\phi}{dx}\Big(N^2x\phi+2(N-\phi)\Big)-N\phi^2(N-2\phi)}{(1+Nx\phi)^2(N-2\phi)^2}\Big|_{x,\phi^*(x)}\end{aligned}$$

Since $\phi > 0$ and the denominator is also positive we need to show

$$\frac{d\phi^*(x)}{dx} < \frac{N\phi^2(N-2\phi)}{N^2x\phi+2(N-\phi)}\Bigg|_{x,\phi^*(x)} \qquad (34)$$

For the rest of the proof, with abuse of notation, we alternatively use ϕ for $\phi^*(x)$, the positive solution of (29). From (32) we have

$$\begin{aligned}\frac{d\phi^*(x)}{dx} &= \frac{-\phi^2(a'\phi+b')}{2a\phi^2+b\phi}\\ &= \frac{-\phi^2(a'\phi+b')}{a\phi^2-c},\end{aligned}$$

Defining

$$\alpha := \frac{N+\gamma-1}{N},$$

we have $a = N(\alpha+\gamma)x,\ b = -N^2\alpha x+2\gamma,\ c = -\alpha N,\ a' = N(\alpha+\gamma),\ b' = -N^2\alpha$, and

$$\begin{aligned}\frac{d\phi^*(x)}{dx} &= \frac{-\phi^2(a'\phi+b')}{a\phi^2-c},\\ &= \frac{N\phi^2(N\alpha-(\alpha+\gamma)\phi)}{(\alpha+\gamma)Nx\phi^2+\alpha N}\\ &= \frac{N\phi^2(N-\beta\phi)}{\beta Nx\phi^2+N}, \qquad (35)\end{aligned}$$

where

$$\beta := 1+\frac{\gamma}{\alpha}.$$

It is not hard to see that $\beta \in (2, N+1)$ for $\gamma > 1$. Plugging (35), (34) becomes equivalent to

$$\begin{aligned} & \frac{N-\beta\phi}{N-2\phi} < \frac{\beta Nx\phi^2+N}{N^2x\phi+2(N-\phi)}\\ \Longleftrightarrow\quad & \frac{N-\beta\phi}{N-2\phi} < \frac{N-\beta\phi+\beta\phi(Nx\phi+1)}{N-2\phi+N(Nx\phi+1)}\\ \Longleftrightarrow\quad & \frac{N-\beta\phi}{N-2\phi} < \frac{\beta\phi}{N}, \qquad (36)\end{aligned}$$

where (36) follows from the fact that for $b, d > 0$,

$$\frac{a}{b} < \frac{c}{d} \quad \Leftrightarrow \quad \frac{a}{b} < \frac{a+c}{b+d}. \qquad (37)$$

Noting that $\beta > 2$ and using (37) with $c = d = 2\phi$, we can see that to prove (36) it is sufficient to show

$$\begin{aligned} & \frac{N-(\beta-2)\phi}{N} \leq \frac{\beta\phi}{N}\\ \Longleftrightarrow\quad & \frac{N+\gamma-1}{2\gamma} \leq \phi,\end{aligned}$$

which holds by the first part. ■

Lemma 4. *Let*

$$C_1(x,\phi) = \frac{1}{2}\log(\ 1+\ Nx\phi\ )$$
$$C_2(x,\phi) = \frac{N}{2(N-1)}\log(\ 1+\ (N-\phi)x\phi\ ),$$

Then for every $x > 0$, there exists a unique $\phi(x)$ satisfying $C_1(x,\phi) = C_2(x,\phi)$. Moreover,

$$1+\frac{(2\phi(x)-N)(1+Nx\phi(x))}{(N-1)(1+x\phi(x)(N-\phi(x)))} > 0. \qquad (38)$$

Proof: Let $f(\phi) = C_2(x,\phi)-C_1(x,\phi)$. We prove there exists a unique solution by showing $f(1) \geq 0, f(N) < 0$, and $f'(\phi) < 0$ for $1 \leq \phi \leq N$. The fact that $f(N) < 0$ is immediate. Condition $f(1) \geq 0$ is equivalent to

$$\Big(1+x(N-1)\Big)^N \geq \Big(1+Nx\Big)^{N-1}.$$

For the above condition to hold it is sufficient that

$$\binom{n}{N}(N-1)^k \geq \binom{N-1}{k}N^k, \qquad (39)$$

which is true since $(1-1/N)^k \geq 1-k/N$ for $N>1$. Finally, we need to show $f'(\phi)<0$ which is equivalent to

$$\frac{N-2\phi}{1+x\phi(N-\phi)} - \frac{N-1}{1+Nx\phi} < 0. \tag{40}$$

Rearranging the terms we have

$$1+Nx\phi - (2\phi + x\phi^2 + Nx\phi^2) < 0,$$

which holds for any $\phi \geq 1$. This shows that there exists a unique solution. Also, note that (40) is same as condition (38) which we wanted to prove. ∎

Lemma 5. *Let $X = X_1,\ldots,X_N \sim N(0,K)$, $X(T) := \{X_j : j \in T\}$. Also, let $A_{m\times|T|}$ be an arbitrary matrix and $I = \begin{pmatrix} I_{m\times m} & 0 \\ 0 & 0 \end{pmatrix}$. Then $\log\left(\frac{|I+K_{AX(T),X(T^c)}|}{|K_{X(T^c)}|}\right)$ is concave in covariance matrix K for all $T \subseteq \{1,\ldots,N\}$.*

Proof: The proof is similar to Bergstrøm's theorem [2, Theorem, 17.10.1]. Let $Y = AX+Z$, where X_θ, $\mathrm{P}(\theta=1)=\lambda = 1-\mathrm{P}(\theta=2)$, $X_1 \sim N(0,K_1), X_2 \sim N(0,K_2)$, $Z=\{Z_1,\ldots,Z_m\}$ are i.i.d. $N(0,1)$. Assume Z, X_1, X_2, θ are independent. The covariance matrix of X is given by $K = \lambda K_1 + (1-\lambda)K_2$. Consider

$$\begin{aligned}
&\frac{\lambda}{2}\log\left(\frac{|I+K_{AX_1(T),X_1(T^c)}|}{|K_{X_1(T^c)}|}\right) \\
&\qquad + \frac{(1-\lambda)}{2}\log\left(\frac{|I+K_{AX_2(T),X_2(T^c)}|}{|K_{X_2(T^c)}|}\right) \\
&= \frac{\lambda}{2}\log\left(\frac{|K_{Y_1(T),X_1(T^c)}|}{|K_{X_1(T^c)}|}\right) + \frac{(1-\lambda)}{2}\log\left(\frac{|K_{Y_2(T),X_2(T^c)}|}{|K_{X_2(T^c)}|}\right) \\
&= \lambda(h(Y_1(T)|X_1(T^c)) - h(Z) \\
&\qquad + (1-\lambda)(h(Y_2(T)|X_2(T^c)) - h(Z)) \qquad (41) \\
&= h(Y_\theta(T)|X_\theta(T^c),\theta) - h(Z) \\
&\leq h(Y(T)|X(T^c)) - h(Z) \\
&= \frac{1}{2}\log\frac{|K_{Y(T),X(T^c)}|}{|K_{X(T^c)}|} \qquad (42) \\
&= \frac{1}{2}\log\frac{|I+K_{AX(T),X(T^c)}|}{|K_{X(T^c)}|}.
\end{aligned}$$

where (41) and (42) come form the fact that $Y(T)$ is a linear combination of $X(T)$ and Z, hence is jointly Gaussian with $X(T^c)$. ∎

Lemma 6. *Let*

$$C_1(K) = \frac{1}{2}\log\Big(1+\sum_{m,l}K_{ml}\Big)$$

$$C_2(K) = \frac{1}{2(N-1)}\sum_{j=1}^{N}\log\Big(1+\sum_{m,l}K_{ml} - \frac{(\sum_l K_{ml})^2}{K_{jj}}\Big)$$

and the elements on the diagonal of K be fixed. Then

$$f(\gamma,K) := (1-\gamma)C_1(K) + \gamma C_2(K)$$

is concave in K for any fixed $\gamma \geq 0$.

Proof: Let $X = X_1,\ldots,X_N \sim N(0,K)$ and $Y = \sum_j X_j + Z$, where Z is independent of $X_1,\ldots,X_N$. Then

$$\begin{aligned}
&f(\gamma,K) = (1-\gamma)C_1(K) + \gamma C_2(K) \\
&\qquad (1-\gamma)h(Y) + \frac{\gamma}{N-1}\sum_{j=1}^{N}h(Y|X_j) \\
&\qquad = (1-\gamma)h(Y) + \frac{\gamma}{N-1}\sum_{j=1}^{N}h(Y)+h(X_j|Y)-h(X_j) \\
&\qquad = h(Y)(1+\frac{\gamma}{N-1}) + \frac{\gamma}{N-1}\sum_{j=1}^{N}h(X_j|Y)-h(X_j).
\end{aligned}$$

We know that $h(Y)$ and $h(X_j|Y)$ are concave in K. If the diagonal of K are fixed then $h(X_j) = \frac{1}{2}\log(2\pi e K_{jj})$ is also fixed and as long as $\gamma \geq 0$, $f(\gamma,K)$ is concave in K. ∎

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