Linear Sum Capacity for Gaussian Multiple Access Channels with Feedback

Ehsan Ardestanizadeh, Michèle A. Wigger, Young-Han Kim, and Tara Javidi

***Abstract*—The capacity region of the N-sender additive white Gaussian noise (AWGN) multiple access channel (MAC) with feedback is not known in general, despite significant contributions by Cover, Leung, Ozarow, Thomas, Pombra, Ordentlich, Kramer, and Gastpar. This paper studies the class of *generalized linear feedback codes* that includes (nonlinear) nonfeedback codes at one extreme and the linear feedback codes by Schalkwijk and Kailath, Ozarow, and Kramer at the other extreme. The *linear sum capacity* $C_{\mathsf{L}}(N, P)$, the maximum sum rate achieved by generalized linear feedback codes, is characterized under symmetric block power constraints P for all the senders. In particular, it is shown that Kramer's linear code achieves this linear sum capacity. The proof involves the dependence balance condition introduced by Hekstra and Willems and extended by Kramer and Gastpar. This condition is not convex in general, and the corresponding nonconvex optimization problem is carefully analyzed via Lagrange dual formulation. Based on the properties of the *conditional maximal correlation*—an extension of the Hirschfeld–Gebelein–Renyi maximal correlation—it is further conjectured that Kramer's linear code achieves not only the *linear* sum capacity, but also the (general) sum capacity, i.e., the maximum sum rate achieved by *arbitrary* feedback codes.**

***Index Terms*—Gaussian MAC with feedback, linear feedback codes, sum capacity.**

I. Introduction

Feedback from the receivers to the senders can improve the performance of the communication systems in various ways. For example, as first shown by Gaarder and Wolf [1], feedback can enlarge the capacity region of memoryless multiple access channels by enabling the distributed senders to establish cooperation via coherent transmissions.

In this paper, we study the sum capacity of the additive white Gaussian noise multiple access channel (AWGN-MAC) with feedback depicted in Figure 1. For $N = 2$ senders, Ozarow [2] established the capacity region which—unlike for the point-to-point channel—turns out to be strictly larger than the one without feedback. The capacity-achieving code proposed by Ozarow is an extension of the Schalkwijk–Kailath code [3], [4] for point-to-point AWGN channels.

For $N \geq 3$, the capacity region is not known in general. On one hand, Thomas [5] proved that feedback can at most double the sum capacity, and later Ordentlich [6] showed that the same bound holds for the entire capacity region even when the noise sequence is not white (cf. Pombra and Cover [7]). On the other hand, Kramer [8] extended Ozarow's linear code to $N \geq 3$ users. Kramer's linear code achieves the sum capacity under the symmetric block power constraints P for all the senders, provided that the power P exceeds a certain threshold (57) that depends on the number of senders.

In this paper, we focus on the class of *generalized linear feedback codes* (or *linear codes* in short), whereby the feedback signals are incorporated linearly into the transmitted signals (see Definition 1 in Section II for the precise definition). This class of generalized linear feedback codes includes the linear feedback codes by Schalkwijk and Kailath [3], Ozarow [2], and Kramer [8] as well as arbitrary (nonlinear) nonfeedback codes.

This paper characterizes the linear sum capacity $C_{\mathsf{L}}(N, P)$, which is the maximum sum rate achieved by generalized linear feedback codes under symmetric block power constraints P. The main contribution is the proof of the converse. We first prove an upper bound on $C_{\mathsf{L}}(N, P)$, which is a multi-letter optimization problem over Gaussian distributions (cf. Cover and Pombra [9]). Next, we derive an equivalent optimization problem over the set of positive semidefinite (covariance) matrices by considering a dependence balance condition, introduced by Hekstra and Willems [10] and extended by Kramer and Gastpar [11]. Lastly, we carefully analyze this nonconvex optimization problem via Lagrange dual formulation [12].

The linear sum capacity $C_{\mathsf{L}}(N, P)$ can be achieved by Kramer's linear code. Hence, this rather simple code, which iteratively refines receiver's knowledge about the messages, is sum rate optimal among the class of generalized linear feedback codes. For completeness, we provide a representation of Kramer's linear code and analyze it via properties of discrete algebraic Riccati recursions (cf. Wu et al. [13]). This analysis differs from the original approaches by Ozarow [2] and Kramer [8].

The complete characterization of $C(N, P)$, the maximum sum rate among all feedback codes, still remains open. We conjecture that $C(N, P) = C_{\mathsf{L}}(N, P)$ based on the observation that linear codes are *greedy optimal* for a multi-letter optimization problem which upper bounds $C(N, P)$. We establish this fact in Section V by introducing and analyzing the properties of *conditional maximal correlation*, which is an extension of the Hirschfeld–Gebelein–Renyi maximal correlation [14] to the case where an additional common random variable is shared.

The rest of the paper is organized as follows. In Section II we formally state the problem and present our main result. Section III provides the proof of the converse and Section IV

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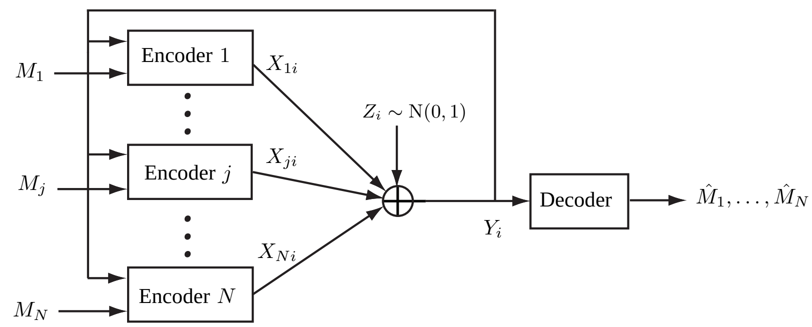

Fig. 1. N-sender AWGN-MAC

gives an alternative proof of achievability via Kramer's linear code. Section V concludes the paper with a discussion on the aforementioned conjecture.

Notation: We follow the notation in [15]. In particular, a random variable is denoted by an upper case letter (e.g. X, Y, Z) and its realization by a lower case letter (e.g. x, y, z). Similarly, a random column vector and its realization are denoted by bold face symbols (e.g. $\mathbf{X}$ and $\mathbf{x}$). Uppercase letters (e.g. A, B, C) also denote matrices, which can be differentiated from a random variable based on the context. The (i, j)-th element of A is denoted by A_{ij} and $(A_k)_{ij}$ is used to represent the (i, j)-th element of a sequence of matrices indexed by k. The transpose of a matrix A is denoted A^T, and its complex transpose by A'. We use the following short notation for covariance matrices: $K_{\mathbf{XY}} := \mathsf{E}(\mathbf{XY}') - \mathsf{E}(\mathbf{X})\,\mathsf{E}(\mathbf{Y}')$ and $K_{\mathbf{X}} := K_{\mathbf{XX}}$. Calligraphic letters (e.g. $\mathcal{A}, \mathcal{B}, \mathcal{C}$) denote discrete sets. Given a set of random variables $\{X_1, \ldots, X_N\}$ and a discrete set $\mathcal{A} \subseteq \mathcal{S} := \{1, \ldots, N\}$, we denote by $X(\mathcal{A})$ the ordered subset $X(\mathcal{A}) := \{X_j : j \in \mathcal{A}\}$. Similarly, for $j \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, n\}$, we define $X_i(\mathcal{A}) := \{X_{ji} : j \in \mathcal{A}\}$ as a subset of $\{X_{ji}\}$. Finally, $\mathsf{L}(\cdot)$ denotes an arbitrary linear function.

II. Problem Setup and the Main Result

Consider the communication problem over an additive white Gaussian noise multiple access channel (AWGN-MAC) with feedback depicted in Figure 1. Each sender $j \in \{1, \ldots, N\}$ wishes to transmit a message $M_j \in \mathcal{M}_j$ reliably to the common receiver. At each time $i = 1, \ldots, n$, the output of the channel is

$$Y_i = \sum_{k=1}^{N} X_{ji} + Z_i \tag{1}$$

where $\{Z_i\}$ is a discrete-time zero-mean white Gaussian noise process with unit average power, i.e., $\mathsf{E}(Z_i^2) = 1$, and independent of $M_1, \ldots, M_N$. We assume that the output symbols are causally fed back to each sender and the transmitted symbol X_{ji} from sender j at time i can depend on both the previous channel output sequence $Y^{i-1} := \{Y_1, Y_2, \ldots, Y_{i-1}\}$ and the message M_j.

We define a $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ code with power constraints $P_1, \ldots, P_N$ as

1) N message sets $\mathcal{M}_j := \{1, \ldots, 2^{nR_j}\}$, $j = 1, \ldots, N$,
2) a set of N encoders, where encoder j at each time i maps the pair (m_j, Y^{i-1}) to a symbol X_{ji} such that the sequence $X_{j1}, \ldots X_{jn}$ satisfies the *block power constraint*
$$\sum_{i=1}^{n} \mathsf{E}(X_{ji}^2(m_j, Y^{i-1})) \le nP_j, \quad m_j \in \mathcal{M}_j,$$
and
3) a decoder map which assigns message estimates $\hat{m}_j \in \mathcal{M}_j$, $j \in \{1, \ldots, N\}$, to each received sequence y^n.

We assume throughout that $M(\mathcal{S}) := (M_1, \ldots, M_N)$ is a random vector uniformly distributed over $\mathcal{M}_1 \times \cdots \times \mathcal{M}_N$. The probability of error is defined as

$$P_e^{(n)} := \mathsf{P}\{\hat{M}(\mathcal{S}) \ne M(\mathcal{S})\}.$$

A rate-tuple $(R_1, \ldots, R_N)$ is called achievable if there exists a sequence of $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ codes such that $P_e^{(n)} \to 0$ as $n \to \infty$. The capacity region $\mathscr{C}$ is defined as the closure of the set of achievable rate-tuples and the sum capacity C is defined as

$$C := \max\left\{ \sum_{j=1}^{N} R_j : \ (R_1, \ldots, R_N) \in \mathscr{C} \right\}.$$

We refer to $R = \sum_{j=1}^{N} R_j$ as the sum rate of a given code.

Definition 1: A $(2^{nR_1}, \ldots, 2^{nR_N}, n)$ code is called a *generalized linear feedback code* if the encoding maps can be decomposed as follows.

1) Nonfeedback (nonlinear) mappings: The message M_j is mapped to a vector $\mathbf{\Theta}_j \in \mathbb{R}^k$ for some $k \in \{1, \ldots, n\}$, which we refer to as the message point.
2) Linear feedback mappings: At each time i, the pair $(\mathbf{\Theta}_j, Y^{i-1})$ is mapped to a symbol X_{ji} such that $X_{ji} = \mathsf{L}_{ji}(\mathbf{\Theta}_j, Y^{i-1})$ is linear in $(\mathbf{\Theta}_j, Y^{i-1})$.

As mentioned earlier, any nonfeedback code is a generalized linear feedback code by picking $k = n$ and $\mathbf{\Theta}_j \in \mathbb{R}^n$ to be the codeword of the j-th user. On the other hand, by picking $k = 1$ we can get the linear codes by Schalkwijk and Kailath [3] and Ozarow [2]. For Kramer's linear code [8], the message points are 2-dimensional and we need $k = 2$. Note that this subclass of linear codes, for which k is independent of n, does not include the nonfeedback codes (cf. [16]).

The linear capacity region $\mathscr{C}_{\mathsf{L}}$ is defined as the closure of the set of rate-tuples achievable by linear codes and the linear sum capacity C_{L} is defined as

$$C_{\mathsf{L}} := \max\left\{\sum_{j=1}^{N} R_j : \ (R_1,\dots,R_N) \in \mathscr{C}_{\mathsf{L}}\right\}.$$

The following theorem characterizes $C_{\mathsf{L}}(N,P)$, the linear sum capacity under symmetric block power constraints P for all the N senders.

Theorem 1: For the AWGN-MAC with symmetric block power constraints $P_j = P$, we have

$$C_{\mathsf{L}}(N,P) = \frac{1}{2}\log(1 + NP\phi(N,P)) \tag{2}$$

where $\phi(N,P) \in \mathbb{R}$ is the unique solution in the interval $[1,N]$ to

$$(1 + NP\phi)^{N-1} = (1 + P\phi(N-\phi))^{N}. \tag{3}$$

Proof: The proof of the converse is provided in Section III. It is known [8] that Kramer's linear code achieves the sum rate (2). For completeness, a simple analysis for Kramer's code is presented in Section IV. ■

Note that $\phi(N,P) \in [1,N]$ captures the ultimate amount of cooperation which can be established among the senders, such that $\phi = 1$ corresponds to no cooperation and $\phi = N$ corresponds to full cooperation. For a fixed N, $\phi(N,P)$ is increasing (more power allows more cooperation) and concave in P as depicted in Figure 2.

Corollary 1: Consider the case of low signal-to-noise ratio (SNR). From (3) we can see that as $P \to 0$, $\phi(N,P) \to 1$ irrespective of the number of senders N, and thus

$$C_{\mathsf{L}}(N,P) - \frac{1}{2}\log(1 + NP) \to 0$$

which means that the linear sum capacity approaches the sum capacity without feedback. Hence, in the low SNR regime almost no cooperation is possible.

Corollary 2: Consider the case of high SNR. Again from (3) we can see that as $P \to \infty$, $\phi(N,P) \to N$ and

$$C_{\mathsf{L}}(N,P) - \frac{1}{2}\log(1 + N^2 P) \to 0.$$

Thus, the linear sum capacity approaches the sum capacity with full cooperation where all the transmitted signals are coherently aligned with combined SNR equal to $N^2 P$.

III. Proof of the Converse

In this section we show that under the symmetric block power constraints P for all senders, the linear sum capacity $C_{\mathsf{L}}(N,P)$ is upper bounded as

$$C_{\mathsf{L}}(N,P) \le \frac{1}{2}\log(1 + NP\phi(N,P)) \tag{4}$$

where $\phi(N,P) \in \mathbb{R}$ is the unique solution in the interval $[1,N]$ to

$$(1 + NP\phi)^{N-1} = (1 + P\phi(N-\phi))^{N}.$$

The proof can be summarized in four steps. First, we derive an upper bound on the linear sum capacity based on Fano's inequality, and we prove that in the resulting

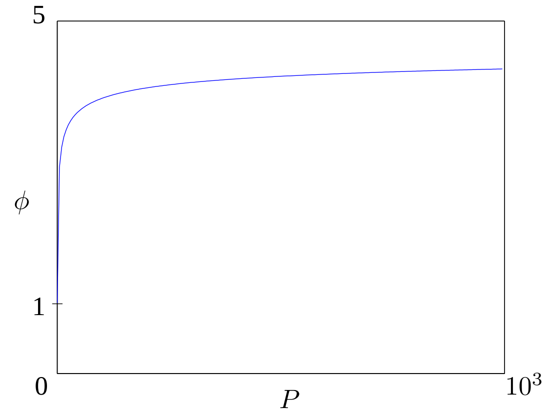

Fig. 2. $\phi(N,P)$ for $N = 5$

multi-letter optimization problem we can limit ourselves to Gaussian distributions (see Lemma 1). Second, we use a dependence balance condition [10], [11] and the Gaussianity of the involved random variables to derive an equivalent optimization problem (see (12)) over positive semidefinite matrices. This optimization problem is nonconvex due to the introduced dependence balance condition. Third, we upper bound the solution to this optimization problem using the Lagrange dual formulation and the symmetry of the involved functions. The so obtained upper bound depends on the choice of the Lagrange multipliers, and for each choice it is again a nonconvex optimization problem but involving only two optimization variables (see Lemma 4). Finally, using a few technical tricks and strong duality, we show that there exists a set of Lagrange multipliers for which this upper becomes equal to the right hand side of (4) (see Lemma 5).

Details are as follows.

Step 1: We provide an upper bound on the linear sum capacity based on Fano's inequality. Then we use linearity of the code and a conditional version of the maximum entropy theorem [5, Lemma 1] to show that it is sufficient to consider only Gaussian distributions.

Lemma 1: The linear sum capacity $C_{\mathsf{L}}(N,P)$, under symmetric block power constraints P for all N senders, is bounded as

$$C_{\mathsf{L}}(N,P) \le \lim_{n\to\infty} C_n(P)$$

where

$$C_n(P) := \max \frac{1}{n}\sum_{i=1}^{n} I(X_{1i},\dots,X_{Ni}; Y_i | Y^{i-1}). \tag{5}$$

Here the maximization is over all inputs X_{ji} of the form

$$X_{ji} = \mathsf{L}_{ji}(\mathbf{V}_j, Y^{i-1}),\ i = 1,\dots,n \tag{6}$$
$$\sum_{i=1}^{n} \mathsf{E}(X_{ji}^2) \le nP,\ j = 1,\dots,N$$

where each $\mathbf{V}_j \in \mathbb{R}^n \sim \mathrm{N}(0, K_{\mathbf{V}_j})$ is Gaussian and independent of Z^n and $\{\mathbf{V}_{j'} : j' \neq j\}$.

Remark 1: Although the functions that will be defined in the rest of the paper depend on the number of senders N, for simplicity of the notation, we avoid including N explicitly, e.g., $C_n(P)$.

Proof: For any achievable rate-tuple $(R_1,\dots,R_N)$, the sum rate R can be upper bounded as follows.

$$\begin{aligned} nR = n\sum_{k=1}^{N} R_j = H(M(\mathcal{S})) & \\ \le I(M(\mathcal{S});Y^n) + n\epsilon_n & \quad (7)\\ \le I(\Theta(\mathcal{S});Y^n) + n\epsilon_n & \quad (8)\\ \le \sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) + n\epsilon_n & \quad (9)\end{aligned}$$

where $\{\epsilon_n\}$ denotes a sequence such that $\epsilon_n \to 0$ as $n\to\infty$. Inequality (7) follows from Fano's inequality [17],

$$H(M(\mathcal{S})|Y^n) \le 1 + nP_e^{(n)} \sum_{k=1}^{N} R_j =: n\epsilon_n,$$

and the fact that $P_e^{(n)} \to 0$ as $n\to\infty$. Inequalities (8) and (9) follow from the data processing inequality and the memoryless property of the channel.

From (9), we upper bound the linear sum capacity as

$$C_{\mathsf{L}}(N,P) \le \lim_{n\to\infty} \max \frac{1}{n}\sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) \qquad (10)$$

where the maximization is over all linear codes which satisfy the the symmetric power constraints P, i.e.,

$$\begin{aligned} &X_{ji} = \mathsf{L}_{ji}(\mathbf{\Theta}_j, Y^{i-1}),\ i=1,\dots,n\\ &\sum_{i=1}^{n} \mathsf{E}(X_{ji}^2) \le nP,\ j=1,\dots,N.\end{aligned}$$

We next prove that message points $\mathbf{\Theta}_1,\dots,\mathbf{\Theta}_N$ can be replaced by Gaussian random variables $\mathbf{V}_1,\dots,\mathbf{V}_N$ with the same covariance matrix. Given a linear code with message points $\mathbf{\Theta}(\mathcal{S})$, let

$$\mathbf{V}(\mathcal{S}) \sim \mathrm{N}(0, K_{\mathbf{\Theta}(\mathcal{S})}).$$

We use $\mathbf{V}(\mathcal{S})$ with the same linear functions as in the given code to generate

$$\tilde{X}_{ji} = \mathsf{L}_{ji}(\mathbf{V}_j, \tilde{Y}^{i-1})$$

where $\tilde{Y}_i$ is the output of the AWGN-MAC corresponding to $\tilde{X}_i(\mathcal{S})$. It is not hard to see that

$$(\tilde{X}_i(\mathcal{S}), \tilde{Y}^i) \sim \mathrm{N}(0, K_{X_i(\mathcal{S}),Y^i}).$$

Therefore, by the conditional maximum entropy theorem [5, Lemma 1] we have

$$I(X_i(\mathcal{S});Y_i|Y^{i-1}) \le I(\tilde{X}_i(\mathcal{S});\tilde{Y}_i|\tilde{Y}^{i-1}). \qquad (11)$$

Combining (10) and (11) completes the proof. ■

Step 2: We show that the optimization problem defining $C_n(P)$ in (5) is equivalent to the following optimization problem

$$\begin{array}{ll} \text{maximize} & \frac{1}{n}\sum_{i=1}^{n} f_1(K_i)\\ \text{subject to} & K_i \succeq 0,\quad i=1,\dots,n\\ & \sum_{i=1}^{n}(K_i)_{jj} \le nP,\quad j=1,\dots,N\\ & \sum_{i=1}^{n} f_1(K_i) - f_2(K_i) \le 0 \end{array} \qquad (12)$$

where

$$f_1(K_i) := \frac{1}{2}\log\Big(1+\sum_{j,j'}(K_i)_{jj'}\Big) \qquad (13)$$

and

$$f_2(K_i) := \frac{1}{2(N-1)}\sum_{j=1}^{N}\log\Bigg[1+\sum_{j',j''}(K_i)_{j'j''} - \frac{\Big(\sum_{j'}(K_i)_{jj'}\Big)^2}{(K_i)_{jj}}\Bigg]. \qquad (14)$$

Before proving the equivalence we state two useful lemmas.

Lemma 2: The functions $f_1(K)$ and $f_2(K)$ in (13) and (14) are concave in K.

Proof: See Appendix A.

From [10], [11] we know the following dependence balance condition.

Lemma 3 ([11], Theorem 1): Let X_{ji} for $i=1,\dots,n$, and $j=1,\dots,N$, be defined by the (causal) functional relationship in (6). Then,

$$\sum_{i=1}^{n} I(X_i(\mathcal{S});Y_i|Y^{i-1}) \le \frac{1}{N-1}\sum_{i=1}^{n}\sum_{j=1}^{N} I(X_i(\mathcal{S}\backslash\{j\});Y_i|Y^{i-1},X_{ji}). \qquad (15)$$

Proof: See Appendix B.

Remark 2: The proof of Lemma 3 relies only on the independence of $\mathbf{V}_j \in \mathbb{R}^n$ from Z^n and $\{\mathbf{V}_{j'} : j' \neq j\}$. Thus, Lemma 3 remains valid also in the more general case where the inputs $X_{ji} = f_{ji}(\mathbf{V}_j, Y^{i-1})$ are obtained using arbitrary functions $\{f_{ji}\}$ and non-Gaussian $\mathbf{V}(\mathcal{S})$.

We now prove the equivalence of the optimization problems (5) and (12). Since random vectors $\{\mathbf{V}_j\}$ in (6) are jointly Gaussian and the functions $\{\mathsf{L}_{ji}\}$ are linear, the random variables $(X^n(\mathcal{S}), Y^n)$ generated according to (6) are also jointly Gaussian and we can replace the mutual information terms in condition (15) with functions of the covariance matrices. Specifically, let $\mathbf{X}_i = (X_{1i},\dots,X_{Ni})^T \sim \mathrm{N}(0,K_i)$ where

$$K_i := K_{\mathbf{X}_i} \succeq 0. \qquad (16)$$

Then

$$\begin{aligned} I(X_{1i},\dots,X_{Ni};Y_i|Y^{i-1}) &= f_1(K_i)\\ \frac{1}{N-1}\sum_{j=1}^{N} I(X_i(\mathcal{S}\backslash\{j\});Y_i|Y^{i-1},X_{ji}) &= f_2(K_i).\end{aligned}$$

Hence, the condition (15) reduces to

$$\sum_{i=1}^{n} f_1(K_i) - f_2(K_i) \leq 0. \tag{17}$$

Recall that the condition (17) follows from the functional relationship (6). Hence, adding the condition (17) to the optimization problem (5), as an additional constraint, does not decrease the maximum. Finally, note that given the functional relationship in (6), the objective function and the power constraints can also be represented only in terms of the covariance matrices $\{K_i\}_{i=1}^n$. Therefore, the functional relationship translates to the constraints that $K_i \succeq 0$ be positive semidefinite for all i, and the equivalence between the optimization problems (5) and (12) follows.

Notice that even though both functions $f_1(K)$ and $f_2(K)$ are concave (see Lemma 2), their difference $f_1(K) - f_2(K)$ is neither concave nor convex. Hence, the optimization problem (12) is nonconvex [12] due to the constraint (17).

Step 3: Using Lagrange multipliers $\lambda, \gamma \geq 0$, we provide a general upper bound $U(\lambda,\gamma)$ for the solution of the optimization problem given in (12). We further simplify this upper bound exploiting symmetry.

For the optimization problem (12), consider the Lagrange dual function [12]

$$L(\lambda,\gamma) = \max_{K_i \succeq 0} \frac{1}{n} \sum_{i=1}^{n} \Bigg[f_1(K_i) + \gamma(f_2(K_i) - f_1(K_i)) + \lambda\Big(\sum_{j=1}^{N} P - (K_i)_{jj}\Big)\Bigg] \tag{18}$$

with equal Lagrange multipliers $\lambda_j = \lambda \geq 0,\ j \in \{1,\ldots,N\}$ for the power constraints $\frac{1}{n}\sum_{i=1}^{n} P - (K_i)_{jj} \geq 0, \quad j \in \{1,\ldots,N\}$, and the Lagrange multiplier $\gamma \geq 0$ for the constraint $\frac{1}{n}\sum_{i=1}^{n} f_2(K_i) - f_1(K_i) \geq 0$.

It is easy to see that for any Lagrange multipliers $\lambda, \gamma > 0$, the solution to the optimization problem (12) is upper bounded by the Lagrange dual function $L(\lambda,\gamma)$, see [12]. Moreover, the right hand side of (18) is an average of some function of K_i, and can further be upper bounded by the maximum

$$U(\lambda,\gamma) := \max_{K \succeq 0} (1-\gamma) f_1(K) + \gamma f_2(K) + \lambda \sum_{j=1}^{N} (P - K_{jj}). \tag{19}$$

Thus, for any $\lambda, \gamma > 0$, the term $U(\lambda,\gamma)$ upper bounds the solution of the optimization problem (12).

Next, we simplify the upper bound $U(\lambda,\gamma)$ exploiting the properties of the functions $f_1(K)$ and $f_2(K)$.

Lemma 4: Let $\lambda, \gamma \geq 0$. Then, the upper bound $U(\lambda,\gamma)$ can be simplified as follows.

$$U(\lambda,\gamma) = \max_{x \geq 0} \max_{0 \leq \phi \leq N} g(\gamma, x, \phi) + \lambda N (P - x). \tag{20}$$

where

$$g(\gamma, x, \phi) := (1-\gamma) C_1(x,\phi) + \gamma C_2(x,\phi). \tag{21}$$

and

$$C_1(x,\phi) := \frac{1}{2}\log(1 + N x \phi)$$
$$C_2(x,\phi) := \frac{N}{2(N-1)}\log(1 + (N-\phi)x\phi). \tag{22}$$

Proof: First, we show that there exists a matrix K of the following form

$$K = x \cdot \begin{pmatrix} 1 & \rho & \rho & \ldots & \rho \\ \rho & 1 & \rho & \ldots & \rho \\ \rho & \rho & 1 & \ldots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \ldots & 1 \end{pmatrix} \tag{23}$$

which achieves the maximum in (19). Towards this end, we shall consider a covariance matrix K' (not necessarily of the from in (23)) that achieves the maximum in (19), and construct a matrix $\bar{K}$ as in (23) with objective function at least as large as the original matrix K':

$$(1-\gamma) f_1(\bar{K}) + \gamma f_2(\bar{K}) + \lambda \sum_{j=1}^{N} (P - \bar{K}_{jj}) \geq (1-\gamma) f_1(K') + \gamma f_2(K') + \lambda \sum_{j=1}^{N} (P - K'_{jj}). \tag{24}$$

Fix a covariance matrix K' that achieves the maximum in (19) and let $\bar{K}$ be the arithmetic average over all $N!$ matrices that can be obtained from the original matrix K' through simultaneous permutation of its rows and columns. That means:

$$\bar{K} := \frac{1}{N!} \sum_{\ell=1}^{N!} \pi_\ell(K'),$$

where $\pi_1, \ldots, \pi_{N!}$ denote all $N!$ possible permutations on the set of indices $\{1,\ldots,N\}$ and where $\pi(K')$ denotes the matrix that is obtained by permuting the rows and the columns of K' according to the permutation π.

It is easily seen that the so obtained matrix $\bar{K}$ has the desired form (23) and it remains to prove the inequality (24). To this end, we first notice that since the function $f_1(K)$ depends on the matrix K only via the sum of its entries:

$$f_1(\bar{K}) = f_1(K'), \tag{25}$$

and similarly,

$$\lambda \sum_{j=1}^{N} K'_{jj} = \lambda \sum_{j=1}^{N} \bar{K}_{jj}. \tag{26}$$

Also, by symmetry it follows that for each permutation π_ℓ, for $\ell = 1, \ldots, N!$:

$$f_2(\pi_\ell(K')) = f_2(K'). \tag{27}$$

Therefore, by concavity of $f_2(K)$ in K (see Lemma 2) and Jensen's inequality we can conclude that

$$f_2(\bar{K}) \geq f_2(K'). \tag{28}$$

Combining (25), (26), and (28) yields the desired inequality (24).

Thus, we continue our analysis with matrices of the form in (23) and by defining

$$\phi = 1 + (N-1)\rho$$

we have

$$\begin{aligned} f_1(K) &= C_1(x,\phi) \\ f_2(K) &= C_2(x,\phi). \end{aligned} \tag{29}$$

Since K is positive semidefinite, $x \geq 0$ and $-1/(N-1) \leq \rho \leq 1$, where the lower bound on ρ comes from the fact that $\sum_{i,j=1}^{N} K_{ij}$ is nonnegative for $K \succeq 0$. Hence, $0 \leq \phi \leq N$ and (19) reduces to (20). ■

The form of K in (23) was also considered in [5], [8]. However, in those cases the objective function was concave. In our case if $\gamma > 1$ the objective function is not necessarily concave and proving this claim needs further treatment based on the symmetry of the functions (see (25)–(27)).

Step 4: We complete the proof of the converse by showing that there exists Lagrange multipliers (λ^*, γ^*) such that $U(\lambda^*, \gamma^*)$ becomes equal to (4).

Lemma 5: There exists $\lambda^*, \gamma^* \geq 0$ such that

$$\begin{aligned} U(\lambda^*, \gamma^*) &= C_1(P, \phi(N,P)) \\ &= \frac{1}{2}\log(1 + NP\phi(N,P)) \end{aligned}$$

where $\phi(N,P) \in \mathbb{R}$ is the unique solution in the interval $[1,N]$ to

$$(1 + NP\phi)^{N-1} = (1 + P\phi(N-\phi))^N .$$

Proof: Consider the optimization problem over (x,ϕ) which defines $U(\lambda,\gamma)$ in (20). Note that $g(\gamma,x,\phi)$ given by (21) is neither concave or convex in (x,ϕ) for $\gamma > 1$. Let

$$U(\gamma) := U(\lambda^*(\gamma),\gamma) = \min_{\lambda \geq 0} U(\lambda,\gamma). \tag{30}$$

where $\lambda^*(\gamma)$ is the minimizer corresponding to γ. We use the following lemma to find $U(\gamma)$.

Lemma 6: The function $g(\gamma,x,\phi)$ is concave in ϕ for fixed $x,\gamma \geq 0$.

Proof: See Appendix C.

By concavity of $g(\gamma,x,\phi)$ in ϕ for fixed γ and x, the inner maximum in (20) happens at $0 < \phi^*(\gamma,x) < N$ such that

$$\begin{aligned} &\frac{\partial g(\gamma,x,\phi)}{\partial \phi} = 0 \\ \Leftrightarrow\ &\frac{(1-\gamma)(N-1)}{1 + \ Nx\phi^*} = \frac{\gamma(2\phi^* - N)}{1 + x\phi^*(N-\phi^*)} \end{aligned} \tag{31}$$

or at the boundaries $\phi^*(\gamma,x) \in \{0,N\}$. Therefore,

$$\begin{aligned} U(\gamma) &= \min_{\lambda \geq 0} \max_{x \geq 0} \max_{0 \leq \phi \leq N} g(\gamma,x,\phi) + \lambda N(P-x) \\ &= \min_{\lambda \geq 0} \max_{x \geq 0} g(\gamma,x,\phi^*(\gamma,x)) + \lambda N(P-x). \end{aligned} \tag{32}$$

for any $\gamma \geq 0$. To evaluate the last expression we use the following lemma.

Lemma 7: Let $\gamma, x \geq 0$ and $\phi^*(\gamma,x) > 0$ be the positive solution to (31). Then, $g(\gamma,x,\phi^*(\gamma,x))$ is increasing and concave in x.

Proof: See Appendix D.

Remark 3: As pointed out earlier, for $\gamma > 1$, $g(\gamma,x,\phi)$ is not concave in both x,ϕ in general. However, this lemma shows that $g(\gamma,x,\phi^*(\gamma,x))$ is concave in x for all $\gamma \geq 0$ and this is sufficient for the rest of the proof.

By concavity of $g(\gamma,x,\phi^*(\gamma,x))$ and Slater's condition [12] we have strong duality as follows.

$$\begin{aligned} &\min_{\lambda \geq 0} \max_{x} g(\gamma,x,\phi^*(\gamma,x)) + \lambda N(P-x) \\ &\quad = \max_{x \leq P} g(\gamma,x,\phi^*(\gamma,x)) \\ &\quad = g(\gamma,P,\phi^*(\gamma,P)) \end{aligned} \tag{33}$$

where the last equality follows from the fact that $g(\gamma,x,\phi^*(\gamma,x))$ is increasing in x (see Lemma 7). Combining (32) and (33) we have

$$U(\gamma) = g(\gamma,P,\phi^*(\gamma,P)). \tag{34}$$

Lastly, we find $\gamma^* \geq 0$ such that $U(\gamma^*) = C_1(P,\phi(N,P))$.

Lemma 8: For a fixed $x \geq 0$, the equation $C_2(x,\phi) - C_1(x,\phi) = 0$ has a unique solution $1 \leq \phi(N,x) \leq N$. Moreover,

$$1 + \frac{(2\phi(N,x) - N)(1 + Nx\phi(N,x))}{(N-1)(1 + x\phi(N,x)(N - \phi(N,x)))} > 0. \tag{35}$$

Proof: See Appendix E.

Let $\phi(N,P) \in [1,N]$ be the unique solution to $C_1(P,\phi) = C_2(P,\phi)$, which is equivalent to the equation (3). Given N and P, we pick $\gamma^*(P,\phi(N,P))$ such that it satisfies (31) for $x = P$ and $\phi^* = \phi(N,P)$. It is easy to check that $\gamma^* := \gamma^*(P,\phi(N,P)) > 0$ is greater than zero by plugging $x = P$ in (35). Since we picked γ^* such that γ^*, P and $\phi(N,P)$ satisfy (31) we conclude that $\phi^*(\gamma^*,P)$, the positive solution of (31), is equal to $\phi(N,P)$. Plugging $\gamma^* > 0$ and $\phi^*(\gamma^*,P)$ into (34) we have

$$\begin{aligned} &U(\gamma^*) \\ &\quad = g(\gamma^*,P,\phi^*(\gamma^*,P)) \\ &\quad = (1-\gamma^*)C_1(P,\phi^*(\gamma^*,P)) + \gamma^* C_2(P,\phi^*(\gamma^*,P)) \\ &\quad = (1-\gamma^*)C_1(P,\phi(N,P)) + \gamma^* C_2(P,\phi(N,P)) && (36) \\ &\quad = C_1(P,\phi(N,P)) && (37) \end{aligned}$$

where (36) and (37) follow from $\phi^*(\gamma^*,P) = \phi(N,P)$ and $C_1(P,\phi) = C_2(P,\phi)$, respectively. Hence, by picking $\lambda^* = \lambda^*(\gamma^*)$ (see (30)), we have $U(\lambda^*,\gamma^*) = C_1(P,\phi(N,P))$. ■

Combining the previous four steps we have $C_{\mathrm{L}}(N,P) \leq C_1(P,\phi(N,P))$, and the proof of the converse is complete.

IV. Achievability via The Kramer Linear Code

In this section we present an equivalent representation for the Kramer linear code [8] and analyze it based on the properties of discrete algebraic Riccati equations (DARE).

A. *Code representation:*

Recall that a linear code has a nonfeedback mapping

$$\mathcal{M}_j \to \Theta_j \in \mathbb{R}^k, \quad j = 1,\ldots,N.$$

We pick $k=2$ such that $\Theta_j \in \mathbb{C} = \mathbb{R}^2$. With slight abuse of notation we represent $\Theta_j \in \mathbb{C}$ as a scalar and reserve the vector notation for $\mathbf{\Theta} := (\Theta_1, \ldots, \Theta_N)^T$. We also assume that the transmitted signals $X_{ji} \in \mathbb{C}$ are complex (each sent over two transmissions).

Message sets: Let $M_j = (M_{j1}, M_{j2})$ be a two dimensional message, where

$$(M_{j1}, M_{j2}) \sim \text{Unif}\left(\{1, \ldots, 2^{nR_j}\} \times \{1, \ldots, 2^{nR_j}\}\right).$$

Nonfeedback mapping: Divide the square with corners at $(\pm 1 \pm i)$ on the complex plane ($i^2 = -1$) into 2^{2nR_j} equal sub-squares and map $m_j = (m_{j1}, m_{j2})$ to the center of the corresponding sub-square. The distance between neighboring points is

$$\Delta = 2 \cdot 2^{-nR_j} \tag{38}$$

in each direction.

Linear feedback mapping: Let the transmissions by all the senders at time i be denoted by the vector $\mathbf{X}_i := (X_{1i}, X_{2i}, \ldots, X_{Ni})^T$. Then, the linear feedback mapping is

$$\begin{aligned} &\mathbf{X}_0 = \mathbf{\Theta}, \\ &\mathbf{X}_i = A \cdot (\mathbf{X}_{i-1} - \hat{\mathbf{X}}_{i-1}(Y_{i-1})), \qquad i > 1 \end{aligned} \tag{39}$$

where

$$A = \begin{pmatrix} \beta_1\omega_1 & 0 & 0 & \ldots & 0 \\ 0 & \beta_2\omega_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \beta_N\omega_N \end{pmatrix}, \tag{40}$$

in which $\omega_1, \ldots, \omega_N$ are distinct points on the unit circle and $\beta_1, \ldots, \beta_N > 1$ are real coefficients, and

$$\hat{\mathbf{X}}_{i-1}(Y_{i-1}) = \frac{\mathsf{E}(\mathbf{X}_{i-1}Y'_{i-1})}{\mathsf{E}(|Y_{i-1}|^2)} \cdot Y_{i-1}$$

is the linear minimum mean square error (MMSE) estimate of $\mathbf{X}_{i-1}$ given Y_{i-1}. Note that the linear feedback mapping (39) is stationary and recursive.

Decoding: At time n, the decoder forms an estimate

$$\hat{\mathbf{\Theta}}_n = \sum_{i=0}^{n-1} A^{-i}\hat{\mathbf{X}}_i \tag{41}$$

and decodes Θ_j to the nearest point of $\hat{\mathbf{\Theta}}(j)$.

Theorem 2: Under the symmetric block power constraints $P_j = P$, the linear code described above achieves any sum rate $R < C_{\mathsf{L}}(N, P)$.

Proof: Proof follows from Lemma 10 and Lemma 11 in Section IV-B. ∎

B. Analysis:

First, using control theoretic tools [18], we analyze the behavior of the sequence of covariance matrices

$$K_n := K_{\mathbf{X}_n}$$

where $\mathbf{X}_n$ is the transmitted vector at time n.

Lemma 9: For the sequence K_n we have

$$K_n \to \bar{K} \succ 0 \quad \text{as } n \to \infty \tag{42}$$

where $\bar{K}$ is the unique positive-definite solution to the following discrete algebraic Riccati equation (DARE)

$$K = AKA' - (AKB)(1 + B'KB)^{-1}(AKB)'. \tag{43}$$

Proof: From (39) we have

$$K_{i+1} = AK_{(\mathbf{X}_i - \hat{\mathbf{X}}_i)}A' \tag{44}$$

where

$$K_{(\mathbf{X}_i - \hat{\mathbf{X}}_i)} = K_{\mathbf{X}_i} - K_{\mathbf{X}_iY_i}K_{Y_i}^{-1}K'_{\mathbf{X}_iY_i} \tag{45}$$

is the error covariance matrix for the linear MMSE estimate of $\mathbf{X}_i$ given Y_i. Since $Y_i = B^T\mathbf{X}_i + Z_i$, where

$$B^T = [11\ldots1]_{(N\times1)} \tag{46}$$

we have

$$K_{(\mathbf{X}_i - \hat{\mathbf{X}}_i)} = K_i - (K_iB)(1 + B'K_iB)^{-1}(K_iB)'. \tag{47}$$

Combining (44) and (47) we have the following Riccati recursion [19] for K_i:

$$K_{i+1} = AK_iA' - (AK_iB)(1 + B'K_iB)^{-1}(AK_iB)'. \tag{48}$$

Since A has no unit-circle eigenvalue and the pair (A, B) is detectable [18], that is, there exists a matrix $C \in \mathbb{R}^{1\times N}$ such that all the eigenvalues of $A - BC$ lie inside the unit circle[1], we can use Lemma II.4 in [20] to show that (42) holds. ∎

Probability of error: The following lemma provides a sufficient condition such that $P_e^{(n)} \to 0$ as $n \to \infty$.

Lemma 10: If $R_j < \log(\beta_j), j = 1, \ldots, N$, then $P_e^{(n)} \to 0$ as $n \to \infty$.

Proof: Let the difference vector be

$$\mathbf{D}_n = \mathbf{\Theta} - \hat{\mathbf{\Theta}}_n.$$

Considering (38), the probability of error can be bounded as

$$\begin{aligned} P_e^{(n)} &\le \mathsf{P}\left(\bigcup_j \left\{|\mathbf{D}_n(j)| > 2^{-nR_j}\right\}\right) \\ &\le \sum_{j=1}^N \mathsf{P}\left(\left\{|\mathbf{D}_n(j)| > 2^{-nR_j}\right\}\right) \qquad (49) \\ &\le \sum_{j=1}^N 2^{2nR_j}\,\mathsf{E}(|\mathbf{D}_n(j)|^2), \qquad (50) \end{aligned}$$

where (49) and (50) follow from the union bound and the Chebyshev inequality, respectively.

To find $\mathsf{E}(|\mathbf{D}_n(j)|^2)$, note that from the encoding rule (39) we have

$$\mathbf{X}_n = A^n\mathbf{\Theta} - \sum_{i=0}^{n-1} A^{n-i}\hat{\mathbf{X}}_i.$$

[1] For a diagonal matrix $A = \text{diag}([\lambda_1 \ldots \lambda_N])$ and a column vector $B = [b_1 \ldots b_N]'$, the pair (A,B) is detectable if and only if all the unstable eigenvalues λ_i, i.e. the ones on or outside the unit-circle, are distinct and the corresponding b_i are nonzero.

Comparing this form of $\mathbf{X}_n$ with the decoding estimate rule (41), we can rewrite $\mathbf{D}_n$ as follows,

$$\mathbf{D}_n = A^{-n}\mathbf{X}_n.$$

Hence, $K_{\mathbf{D}_n} = A^{-n}K_n(A')^{-n}$ and the diagonal elements of $K_{\mathbf{D}_n}$ are

$$\mathsf{E}(|\mathbf{D}_n(j)|^2) = \beta_j^{-2n}K_n(j,j).$$

Plugging $\mathsf{E}(|\mathbf{D}_n(j)|^2)$ into (50) we get

$$P_e^{(n)} \leq \sum_{j=1}^{N} K_n(j,j)\cdot 2^{2n(R_j-\log(\beta_j))}. \tag{51}$$

From Lemma 9 we know that $\limsup_{n\to\infty} K_n(j,j) < \infty$. Therefore, it follows from (51) that $P_e \to 0$ as $n \to \infty$ if $R_j < \log(\beta_j)$ for $j = 1,\dots,N$. ■

Asymptotic power allocation: For the linear code described above, the asymptotic power of user j is

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathsf{E}(X_{ji}^2) &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(K_i)_{jj}\\ &= \bar{K}_{jj}\end{aligned}$$

where $\bar{K}$ is the unique solution to (43) and the last equality follows from the Cesàro mean theorem and Lemma 9.

Thus, a rate-tuple $(R_1,\dots,R_N)$ is achievable with the code described above if for some $(\beta_1,\dots,\beta_N)$ satisfying

$$\log(\beta_j) > R_j$$

we can find a set of $(\omega_1,\dots,\omega_N)$ such that the corresponding asymptotic power allocation $(\bar{K}_{11},\dots,\bar{K}_{NN})$ strictly satisfies the power constraints, i.e.,

$$\bar{K}_{jj} < P_j \quad j = 1,\dots,N.$$

The strict condition above makes sure that the power constraint is satisfied for sufficiently large n.

The following lemma shows that for every sum rate $R < C_{\mathsf{L}}(P,N)$, there exists a choice of the coefficients $\{\beta_j\}$ and $\{\omega_j\}$ such that $\sum_{j=1}^{N}\log(\beta_j) > R$ and the corresponding asymptotic matrix $\bar{K}$ strictly satisfies the symmetric power constraints, i.e., $\bar{K}_{jj} < P$. Thus, the lemma establishes the achievability of the sum capacity $C_{\mathsf{L}}(N,P)$ and concludes our analysis.

Lemma 11: Given a sum rate $R < C_{\mathsf{L}}(N,P)$, let A be of the form (40) with coefficients

$$\beta_j = \beta, \quad j = 1,\dots,N,$$

for some choice of $\beta > 1$ satisfying

$$R < N\log(\beta) < C_{\mathsf{L}}(P,N), \tag{52}$$

and with coefficients

$$\omega_j = e^{2\pi i\frac{(j-1)}{N}}, \quad j = 1,\dots,N.$$

Then, the unique positive definite solution $\bar{K} \succ 0$ of the discrete algebraic Riccati equation (43) satisfies

$$\bar{K}_{jj} < P.$$

Before presenting the proof, we show that for this symmetric choice of A, the matrix $\bar{K}$ is completely characterized by β as follows.

Lemma 12: Let A and B be of the form (40) and (46) with $\beta_j = \beta$ and $\omega_j = e^{2\pi i\frac{(j-1)}{N}}$. Then the unique positive-definite solution $\bar{K} \succ 0$ of the following discrete algebraic Riccati equation

$$K = AKA' - AKB(1+B'KB)^{-1}(AKB)',$$

is circulant with real eigenvalues satisfying

$$\lambda_i = \frac{1}{\beta^2}\lambda_{i-1},$$

for $i = 2,\dots,N$. The largest eigenvalue λ_1 satisfies

$$1 + N\lambda_1 = \beta^{2N} \tag{53}$$

$$\Big(1+\lambda_1\big(N-\frac{\lambda_1}{\bar{K}_{jj}}\big)\Big) = \beta^{2(N-1)}. \tag{54}$$

Proof: See Appendix F.

We use this lemma to prove Lemma 11.

Proof of Lemma 11: From (53) we have

$$\frac{1}{2}\log(1+N\lambda_1) = N\log(\beta).$$

and thus by (52)

$$\frac{1}{2}\log(1+N\lambda_1) < C_{\mathsf{L}}(N,P) = \frac{1}{2}\log\left(1+NP\phi(N,P)\right).$$

We can hence conclude that

$$\lambda_1 < P\phi(N,P). \tag{55}$$

On the other hand, from (53) and (54) we have

$$\Big(1+N\lambda_1\Big)^{N-1} = \left(1+\lambda_1\Big(N-\frac{\lambda_1}{\bar{K}_{jj}}\Big)\right)^{N},$$

and hence by the definition of the function $\phi(N,\cdot)$ in Theorem 1,

$$\lambda_1 = \bar{K}_{jj}\phi(N,\bar{K}_{jj}). \tag{56}$$

Combining (55) and (56) we get

$$\bar{K}_{jj}\phi(N,\bar{K}_{jj}) < P\phi(N,P)$$

and the monotonicity of $\phi(N,\cdot)$ completes the proof.

V. Discussion

It is still unknown whether the linear sum capacity $C_{\mathsf{L}}(N,P)$ is in general equal to the sum capacity $C(N,P)$ under symmetric power constraints P for all N senders. However, we know [8] that they coincide if the power P exceeds the threshold $P_c(N) \geq 0$, which is the unique solution to

$$(1+N^2P/2)^{N-1} = (1+N^2P/4)^N. \tag{57}$$

We show that the condition (57) corresponds to the case where the the linear sum capacity $C_{\mathsf{L}}(N,P)$ coincides with the following cutset upper bound [5] on the sum capacity,

$$C(N,P) \leq \max_{\phi}\min\Big\{C_1(P,\phi), C_2(P,\phi)\Big\}. \tag{58}$$

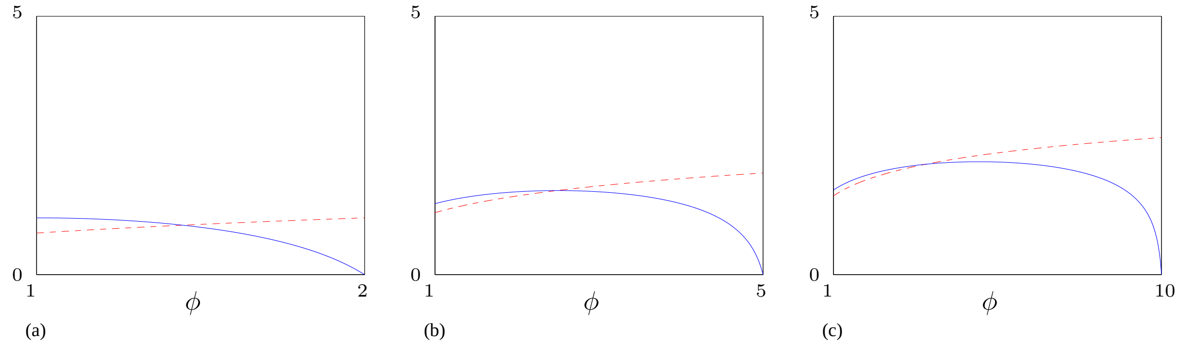

Fig. 3. $C_1(P,\phi)$ (dashed) and $C_2(P,\phi)$ for $P=2$ and (a) $N=2$ (b) $N=5$ (c) $N=10$

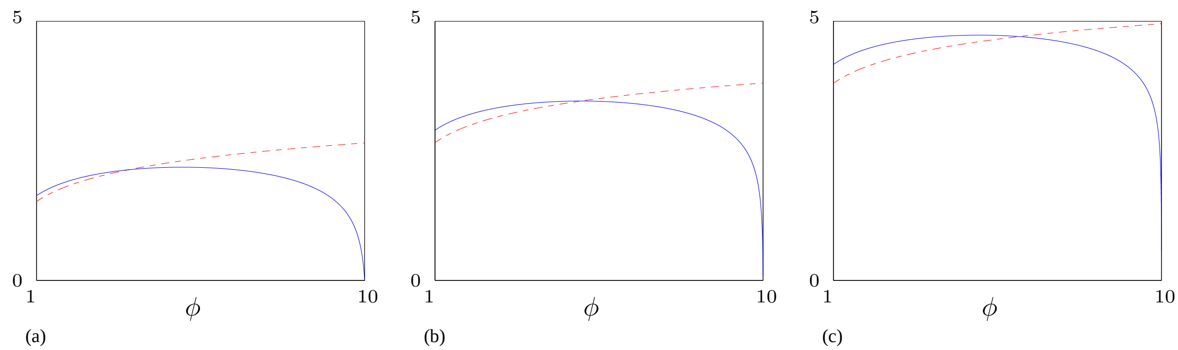

Fig. 4. $C_1(P,\phi)$ (dashed) and $C_2(P,\phi)$ for $N=10$ and (a) $P=2$ (b) $P=20$ (c) $P=200$

Here, the functions $C_1(P,\phi), C_2(P,\phi)$ are same as in (22).

Towards this end, note that $\phi(N,P)$ defined in Theorem 1, is the unique solution to $C_1(P,\phi)=C_2(P,\phi)$ for fixed N and P, and the linear sum capacity is

$$C_{\mathsf{L}}(N,P)=C_1(P,\phi(N,P))=C_2(P,\phi(N,P)).$$

Since the functions $C_1(P,\phi)$ and $C_2(P,\phi)$ are concave in ϕ (see Appendix A) and $C_1(P,\phi)$ is increasing in ϕ, the intersection point of the two functions and the max-min in (58) coincides if and only if $C_2(P,\phi)$ is nonincreasing at $\phi(N,P)$ (see Fig. 3 (a,b) and Fig. 4 (b,c)), that is,

$$\frac{\partial C_2(P,\phi)}{\partial\phi}\Big|_{\phi(N,P)}\le 0. \tag{59}$$

Considering (22), the condition (59) is equivalent to

$$\phi(N,P)\ge N/2$$

and plugging $\phi(N,P)=N/2$ into (3) gives (57).

For $P<P_c$, we conjecture that we still have $C(N,P)=C_{\mathsf{L}}(N,P)$ based on the properties of Hirschfeld–Gebelein–Rényi maximal correlation [14]. In the following we provide some insights.

Let $\rho^*(\Theta_1,\Theta_2)$ denote the maximal correlation between two random variables Θ_1 and Θ_2 as defined in [14]:

$$\rho^*(\Theta_1,\Theta_2)=\sup_{g_1,g_2}\mathsf{E}\left(g_1(\Theta_1)g_2(\Theta_2)\right) \tag{60}$$

where the supremum is over all functions g_1,g_2 such that

$$\mathsf{E}(g_1)=\mathsf{E}(g_2)=0 \quad\text{and}\quad \mathsf{E}(g_1^2)=\mathsf{E}(g_2^2)=1.$$

We extend this notion of maximal correlation to *conditional maximal correlation* as follows. Let the random variables Θ_1,Θ_2,Y be given. The conditional maximal correlation between Θ_1 and Θ_2 given a common random variable Y is defined as

$$\rho^*(\Theta_1,\Theta_2|Y)=\sup_{g_1,g_2}\mathsf{E}\left(g_1(\Theta_1,Y)g_2(\Theta_2,Y)\right) \tag{61}$$

where the supremum is over all functions g_1,g_2 such that

$$\mathsf{E}(g_1|Y)=\mathsf{E}(g_2|Y)=0 \quad\text{and}\quad \mathsf{E}(g_1^2)=\mathsf{E}(g_2^2)=1.$$

The assumption $\mathsf{E}(g_1|Y)=\mathsf{E}(g_2|Y)=0$ is crucial; otherwise, g_1 and g_2 can be picked as Y and $\rho^*(\Theta_1,\Theta_2|Y)=1$ trivially. For conditional maximal correlation we have the following lemma.

Lemma 13: If (Θ_1,Θ_2,Y) are jointly Gaussian, then

$$\rho^*(\Theta_1,\Theta_2|Y)=\rho(\Theta_1,\Theta_2|Y)$$

and linear functions $g_1^{\mathsf{L}},g_2^{\mathsf{L}}$ of the form

$$\begin{aligned} g_1^{\mathsf{L}}(\Theta_1,Y)&=\frac{\Theta_1-\mathsf{E}(\Theta_1|Y)}{\sqrt{\mathsf{E}\left((\Theta_1-\mathsf{E}(\Theta_1|Y))^2\right)}}\\ g_2^{\mathsf{L}}(\Theta_2,Y)&=\frac{\Theta_2-\mathsf{E}(\Theta_2|Y)}{\sqrt{\mathsf{E}\left((\Theta_2-\mathsf{E}(\Theta_2|Y))^2\right)}} \end{aligned} \tag{62}$$

attain the supremum in $\rho^*(\Theta_1,\Theta_2|Y)$.

Proof: See Appendix G.

Based on the conditional maximal correlation, we now present an upper bound on the (general) sum capacity. For simplicity, we focus on $N=2$ and equal per-symbol power constraints $\mathsf{E}(X_{ji}^2) \leq P$, for $j=1,2$. Also, without loss of generality, we assume that the message $M_j \in \{1,\dots,2^{nR}\}$ is mapped to a message point $\Theta_j \in \mathbb{R}$ and X_{ji} is a function of (Θ_j, Y^{i-1}). Note that by picking the identity mapping we have $\Theta_j = M_j$.

Lemma 14: A sum rate R, achievable by a code with block length n and per-symbol power constraints $\mathsf{E}(X_{ji}^2) \leq P$, is upper bounded as

$$R \leq \frac{1}{n}\sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_i|Y^{i-1}) + \epsilon_n \tag{63}$$

$$\leq \frac{1}{2n}\sum_{i=1}^{n} \log\Big(1+2P\Big(1+\rho^*\big(\Theta_{1i},\Theta_{2i}\big|Y^{i-1}\big)\Big)\Big) + \epsilon_n \tag{64}$$

where $\epsilon_n \to 0$ as $n \to \infty$.

Proof: See Appendix H.

Consider the multi-letter maximization problem where the objective function is the right hand side of (63). For the special class of Gaussian message points $(\Theta_1,\Theta_2) \sim \mathrm{N}(0, K_{\Theta_1,\Theta_2})$, we show that linear functions are greedy optimal for this maximization problem.

First note that the first term $I(X_{11}, X_{21}; Y_1)$ is maximized by linear functions, because (Θ_1,Θ_2) are Gaussian. Now, suppose that we have used linear functions up to time $i-1$ and therefore $(\Theta_1,\Theta_2,Y^{i-1})$ are Gaussian. Then, by Lemma 13 we know that $\rho^*(\Theta_1,\Theta_2|Y^{i-1}) = \rho(\Theta_1,\Theta_2|Y^{i-1})$ and $X_{ji} = \mathsf{L}_{ji}(\Theta_j, Y^{i-1})$, where L_{ji} is of the form (62), achieves the conditional maximal correlation. Hence, the i-th term $I(X_{1i}, X_{2i}; Y_i|Y^{i-1})$ which is upper bounded by (see Appendix H)

$$\frac{1}{2}\log\Big(1+2P\Big(1+\rho^*\big(\Theta_{1i},\Theta_{2i}\big|Y^{i-1}\big)\Big)\Big)$$

is maximized by linear functions of the form (62). A similar argument holds for any number of senders N, where we have the the following upper bound,

$$R \leq \frac{1}{n}\sum_{i=1}^{n} I(X(\mathcal{S}); Y_i|Y^{i-1})$$
$$\leq \frac{1}{2n}\sum_{i=1}^{n} \log\Big(1+NP+P\sum_{j\neq k}\rho^*\big(\Theta_{ji},\Theta_{ki}\big|Y^{i-1}\big)\Big).$$

Therefore, in establishing the sum rate optimality of linear codes, the missing step is as follows. We need to show that without loss of optimality we can consider only Gaussian message points and that linear functions are not only greedy optimal but also globally optimal for maximizing the right hand side of (63). Note that using functions which might hurt the current mutual information term $I(X_i(\mathcal{S}); Y_i|Y^{i-1})$ at time i, can be potentially advantageous for the future terms $I(X_k(\mathcal{S}); Y_k|Y^{k-1}), k > i$. Hence, this last step requires an analysis which captures the effect of the functions used at each time i, on the joint distribution of all the random variables $(\Theta(\mathcal{S}), X^n(\mathcal{S}), Y^n)$ in the entire block.

Appendix A
Proof of Lemma 2

Using similar argument as in Bergstrøm's theorem [17, Theorem, 17.10.1], we show $f_1(K)$ and $f_2(K)$ are both concave in K, where

$$f_1(K) = \frac{1}{2}\log\Big(1+\sum_{j,j'} K_{jj'}\Big)$$

and

$$f_2(K) = \frac{1}{2(N-1)}\sum_{j=1}^{N}\log\left[1+\sum_{j',j''}K_{j'j''} - \frac{\left(\sum_{j'}K_{jj'}\right)^2}{K_{jj}}\right].$$

Let $X(\mathcal{S}) = X^{(t)}(\mathcal{S})$, $\mathsf{P}(t=1)=\lambda = 1-\mathsf{P}(t=2)$, $X^{(1)}(\mathcal{S}) \sim N(0,K_1), X^{(2)}(\mathcal{S}) \sim N(0,K_2)$ and $Y = Y^{(t)} = \sum_{j=1}^{N} X_j^{(t)} + Z$, where $Z \sim \mathrm{N}(0,1)$. Assume $Z, X^{(1)}, X^{(2)}, t$ are independent. Under these assumptions, the covariance matrix of $X(\mathcal{S})$ is given by $K = \lambda K_1 + (1-\lambda)K_2$ and

$$f_1(K) = \frac{1}{2}\log(\mathrm{Var}(Y)) \tag{65}$$

and

$$f_2(K) = \frac{1}{2(N-1)}\sum_{j=1}^{N}\log(\mathrm{Var}(Y|X_j)). \tag{66}$$

Note that since $X(\mathcal{S})$ and Y are jointly Gaussian $\mathrm{Var}(Y|X_j)$ is a constant independent of X_j. Consider

$$\frac{\lambda}{2}\log\mathrm{Var}(Y^{(1)}|X_j^{(1)}) + \frac{(1-\lambda)}{2}\log\mathrm{Var}(Y^{(2)}|X_j^{(2)})$$
$$= \frac{\lambda}{2}\log\left(\frac{|K_{Y^{(1)},X_j^{(1)}}|}{|K_{X_j^{(1)}}|}\right) + \frac{(1-\lambda)}{2}\log\left(\frac{|K_{Y^{(2)},X_j^{(2)}}|}{|K_{X_j^{(2)}}|}\right)$$
$$= \lambda(h(Y^{(1)}|X_j^{(1)}) - h(Z)$$
$$\quad + (1-\lambda)(h(Y^{(2)}|X_j^{(2)}) - h(Z)) \tag{67}$$
$$= h(Y^{(t)}|X_j^t, t) - h(Z)$$
$$\leq h(Y|X_j) - h(Z)$$
$$= \frac{1}{2}\log\frac{|K_{Y,X_j}|}{|K_{X_j}|} \tag{68}$$
$$= \frac{1}{2}\log\mathrm{Var}(Y|X_j).$$

where (67) and (68) come form the fact that $Y^{(t)}$ is jointly Gaussian with $X_j^{(t)}$. Thus $\mathrm{Var}(Y|X_j)$ is concave in K for all j. The same argument holds for $h(Y)$. Then, concavity of $f_1(K)$ and $f_2(K)$ in K follows.

Appendix B
Proof of Lemma 3

Let

$$X_{ji} = \mathsf{L}_{ji}(\mathbf{V}_j, Y^{i-1}), i=1,\dots,n, j=1,\dots,N$$

such that $\mathbf{V}_j \in \mathbb{C}^n$ is independent of $\{\mathbf{V}_{j'} : j' \neq j\}$ and Z^n. We show that

$$\sum_{i=1}^{n} \Big(I(X_i(\mathcal{S}); Y_i | Y^{i-1}) \leq \frac{1}{N-1} \sum_{i=1}^{n} \sum_{j=1}^{N} I(X_i(\mathcal{S}\backslash\{j\}); Y_i | Y^{i-1}, X_{ji}) \Big). \tag{69}$$

By independence of $\mathbf{V}_j$'s we have

$$h(\mathbf{V}(\mathcal{S})) = \sum_{j=1}^{N} h(\mathbf{V}_j). \tag{70}$$

Consider

$$\begin{aligned}
0 &\leq I(\mathbf{V}(\mathcal{S}); Y^n) - \sum_{j=1}^{N} I(\mathbf{V}_j; Y^n) \qquad (71)\\
&= \sum_{i=1}^{n} \Big[I(\mathbf{V}(\mathcal{S}); Y_i | Y^{i-1}) - \sum_{j=1}^{N} I(\mathbf{V}_j; Y_i | Y^{i-1}) \Big] \\
&= \sum_{i=1}^{n} \Big[I(\mathbf{V}(\mathcal{S}), X_i(\mathcal{S}); Y_i | Y^{i-1}) \\
&\qquad - \sum_{j=1}^{N} I(\mathbf{V}_j, X_{ji}; Y_i | Y^{i-1}) \Big] \\
&\leq \sum_{i=1}^{n} \Big[I(X_i(\mathcal{S}); Y_i | Y^{i-1}) - \sum_{j=1}^{N} I(X_{ji}; Y_i | Y^{i-1}) \Big]. \qquad (72)
\end{aligned}$$

where inequality (71) follows from (70) and the fact that conditioning reduces entropy.

Inequality (72) follows from the facts that mutual information is positive and that the following Markov chain holds

$$\mathbf{V}(\mathcal{S}) \to (X_i(\mathcal{S}), Y^{i-1}) \to Y_i.$$

Adding $(N-1)\sum_{i=1}^{n} I(X_i(\mathcal{S}); Y_i | Y^{i-1})$ to both sides in (72) and rearranging terms we have

$$\sum_{i=1}^{n} \Big(I(X_i(\mathcal{S}); Y_i | Y^{i-1}) \leq \frac{1}{N-1} \sum_{i=1}^{n} \sum_{j=1}^{N} I(X_i(\mathcal{S}\backslash\{j\}); Y_i | Y^{i-1}, X_{ji}) \Big).$$

Appendix C
Proof of Lemma 6

We show

$$g(\gamma, x, \phi) = (1-\gamma)C_1(x, \phi) + \gamma C_2(x, \phi).$$

where

$$\begin{aligned}
C_1(x, \phi) &= \frac{1}{2}\log(1 + Nx\phi) \\
C_2(x, \phi) &= \frac{N}{2(N-1)}\log(1 + (N-\phi)x\phi),
\end{aligned} \tag{73}$$

is concave in ϕ for fixed $x, \gamma \geq 0$.

Note that $C_1(x, \phi) = f_1(K)$ and $C_2(x, \phi) = f_2(K)$ for symmetric K given in (23) (see (22)). We prove for general $K \succeq 0$ with fixed diagonal elements $(1-\gamma)f_1(K) + \gamma f_2(K)$ is concave in K for any $\gamma \geq 0$ and concavity of $g(\gamma, x, \phi)$ in ϕ for fixed x, γ immediately follows.

Let $X = X_1, \ldots, X_N \sim N(0, K)$ and $Y = \sum_{j=1}^{N} X_j + Z$, where $Z \sim \mathrm{N}(0, 1)$ is independent of $X_1, \ldots, X_N$. Then

$$\begin{aligned}
&(1-\gamma)f_1(K) + \gamma f_2(K) \\
&\quad = (1-\gamma)h(Y) + \frac{\gamma}{N-1}\sum_{j=1}^{N} h(Y|X_j) \\
&\quad = (1-\gamma)h(Y) + \frac{\gamma}{N-1}\sum_{j=1}^{N} \Big(h(Y) \\
&\qquad\qquad + h(X_j|Y) - h(X_j) \Big) \\
&\quad = h(Y)\Big(1 + \frac{\gamma}{N-1}\Big) + \frac{\gamma}{N-1}\sum_{j=1}^{N} h(X_j|Y) - h(X_j).
\end{aligned}$$

By Lemma 2, we know that $h(Y)$ and $h(X_j|Y)$ are concave in K. If the diagonal of K are fixed then $h(X_j) = \frac{1}{2}\log(2\pi e K_{jj})$ is also fixed and as long as $\gamma \geq 0$, $(1-\gamma)f_1(K) + \gamma f_2(K)$ is concave in K.

Appendix D
Proof of Lemma 7

Let $\gamma, x \geq 0$ and $\phi^*(\gamma, x) > 0$ be the positive solution to

$$\frac{(1-\gamma)(N-1)}{1 + Nx\phi} = \frac{\gamma(2\phi - N)}{1 + x\phi(N-\phi)}. \tag{74}$$

Then $g(\gamma, x, \phi^*(\gamma, x))$, where

$$g(\gamma, x, \phi) = (1-\gamma)C_1(x, \phi) + \gamma C_2(x, \phi).$$

is increasing and concave in x.

Let

$$g(x, \phi) := (1-\gamma)C_1(x, \phi) + \gamma C_2(x, \phi). \tag{75}$$

Note that $g(x, \phi)$ is same as $g(\gamma, x, \phi)$, but for simplicity we do not include γ explicitly.

Similarly, let

$$\phi^*(x) := \phi^*(\gamma, x)$$

Equation (74) can be written as

$$a\phi^2 + b\phi + c = 0 \tag{76}$$

where

$$\begin{aligned}
a &= (N + \gamma - 1 + \gamma N)x \\
b &= -N(N + \gamma - 1)x + 2\gamma \\
c &= -(N + \gamma - 1).
\end{aligned}$$

Since $ac < 0$, there is a unique positive solution $\phi^*(x) > 0$, where

$$0 < \phi^*(x) = \frac{-b + \sqrt{b^2 - 4ac}}{2a}. \tag{77}$$

We wish to show that the first derivative of

$$f(x) := g(x, \phi^*(x)).$$

is positive and the its second derivative is negative.

First derivative: Note that (see (31)) for $\phi^*(x)$ which satisfy (74) we have

$$\frac{\partial g(x,\phi)}{\partial \phi}\Big|_{x,\phi^*(x)} = 0.$$

Hence, we have

$$\begin{aligned}\frac{df(x)}{dx} &= \frac{\partial g(x,\phi)}{\partial x} + \frac{\partial g(x,\phi)}{\partial \phi}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)} \\ &= \frac{\partial g(x,\phi)}{\partial x}\Big|_{x,\phi^*(x)}\end{aligned}$$

Plugging $C_1(x,\phi)$ and $C_2(x,\phi)$ from (73) in $g(x,\phi)$, we have

$$\begin{aligned}\frac{df(x)}{dx} &= (1-\gamma)\frac{N\phi}{1+Nx\phi} \\ &\quad + \gamma\frac{N\phi(N-\phi)}{(N-1)(1+x\phi(N-\phi))}\Big|_{x,\phi^*(x)} \\ &= \frac{N\phi}{1+Nx\phi} \\ &\quad \times \left(1-\gamma+\gamma\frac{(N-\phi)(1+Nx\phi)}{(N-1)(1+x\phi(N-\phi))}\right)\Big|_{x,\phi^*(x)} \\ &= \frac{N(\gamma-1)(\phi^*(x))^2}{(1+Nx\phi^*(x))(N-2\phi^*(x))} \qquad (78) \\ &\geq 0, \qquad (79)\end{aligned}$$

where equality (78) follows from the fact that $\phi^*(x)$ satisfies (74), and inequality (79) follows from the fact that $(1-\gamma)$ and $(2\phi^*(x)-N)$ have the same sign (see (74)).

Second derivative: For $0 \leq \gamma \leq 1$, the concavity is immediate since $C_1(x,\phi) = f_1(K)$ and $C_2(x,\phi) = f_2(K)$ for symmetric K given in (23) (see (22)) and $f_1(K)$ and $f_2(K)$ are concave in K (see Appendix A).

To prove the concavity of $f(x)$ for $\gamma > 1$, we show that

$$\frac{d^2 f(x)}{dx^2} < 0.$$

From (78) we have

$$\frac{df(x)}{dx} = N(\gamma-1)\tilde{f}(x),$$

where

$$\begin{aligned}\tilde{f}(x) &:= h(x,\phi^*(x)) \\ h(x,\phi) &:= \frac{\phi^2}{(1+Nx\phi)(N-2\phi)}.\end{aligned}$$

Therefore it is enough to show that

$$\frac{d\tilde{f}(x)}{dx} < 0.$$

Consider

$$\begin{aligned}\frac{d\tilde{f}(x)}{dx} &= \frac{\partial h(x,\phi)}{\partial x} + \frac{\partial h(x,\phi)}{\partial \phi}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)} \\ &= \frac{-N\phi^3}{(1+Nx\phi)^2(N-2\phi)} \\ &\quad + \frac{\phi(N^2x\phi+2(N-\phi))}{(1+Nx\phi)^2(N-2\phi)^2}\frac{d\phi}{dx}\Big|_{x,\phi^*(x)} \\ &= \phi\cdot\frac{\frac{d\phi}{dx}\Big(N^2x\phi+2(N-\phi)\Big)-N\phi^2(N-2\phi)}{(1+Nx\phi)^2(N-2\phi)^2}\Big|_{x,\phi^*(x)}\end{aligned}$$

Since $\phi > 0$ and the denominator is also positive we need to show

$$\frac{d\phi^*(x)}{dx} < \frac{N\phi^2(N-2\phi)}{N^2x\phi+2(N-\phi)}\Big|_{x,\phi^*(x)} \qquad (80)$$

For the rest of the proof, with abuse of notation, we alternatively use ϕ for $\phi^*(x)$, the positive solution of (76). Taking the derivative of (76) with respect to x we have

$$\begin{aligned}\frac{d\phi^*(x)}{dx} &= \frac{-\phi^2(a'\phi+b')}{2a\phi^2+b\phi} \qquad (81) \\ &= \frac{-\phi^2(a'\phi+b')}{a\phi^2-c}.\end{aligned}$$

where

$$\begin{aligned}a' &= N+\gamma-1+\gamma N \\ b' &= -N(N+\gamma-1). \qquad (82)\end{aligned}$$

are derivatives of a, b with respect to x. Defining

$$\alpha := \frac{N+\gamma-1}{N}$$

we have $a = N(\alpha+\gamma)x,\ b = -N^2\alpha x + 2\gamma,\ c = -\alpha N,\ a' = N(\alpha+\gamma),\ b' = -N^2\alpha$, and

$$\begin{aligned}\frac{d\phi^*(x)}{dx} &= \frac{-\phi^2(a'\phi+b')}{a\phi^2-c}, \\ &= \frac{N\phi^2(N\alpha-(\alpha+\gamma)\phi)}{(\alpha+\gamma)Nx\phi^2+\alpha N} \\ &= \frac{N\phi^2(N-\beta\phi)}{\beta Nx\phi^2+N}, \qquad (83)\end{aligned}$$

where

$$\beta := 1+\frac{\gamma}{\alpha}.$$

It is not hard to see that $\beta \in \big(2, N+1\big)$ for $\gamma > 1$. Considering (83), (80) becomes equivalent to

$$\begin{aligned} & \frac{N-\beta\phi}{N-2\phi} < \frac{\beta Nx\phi^2+N}{N^2x\phi+2(N-\phi)} \\ \iff & \frac{N-\beta\phi}{N-2\phi} < \frac{N-\beta\phi+\beta\phi(Nx\phi+1)}{N-2\phi+N(Nx\phi+1)} \\ \iff & \frac{N-\beta\phi}{N-2\phi} < \frac{\beta\phi}{N}, \qquad (84)\end{aligned}$$

where (84) follows from the fact that for $b, d > 0$,

$$\frac{a}{b} < \frac{c}{d} \quad \Leftrightarrow \quad \frac{a}{b} < \frac{a+c}{b+d}. \qquad (85)$$

Considering (85) again with $c = d = 2\phi$ and noting that $\beta > 2$, we can see that to prove (84) it is sufficient to show

$$\begin{aligned} & \frac{N-(\beta-2)\phi}{N} \leq \frac{\beta\phi}{N} \\ \iff & \frac{N+\gamma-1}{2\gamma} \leq \phi. \qquad (86)\end{aligned}$$

To show the last condition, consider

$$\phi^*(0) = \frac{N+\gamma-1}{2\gamma}. \qquad (87)$$

$$\frac{d\phi^*}{dx}\Big|_{x=0} > 0. \qquad (88)$$

where (87) follows from (76). Condition (88) follows from (81) and the facts that $2a\phi^*(0) + b > 0$ by (77) and that for $\gamma > 1$, $a'\phi^*(0) + b' < 0$. Therefore, condition (86) holds.

APPENDIX E
PROOF OF LEMMA 8

We show that for a fixed $x \geq 0$, $C_2(x, \phi) - C_1(x, \phi) = 0$ has a unique solution $1 \leq \phi(N, x) \leq N$. Moreover,

$$1 + \frac{(2\phi(N,x) - N)(1 + Nx\phi(N,x))}{(N-1)(1 + x\phi(N,x)(N - \phi(N,x)))} > 0. \tag{89}$$

Let $f(\phi) = C_2(x, \phi) - C_1(x, \phi)$. We prove there exists a unique solution by showing $f(1) \geq 0$, $f(N) < 0$, and $f'(\phi) < 0$ for $1 \leq \phi \leq N$. The fact that $f(N) < 0$ is immediate. Condition $f(1) \geq 0$ is equivalent to

$$\Big(1 + x(N-1)\Big)^N \geq \Big(1 + Nx\Big)^{N-1}.$$

For the above condition to hold it is sufficient that

$$\binom{n}{N}(N-1)^k \geq \binom{N-1}{k} N^k, \tag{90}$$

which is true since $(1 - 1/N)^k \geq 1 - k/N$ for $N > 1$.

Finally, we need to show $f'(\phi) < 0$ which is equivalent to

$$\frac{N - 2\phi}{1 + x\phi(N - \phi)} - \frac{N-1}{1 + Nx\phi} < 0. \tag{91}$$

Rearranging the terms we have

$$1 + Nx\phi - (2\phi + x\phi^2 + Nx\phi^2) < 0,$$

which holds for any $\phi \geq 1$. This completes the proof of the uniqueness. Moreover, the condition (89) follows from plugging $\phi(N, x)$ in (91).

APPENDIX F
PROOF OF LEMMA 12

Let A and B be of the form (40) and (46) with $\beta_j = \beta$ and $\omega_j = e^{2\pi i \frac{(j-1)}{N}}$. We show that the unique positive-definite solution $\bar{K} \succ 0$ of the following discrete algebraic Riccati equation

$$K = AKA' - AKB(1 + B'KB)^{-1}(AKB)',$$

is circulant with real eigenvalues satisfying

$$\lambda_i = \frac{1}{\beta^2}\lambda_{i-1},$$

for $i = 2, \ldots, N$ and the largest eigenvalue λ_1 satisfies

$$1 + N\lambda_1 = \beta^{2N}$$
$$\Big(1 + \lambda_1\big(N - \frac{\lambda_1}{\bar{K}_{jj}}\big)\Big) = \beta^{2(N-1)}.$$

We know that any circulant matrix can be written as $Q\Lambda Q'$, where Q is the N point DFT matrix with

$$Q_{jk} = \frac{1}{\sqrt{N}} e^{-2\pi i (j-1)(k-1)/N}, \tag{92}$$

and $\Lambda = \mathrm{diag}([\lambda_1, \ldots, \lambda_N])$ is the matrix with eigenvalues on its diagonal. We show that the circulant matrix $\bar{K} = Q\Lambda Q'$ with positive $\lambda_j > 0$, such that $\lambda_j = \lambda_{j-1}/\beta^2$ for $j \geq 2$, satisfies the Riccati equation (43). Plugging $Q\Lambda Q'$ into (43) and rearranging we get

$$\begin{aligned}\Lambda = (Q'AQ)\Lambda(Q'AQ)' - ((Q'AQ)\ \Lambda\ (Q'B))\\ (1 + B'Q\Lambda Q'B)^{-1}((Q'AQ)\ \Lambda\ (Q'B))'.\end{aligned}$$

For this symmetric choice of A we have

$$Q'AQ = \beta \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \end{pmatrix}, Q'B = \begin{pmatrix} \sqrt{N} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Hence,

$$(Q'AQ)\Lambda(Q'AQ)' = \beta^2 \begin{pmatrix} \lambda_2 & 0 & \ldots & 0 \\ 0 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_1 \end{pmatrix}$$

$$(Q'AQ)\Lambda(Q'B) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \beta\lambda_1\sqrt{N} \end{pmatrix}$$

and the Riccati equation become N diagonal equations. The first $N-1$ equations are

$$\lambda_j = \beta^2\lambda_{j+1}, \quad j = 1, \ldots, N-1 \tag{93}$$

and the N-th equation is

$$\lambda_N = \beta^2\lambda_1 - \frac{\beta^2\lambda_1^2 N}{1 + N\lambda_1}. \tag{94}$$

From (93) we see that λ_1 is the largest eigenvalue and $\lambda_N = \beta^{-2(N-1)}\lambda_1$. Combining this with (94) we get

$$(1 + N\lambda_1) = \beta^{2N}. \tag{95}$$

Hence, λ_1 is real and so are $\lambda_j, j = 2, \ldots, N$. Note that from the form of Q in (92), $\lambda_1 = \sigma_1$ where

$$\sigma_j := \sum_{k=1}^{N} \bar{K}_{jk}.$$

Moreover, since $\bar{K}$ is circulant $\sigma_j = \sigma_1$ for all j, and $(1 + B'\bar{K}B) = 1 + N\lambda_1$. Hence, the diagonal equations of the original Riccati equation

$$K_{jj} = \beta^2 K_{jj} - \beta^2 \frac{\sigma_j^2}{(1 + B'\bar{K}B)} \tag{96}$$

are equivalent to

$$\beta^2 = \frac{1 + N\lambda_1}{1 + \lambda_1\Big(N - \frac{\lambda_1}{\bar{K}_{jj}}\Big)}$$

and by (95) we have

$$\Big(1 + \lambda_1\big(N - \frac{\lambda_1}{\bar{K}_{jj}}\big)\Big) = \beta^{2(N-1)}.$$

Appendix G
Proof of Lemma 13

We show that for jointly Gaussian (Θ_1, Θ_2, Y), we have

$$\rho^*(\Theta_1, \Theta_2|Y) = \rho(\Theta_1, \Theta_2|Y)$$

and linear functions $g_1^{\mathsf{L}}, g_2^{\mathsf{L}}$ of the form

$$\begin{aligned} g_1^{\mathsf{L}}(\Theta_1, Y) &= \frac{\Theta_1 - \mathsf{E}(\Theta_1|Y)}{\sqrt{\mathsf{E}\left((\Theta_1 - \mathsf{E}(\Theta_1|Y))^2\right)}} \\ g_2^{\mathsf{L}}(\Theta_2, Y) &= \frac{\Theta_2 - \mathsf{E}(\Theta_2|Y)}{\sqrt{\mathsf{E}\left((\Theta_2 - \mathsf{E}(\Theta_2|Y))^2\right)}} \end{aligned} \tag{97}$$

attain the supremum in $\rho^*(\Theta_1, \Theta_2|Y)$.

Let (U, V) be two zero-mean jointly Gaussian random variables with correlation $\rho(U, V)$. It is well known [21] that

$$\rho^*(U, V) = \rho(U, V). \tag{98}$$

Hence, the maximal correlation $\rho^*(U, V)$ is attained by linear functions $g_1^{\mathsf{L}}(U) = \frac{U}{\sqrt{\mathsf{E}(U^2)}}$ and $g_2^{\mathsf{L}}(V) = \frac{V}{\sqrt{\mathsf{E}(V^2)}}$.

Since (Θ_1, Θ_2, Y) is Gaussian we know that given $Y = y$, $(\Theta_1, \Theta_2)\big|_{Y=y}$ is Gaussian with some correlation

$$\rho(\Theta_1, \Theta_2 \big| Y = y) = \rho \quad \text{for all } y \tag{99}$$

independent of y. From (98) and (99), we have

$$\rho^*(\Theta_1, \Theta_2 \big| Y = y) = \rho(\Theta_1, \Theta_2|Y = y) = \rho \quad \text{for all } y \tag{100}$$

and therefore

$$\begin{aligned} &\rho^*(\Theta_1, \Theta_2|Y) \\ &\quad = \sup_{g_1, g_2} \mathsf{E}\Big(g_1(\Theta_1, Y) g_2(\Theta_2, Y)\Big) \\ &\quad = \sup_{g_1, g_2} \mathsf{E}_Y\left(\mathsf{E}\Big(g_1(\Theta_1, Y = y) g_2(\Theta_2, Y = y)\Big)\right) \\ &\quad \le \mathsf{E}_Y\left(\sup_{g_1, g_2} \mathsf{E}\Big(g_1(\Theta_1, Y = y) g_2(\Theta_2, Y = y)\Big)\right) \qquad (101) \\ &\quad = \mathsf{E}_Y\Big(\rho^*(\Theta_1, \Theta_2 \big| Y = y)\Big) \\ &\quad = \rho \qquad (102) \end{aligned}$$

where inequality (101) follows from Jensen's inequality and equality (102) follows from (100). It is easy to check that for linear functions $g_1^{\mathsf{L}}, g_2^{\mathsf{L}}$ of the form (97) we have

$$\mathsf{E}(g_1|Y) = \mathsf{E}(g_2|Y) = 0, \mathsf{E}(g_1^2) = \mathsf{E}(g_2^2) = 1 \tag{103}$$

and

$$\mathsf{E}\left(g_1(\Theta_1, Y) g_2(\Theta_2, Y)\right) = \rho. \tag{104}$$

From (102), (103) and (104) we conclude that linear functions of the form (97) achieves the supremum in $\rho^*(\Theta_1, \Theta_2|Y)$.

Appendix H
Proof of Lemma 14

For a sum rate R achievable by a code with block length n and per-symbol power constraints $\mathsf{E}(X_{ji}^2) \le P$ for $j = 1, 2$, we show

$$R \le \frac{1}{2n} \sum_{i=1}^{n} \log\Big(1 + 2P\Big(1 + \rho^*\big(\Theta_{1i}, \Theta_{2i} \big| Y^{i-1}\big)\Big)\Big) + \epsilon_n \tag{105}$$

where $\epsilon_n \to 0$ as $n \to \infty$ and $\rho^*\big(\Theta_{1i}, \Theta_{2i}\big|Y^{i-1}\big)$ is the conditional maximal correlation between message points Θ_1 and Θ_2 given the previous channel outputs Y^{i-1}.

By standard arguments based on Fano's inequality, we have

$$\begin{aligned} R &\le \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, X_{2i}; Y_i|Y^{i-1}) + \epsilon_n && (106) \\ &= \frac{1}{n} \sum_{i=1}^{n} I(\tilde{X}_{1i}, \tilde{X}_{2i}; \tilde{Y}_i|Y^{i-1}) + \epsilon_n && (107) \\ &\le \frac{1}{n} \sum_{i=1}^{n} I(\tilde{X}_{1i}, \tilde{X}_{2i}; \tilde{Y}_i) + \epsilon_n && (108) \end{aligned}$$

where

$$\begin{aligned} \tilde{X}_{ji} &:= X_{ji} - \mathsf{E}(X_{ji}|Y^{i-1}) \\ \tilde{Y}_i &:= \tilde{X}_{1i} + \tilde{X}_{2i} + Z_i \end{aligned}$$

and $\epsilon_n \to 0$ as $n \to \infty$. The equality (107) holds since $\mathsf{E}(X_{ji}|Y^{i-1})$ is a function of Y^{i-1}, and the inequality (108) follows from the data processing inequality and the fact that Markov chain $\tilde{Y}_i - (\tilde{X}_{1i}, \tilde{X}_{2i}) - Y^{i-1}$ holds.

Notice that by the definition of $\tilde{X}_{ji}$ we have

$$\mathsf{E}(\tilde{X}_{ji}|Y^{i-1}) = 0, \quad i = 1, \ldots, n, \ j = 1, 2. \tag{109}$$

Therefore,

$$\begin{aligned} \mathsf{E}(\tilde{X}_{ji}^2) &= \mathsf{E}\left(\mathsf{E}\left((X_{ji} - \mathsf{E}(X_{ji}|Y^{i-1}))^2 \Big| Y^{i-1}\right)\right) \\ &= \mathsf{E}\left(\mathsf{E}(X_{ji}^2 \big| Y^{i-1}) - \mathsf{E}^2(X_{ji}|Y^{i-1})\right) \\ &\le \mathsf{E}\left(\mathsf{E}(X_{ji}^2 \big| Y^{i-1})\right) \\ &= \mathsf{E}(X_{ji}^2) \\ &\le P. \end{aligned} \tag{110}$$

where the last inequality follows by the assumption of per-symbol power constraints P. Using (108), (109) and (110) we can further upper bound the sum rate R as follows.

$$\begin{aligned} R &\le \frac{1}{n} \sum_{i=1}^{n} I(\tilde{X}_{1i}, \tilde{X}_{2i}; \tilde{Y}_i) + \epsilon_n \\ &\le \frac{1}{2n} \sum_{i=1}^{n} \log\Big(1 + 2P + 2P\,\mathsf{E}\left(\rho(\tilde{X}_{1i}, \tilde{X}_{2i})\right)\Big) + \epsilon_n && (111) \\ &\le \frac{1}{2n} \sum_{i=1}^{n} \log\Big(1 + 2P\Big(1 + \rho^*\big(\Theta_{1i}, \Theta_{2i} \big| Y^{i-1}\big)\Big)\Big) + \epsilon_n && (112) \end{aligned}$$

where

$$\rho(\tilde{X}_{1i}, \tilde{X}_{2i}) = \mathsf{E}\left(\frac{\tilde{X}_{1i}}{\sqrt{\mathsf{E}(\tilde{X}_{1i}^2)}} \cdot \frac{\tilde{X}_{2i}}{\sqrt{\mathsf{E}(\tilde{X}_{2i}^2)}}\right)$$

is the correlation coefficient between $\tilde{X}_{1i}$ and $\tilde{X}_{2i}$. The inequality (111) follows from the maximum entropy theorem [17] and equal per-symbol power constraints $\mathsf{E}(X_{ji}^2) \le P$. The inequality (112) follows from the definition of conditional maximal correlation (61), and the fact that $\tilde{X}_{ji}$ is some function of (Θ_j, Y^{i-1}) satisfying the condition (109).

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