Integrated Sensing and Communication in the Finite Blocklength Regime

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***Abstract*—A point-to-point integrated sensing and communication (ISAC) system is considered where a transmitter conveys a message to a receiver over a discrete memoryless channel (DMC) and simultaneously estimates the state of the channel through the backscattered signals of the emitted waveform. We derive achievability and converse bounds on the rate-distortion-error tradeoff in the finite blocklength regime, and also characterize the second-order rate-distortion-error region for the proposed setup. Numerical analysis shows that our proposed joint ISAC scheme significantly outperforms traditional time-sharing based schemes where the available resources are split between the sensing and communication tasks.**

I. Introduction

Integrating sensing capabilities into a communication network is a promising approach to resolve the challenges of the upcoming sixth generation (6G) wireless communication system [1]–[5]. In fact, network sensing functionality is a key enabler to allow sensory data collection from the environment, which is required in applications such as industrial robots and autonomous vehicles. A recent paradigm, called *integrated sensing and communication (ISAC)*, suggests to fully integrate the sensing functionality into the communication functionality [6]–[8]. In other words, ISAC systems jointly perform both the sensing and communication tasks using common hardware, antenna(s) and spectrum.. The benefits of such a joint approach are reductions in hardware and signaling costs and improvements in energy consumption and spectral efficiency [9], [10].

Despite a consider amount of interesting ISAC research efforts, the fundamental performance limits, and thus the inherent tradeoffs between sensing and communication performances of optimal systems, remain unsolved. In particular, while [12]–[20] determined the information-theoretic fundamental performance limits for the asymptotic infinite blocklength regime, the focus of this article lies on the performances of real codes at finite blocklengths.

Specifically, in this work we consider a point-to-point ISAC system in which the transmitter conveys a message to a receiver over a discrete memoryless state-dependent channel, and in addition, based on a generalized feedback signal, it estimates the memoryless state sequence of the channel so as to minimize a given distortion criterion. We derive achievability and converse bounds on the optimal tradeoff between the communication rate and decoding error and the sensing distortion. Our achievability and converse bounds are close, and coincide up to third-order terms in the asymptotic regimes of infinite blocklengths. For this asymptotic regime we thus refine the capacity-distortion result in [14], [16] to the optimal scaling of the rate as a function of the allowed distortion and decoding error probability. The finite-blocklength behavior of ISAC has already been studied in [21], however for a Gaussian channel model where a single state (the channel coefficient) governs the entire transmission and the receiver wishes to estimate this state with smallest possible squared-error. In our setup, the state is described by a memoryless sequence impacting the various channel uses and the goal of the estimation is to reconstruct this sequence with minimum distortion.

II. Problem Setup

Consider the point-to-point setup in Figure 1 where a transmitter wishes to communicate a message M, which is uniformly distributed over a set $\{1,\dots,\mathsf{M}\}$, to a receiver over a state-dependent memoryless channel and at the same time wishes to estimate the channel state sequence based on a generalized feedback signal. So, if $M=m$, at a given time $i\in\{1,\dots,n\}$ and after observing the feedback sequence Z_{i-1}, the transmitter sends an input symbol

$$X_i = f_i^{(n)}(m, Z^{i-1}) \tag{1}$$

where for any $i\in\{1,\dots,n\}$ the encoding function $f_i^{(n)}$ is defined on appropriate domains. The transmitter also estimates the channel state S^n as

$$\hat{S}^n = h^{(n)}(Z^n, X^n), \tag{2}$$

based on a block-estimation function $h^{(n)}$, defined on appropriate domains.

We consider the discrete memoryless state-dependent channel with finite input alphabet $\mathcal{X}$, finite channel state alphabet $\mathcal{S}$, finite feedback alphabet $\mathcal{Z}$, finite output alphabet $\mathcal{Y}$ and the channel transition law

$$P_{Y^nZ^n|X^nS^n}(y^n,z^n|x^n,s^n)=\prod_{i=1}^{n}W(y_i,z_i|x_i,s_i) \tag{3}$$

for a given conditional pmf $W(\cdot,\cdot|\cdot,\cdot)$.

After observing the channel outputs Y^n, the receiver decodes the message M as

$$\hat{M} = g^{(n)}(Y^n), \tag{4}$$

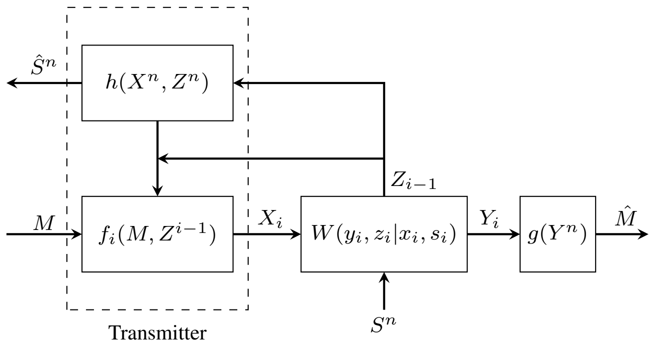

Fig. 1: ISAC System model.

where $g^{(n)}$ is a decoding function on appropriate domains. The quality of the state estimation at the transmitter is measured by the expected average per-block distortion

$$\Delta^{(n)} := \mathbb{E}[d(S^n; \hat{S}^n)] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[d(S)_i, \hat{S}_i)] \tag{5}$$

for a given bounded per-symbol distortion function $d(\cdot, \cdot)$.

The decoding error probability is defined as:

$$\epsilon^{(n)} := \mathbb{P}[\hat{M} \neq M]. \tag{6}$$

Definition 1: Given a blocklength n, the rate-distortion-error triple $(\mathsf{R}, \mathsf{D}, \epsilon)$ is said to be achievable, if there exist encoding, decoding, and estimation functions $\{f^{(n)}, g^{(n)}, h^{(n)}\}$ satisfying

$$\frac{1}{n}\log_2(\mathsf{M}) \geq \mathsf{R}, \tag{7}$$
$$\epsilon^{(n)} \leq \epsilon, \tag{8}$$
$$\Delta^{(n)} \leq \mathsf{D}. \tag{9}$$

III. Optimal Estimator

For the described memoryless setup, the optimal state estimator is a symbolwise estimator applied to the transmitter's observations X^n and Z^n:

$$\hat{S}^n = [(\hat{s}^*(X_1, Z_1), \hat{s}^*(X_2, Z_2), \ldots, \hat{s}^*(X_n, Z_n)], \tag{10}$$

where

$$\hat{s}^*(x, z) := \arg\min_{s'' \in \hat{\mathcal{S}}} \sum_{s \in \mathcal{S}} P_{S|XZ}(s|x, z) d(s, s'), \tag{11}$$

with

$$P_{S|XZ}(s|x, z) = \frac{P_S(s)P_{Z|SX}(z|s, x)}{\sum_{\tilde{s} \in \mathcal{S}} P_S(\tilde{s})P_{Z|SX}(z|\tilde{s}, x)}. \tag{12}$$

The proof of optimality of this symbolwise estimator relies on the Markov chain relation

$$(X^{i-1}, X_{i+1}^n, Z^{i-1}, Z_{i+1}^n) -\!\!\circ\!\!- (X_i, Z_i) -\!\!\circ\!\!- S_i, \tag{13}$$

see [16, Appendix A] for more details.

IV. Main Results

Given two random variables X and Y having joint probability mass function (pmf) $P_{XY}(x, y)$, define their information density

$$i(X; Y) := \log \frac{P_{Y|X}(y|x)}{P_Y(y)}, \tag{14}$$

and notice that the expectation of the information density equals the mutual information $I(X; Y) = \mathbb{E}[i(X, Y)]$. Denote the higher central moments of the information density as

$$\begin{aligned} \mathsf{V} &:= \mathrm{Var}[i(X; Y)] \\ &= \sum_{x,y} P_X(x)P_{Y|X}(y|x)\log^2 \frac{P_{Y|X}(y|x)}{P_Y(y)} - I(X; Y)^2, \quad (15) \\ \mathsf{T} &:= \mathbb{E}[|i(X; Y) - \mathbb{I}(X; Y)|^3] \\ &= \sum_{x,y} P_X(x)P_{Y|X}(y|x)\left|\log \frac{P_{Y|X}(y|x)}{P_Y(y)} - I(X; Y)\right|^3. \quad (16) \end{aligned}$$

Our main results are the following theorems on the rate-distortion-error tradeoff.

Theorem 1 (Achievability Bound): Given a blocklength n, the rate-distortion-error tradeoff $(\mathsf{R}, \mathsf{D}, \epsilon)$ is achievable if there exists a P_X and a constant $\mathsf{K} > 0$ such that the following two conditions are satisfied,

$$\mathsf{R} \leq I(X; Y) - \sqrt{\frac{\mathsf{V}}{n}}\mathbb{Q}^{-1}(\epsilon - \beta_u) - \mathsf{K}\frac{\log(n)}{n}, \tag{17}$$

$$\mathsf{D} \geq \sum_{x \in \mathcal{X}} \sum_{s \in \mathcal{S}} \sum_{z \in \mathcal{Z}} d(s, \hat{s}^*(x, z))P_X(x)P_S(s)P_{Z|XS}(z|x, s), \tag{18}$$

with

$$\beta_u := \frac{1}{n^{\mathsf{K}}} + \frac{0.7975\mathsf{T}}{\sqrt{n\mathsf{V}^3}}, \tag{19}$$

and where the mutual information $I(X; Y)$ and the two central moments V and T are defined based on the joint pmf $P_{XY}(x, y) = P_X(x)P_{Y|X}(y|x)$.

Proof: See Section VI. ∎

We also have the following converse bounds.

Theorem 2 (Converse Bound): Given the blocklength n, a rate-distortion-error triple $(\mathsf{R}, \mathsf{D}, \epsilon)$ is not achievable if for all $\delta > 0$ and pmfs P_X satisfying (18) the following lower bound holds:

$$\mathsf{R} \geq I(X; Y) - \sqrt{\frac{\mathsf{V}}{n}}\mathbb{Q}^{-1}(\epsilon + \beta_l) + \frac{\log(n)}{2n} - \frac{\log \delta}{n}, \tag{20}$$

where

$$\beta_l := \frac{6\mathsf{T}}{\sqrt{n\mathsf{V}^3}} + \frac{\delta}{\sqrt{n}}. \tag{21}$$

Proof: The proof of the bound in (20) follows similar steps as the proof of [22, Lemma 58, Theorem 28], where one has to integrate the optimal estimator in (10). ∎

Proposition 1: For sufficiently large n, the rate-distortion-error triple $(\mathsf{R}, \mathsf{D}, \epsilon)$ is achievable if, and only if, there is a P_X such that condition (18) holds and

$$\mathsf{R} = I(X; Y) - \sqrt{\frac{\mathsf{V}}{n}}\mathbb{Q}^{-1}(\epsilon) + O\left(\frac{\log n}{n}\right). \tag{22}$$

Proof: By the differentiability $\mathbb{Q}^{-1}$ and by the forms of β_u and β_l in (19) and (21), we have

$$\mathbb{Q}^{-1}(\epsilon - \beta_u) = \mathbb{Q}^{-1}(\epsilon) + O\left(\frac{1}{\sqrt{n}}\right), \tag{23}$$

$$\mathbb{Q}^{-1}(\epsilon + \beta_l) = \mathbb{Q}^{-1}(\epsilon) + O\left(\frac{1}{\sqrt{n}}\right). \tag{24}$$

Substituting (23) into (17), and (24) into (20) proves the proposition. ■

Remark 1: Equality (22) agrees with [22, Theorem 49] which determines the second-order coding rate of a DMC in the finite blocklength regime.

V. Comparisons and Examples

In this section, we evaluate Theorems 1 and 2 numerically for a binary example and compare them also with the performance of two baseline schemes that are frequently employed in practice.

A. Time-Sharing Schemes

Many practical systems employ a basic resource-sharing approach where a fraction of the resources (here $(1-\gamma)n$ channel uses) are dedicated only to the communication task and the remaining resources (here γn channel uses) to the sensing task, each one completely ignoring the other task. A slightly improved scheme uses the resources for the communication task also for some basic sensing, but using the waveform that is best for communication, and similarly uses the resources for the sensing task also for communication, but using the best waveform for sensing.

1) Basic Resource-Sharing Scheme: Given time-sharing parameter $\gamma \in [0,1]$, the performance of the basic resource-sharing scheme described above achieves rate

$$\mathsf{R} = (1-\gamma) R_{\max} \tag{25}$$

and distortion

$$\mathsf{D} = \gamma \mathsf{D}_{\min} + (1-\gamma)\mathsf{D}_{\text{trivial}}, \tag{26}$$

where $R_{\max}$ is the largest achievable rate:

$$R_{\max} := \max_{P_X} \left[I(X;Y) - \sqrt{\frac{\mathsf{V}}{n}} \mathbb{Q}^{-1}(\epsilon - \beta_u) - \mathsf{K}\frac{\log(n)}{n} \right] \tag{27}$$

and $\mathsf{D}_{\min}$ denotes the best possible distortion while $\mathsf{D}_{\text{trivial}}$ denotes the distortion achieved by the optimal trivial estimator that does not exploit the feedback:

$$\mathsf{D}_{\min} := \min_{P_X} \sum_{x\in\mathcal{X}} \sum_{s\in\mathcal{S}} \sum_{z\in\mathcal{Z}} P_X(x) P_S(s) P_{Z|SX}(z|s,x) \hat{s}^*(x,z), \tag{28}$$

$$\mathsf{D}_{\text{trivial}} := \min_{s'\in\mathcal{S}} \sum_{s\in\mathcal{S}} P_S(s) d(s,s'). \tag{29}$$

2) Improved Resource-Sharing Scheme: For a given time-sharing parameter $\gamma \in [0,1]$, the improved resource-sharing scheme achieves rate

$$\mathsf{R} = \gamma \mathsf{R}_{\text{sense}} + (1-\gamma)\mathsf{R}_{\max} \tag{30}$$

and distortion

$$\mathsf{D} = \gamma \mathsf{D}_{\min} + (1-\gamma)\mathsf{D}_{\text{comm}}, \tag{31}$$

where

$$\mathsf{D}_{\text{comm}} := \sum_{x\in\mathcal{X}} P_X^\star(x) \sum_{s\in\mathcal{S}} \sum_{z\in\mathcal{Z}} P_S(s) P_{Z|SX}(z|s,x) \hat{s}^*(x,z) \tag{32}$$

for $P_X^\star$ the optimizer in (27) and

$$\mathsf{R}_{\text{sense}} := I(X;Y) - \sqrt{\frac{\mathsf{V}}{n}} \mathbb{Q}^{-1}(\epsilon - \beta_u) - \mathsf{K}\frac{\log(n)}{n} \tag{33}$$

evaluated for $P_{XY} = P_X' P_{Y|X}$ with P_X' the optimizer of (28).

B. Binary Channel with Multiplicative Bernoulli State

Consider the channel

$$Y = SX, \tag{34}$$

with binary alphabets $\mathcal{X} = \mathcal{S} = \mathcal{Y} \in \{0,1\}$ and where the state is Bernoulli-q with $q \in (0,1)$ and the feedback is perfect, i.e., $Z = Y$. We consider the Hamming distortion measure $d(s,\hat{s}) = s \oplus \hat{s}$.

To compare the performance specified in Theorems 1 and 2 with each other and with the performance of the two baseline time-sharing schemes, we parametrize the binary input distribution P_X by $\alpha := \mathbb{P}[X=1]$. We also notice that the channel in (34) is equivalent to a Z-Channel: input 0 always leads to the output symbol 0 and input 1 leads to output 0 with probability $1-q$ and to output 1 with probability q. The mutual information between input and output of the channel is then obtained as

$$I(X;Y) = H_b(q\alpha) - \alpha H_b(q) \tag{35a}$$

and for the second and third central moments of the information density we have

$$\begin{aligned}\mathsf{V}_\alpha = \alpha&\left(q \log^2 \frac{1}{\alpha} + (1-q)\log^2 \frac{1-q}{1-q\alpha}\right) \\ &+ (1-\alpha)\log^2 \frac{1}{1-q\alpha} - I(X;Y)^2,\end{aligned} \tag{35b}$$

$$\begin{aligned}\mathsf{T}_\alpha = \alpha q &\left| \log\frac{1}{\alpha} - I(X;Y)\right|^3 \\ &+ (1-q)\left| \log \frac{1-q}{1-q\alpha} - I(X;Y)\right|^3 \\ &+ (1-\alpha)\left|\log\frac{1}{1-q\alpha} - I(X;Y)\right|^3,\end{aligned} \tag{35c}$$

where $H_b(x) = -x\log(x) - (1-x)\log(1-x)$ is the binary entropy function. We can then substitute $I(X;Y)$ and V, T from (35) into (17) and (20) to obtain the desired bounds on the rate.

To calculate the distortion bound (18), notice that whenever $x=1$, then $z=y=s$ and thus the distortion is zero. On the other hand, when $x=0$ then $y=0$ and the transmitter does not receive any information about the state of the channel. In this case, the optimal estimator is to choose the most likely state symbol, i.e. $\hat{s} = 0$ if $q < 1/2$ and $\hat{s} = 1$ if $q \geq 1/2$. We combine these observations to obtain the following bound:

$$\mathsf{D} \geq P_X(0) \sum_{s,y} d(s, \hat{s}^*(x=0,y)) P_S(s) P_{Y|XS}(y|x=0,s) \quad (36)$$

$$= P_X(0) \sum_{s \in \mathcal{S}} d(s, \hat{s}^*(x=0,y=0)) P_S(s) \quad (37)$$

$$= (1-\alpha) \min\{q, 1-q\}. \quad (38)$$

In other words, a distortion constraint imposes the following bound on α:

$$\alpha \geq 1 - \frac{\mathsf{D}}{\min\{q,1-q\}}. \quad (39)$$

Thus, for this example Theorem 1 states that for any $\mathsf{D} > 0$, all triples $(\mathsf{R}, \mathsf{D}, \epsilon)$ are achievable if

$$\mathsf{R} \leq \max_{\alpha, \mathsf{K} \geq 0} I(X;Y) - \sqrt{\frac{\mathsf{V}}{n}} \mathbb{Q}^{-1}(\epsilon - \beta_u) - \mathsf{K}\frac{\log(n)}{n}, \quad (40)$$

where the maximization is over all $\alpha \in [0,1]$ satisfying $1 \geq \alpha \geq 1 - \frac{\mathsf{D}}{\min\{q,1-q\}}$. Theorem 2 states that for any $D > 0$ all triples $(\mathsf{R}, \mathsf{D}, \epsilon)$ satisfying

$$\mathsf{R} \geq \max_{\alpha, \delta > 0} I(X;Y) - \sqrt{\frac{\mathsf{V}}{n}} \mathbb{Q}^{-1}(\epsilon + \beta_l) + \frac{\log(n)}{2n} - \frac{\log \delta}{n} \quad (41)$$

are not achievable. Here, the maximization is again over values $\alpha \in \left[\min\left\{0, 1 - \frac{\mathsf{D}}{\min\{q,1-q\}}\right\}, 1\right]$.

Notice that for this channel (which is a Z-channel) the capacity is equal to [23]

$$\mathsf{C} = \log\left(1 + q(1-q)^{\frac{1-q}{q}}\right), \quad (42)$$

and is achieved for

$$P_X^\star(1) = \alpha^\star = \frac{1}{q\left(1 + 2^{\frac{H_b(q)}{q}}\right)}. \quad (43)$$

The distortion achieved with this capacity-achieving $\alpha^\star$ is $\mathsf{D}_{\text{comm}} = (1-\alpha^\star)\min\{q, 1-q\}$.

C. Numerical Analysis

Fig. 2 illustrates the achievability and converse bounds on the rate-distortion-error tradeoff presented in (40) and (41) for $\epsilon = 10^{-3}$, $q = 0.4$, $\mathsf{K} = 0.5$. As can be seen from this figure the bounds are tight for large values of n. Notice that for $q = 0.4$ the capacity of the channel is $\mathsf{C} = 0.246$ and the achieved distortion is $\mathsf{D}_{\text{comm}} = 0.2432$.

Fig. 3 compares the rate-distortion-error tradeoff achieved by our scheme with the tradeoff achieved under the basic and improved resource-sharing schemes. As can be seen from this figure, our scheme outperforms the other two baseline schemes.

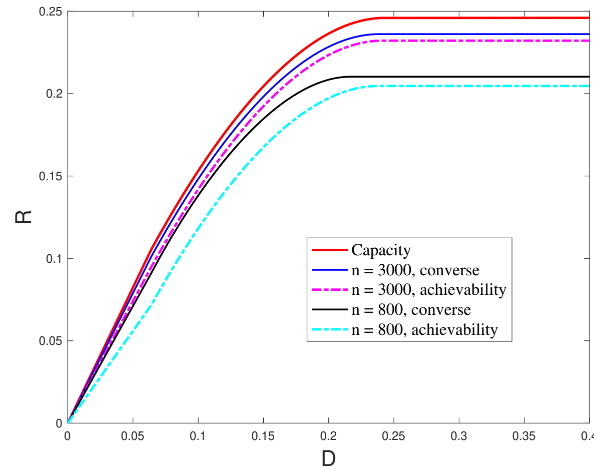

Fig. 2: Achievability and converse bounds on the rate-distortion-error trade-off of Theorems 1 and 2 for $\epsilon = 10^{-3}$, $q = 0.4$, $\mathsf{K} = 0.5$ and different values of n.

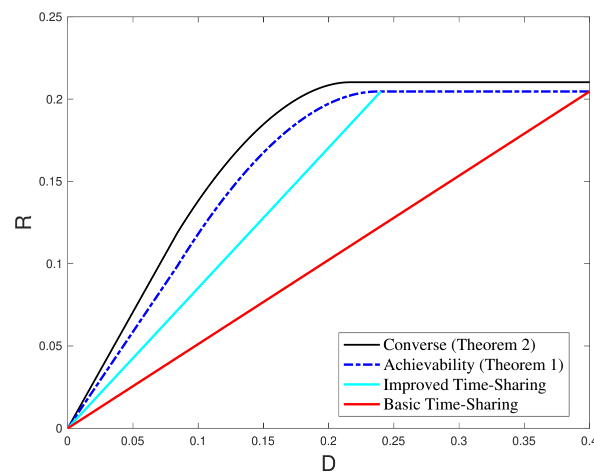

Fig. 3: Comparison of the rate-distortion-error trade-off in Theorems 1 and 2 with the basic and improved resource-sharing schemes for $\epsilon = 10^{-3}$, $q = 0.4$, $\mathsf{K} = 0.5$ and $n = 700$.

VI. Proof of Theorem 1

A. Codebook Generation

Fix P_X. The codebook $\mathcal{C} = \{x^n(m)\}_{m=1}^{\mathsf{M}}$ is generated by randomly and independently choosing each entry according to P_X.

B. Encoding

To send a message m, the transmitter encodes this message via the codeword $x^n(m)$ and sends it over the channel.

C. Estimation

After observing the feedback sequence $Z^n = z^n$, the transmitter estimates the channel state through (10).

D. Decoding

Given the channel outputs $Y^n = y^n$, the receiver estimates the message M by choosing the index $\hat{m}$ that corresponds to the codeword $x^n(\hat{m})$ that maximizes the information density:

$$\hat{m} := \arg\max_m i(x^n(m); y^n). \quad (44)$$

The receiver then produces the guess $\hat{M} = \hat{m}$.

E. Error Analysis

To analyze $\mathbb{P}[\hat{M} \neq M]$, we use the threshold-based metric bound in [22]. For any $\gamma \in \mathbb{R}$, we have

$$\mathbb{P}[\hat{M} \neq M] \leq \mathbb{P}[i(X^n; Y^n) \leq \gamma] + \mathsf{M} \cdot \mathbb{P}[i(\bar{X}^n; Y^n) \geq \gamma], \tag{45}$$

where $\bar{X}^n \sim P_{X^n}$ and is independent of X^n and Y^n. We will set

$$\gamma := \log \mathsf{M} + \mathsf{K} \log n, \tag{46}$$

for some $\mathsf{K} > 0$, and employ the Berry-Esseen theorem and Bayes' formula to evaluate the two terms on the right-hand side of (45).

By the strengthening of the Berry-Esseen theorem in [24], and because $\mathbb{E}[i(X^n; Y^n)] = nI(X;Y)$, we have with the definition in (46)

$$\mathbb{P}[i(X^n; Y^n) \leq \gamma] \leq \mathbb{Q}\left(\frac{-\log \mathsf{M} + nI(X;Y) - \mathsf{K}\log(n)}{\sqrt{n\mathsf{V}}}\right) + \frac{0.7975\mathsf{T}}{\sqrt{n\mathsf{V}^3}}. \tag{47}$$

To bound $\mathbb{P}[i(\bar{X}^n; Y^n) \geq \gamma]$, we first use Bayes' formula to write

$$\begin{aligned} P_{X^n}(\overline{x}^n) &= \frac{P_{Y^n}(y^n) P_{X^n|Y^n}(\overline{x}^n|y^n)}{P_{Y^n|X^n}(y^n|\overline{x}^n)} && (48)\\ &= P_{\bar{X}^n|Y^n}(\overline{x}^n|y^n) 2^{-i(\bar{x}^n;y^n)}. && (49) \end{aligned}$$

For any $y^n \in \mathbb{R}^n$, we then have

$$\begin{aligned} &\sum_{\bar{x}^n \in \mathcal{X}} \mathbb{1}\left\{i(\bar{x}^n; y^n) > \gamma\right\} P_{X^n}(\overline{x}^n) \\ &= \sum_{\bar{x}^n \in \mathcal{X}} 2^{-i(\bar{x}^n;y^n)} \mathbb{1}\left\{\frac{P_{Y^n|X^n}(y^n|\overline{x}^n)}{P_{Y^n}(y^n)} > 2^{\gamma}\right\} \\ &\qquad \cdot P_{X^n|Y^n}(\overline{x}^n|y^n) && (50)\\ &\leq \sum_{\bar{x}^n \in \mathcal{X}} 2^{-i(\bar{x}^n;y^n)} \frac{P_{Y^n|X^n}(y^n|\overline{x}^n)}{P_{Y^n}(y^n)} 2^{-\gamma} P_{X^n|Y^n}(\overline{x}^n|y^n) \\ &= \sum_{\bar{x}^n \in \mathcal{X}} P_{X^n|Y^n}(\overline{x}^n|y^n) 2^{-\gamma} \\ &= 2^{-\gamma}. && (51) \end{aligned}$$

As a consequence,

$$\mathbb{P}[i(\bar{X}^n; Y^n) \geq \gamma] \leq 2^{-\gamma} \tag{52}$$

and

$$\mathsf{M}\mathbb{P}[i(\bar{X}^n; Y^n) \geq \gamma] \leq 2^{-\gamma + \log \mathsf{M}} = n^{-\mathsf{K}}. \tag{53}$$

Combining (45), (47), and (53), we obtain

$$\mathbb{P}[\hat{M} \neq M] \leq \mathbb{Q}\left(\frac{-\log \mathsf{M} + nI(X;Y) - \mathsf{K}\log(n)}{\sqrt{n\mathsf{V}}}\right) + \beta_u, \tag{54}$$

where β_u is defined in (19).

Thus, the probability of error stays below ϵ whenever

$$\epsilon - \beta_u \geq \mathbb{Q}\left(\frac{-\log \mathsf{M} + nI(X;Y) - \mathsf{K}\log(n)}{\sqrt{n\mathsf{V}}}\right), \tag{55}$$

or equivalently when

$$\log \mathsf{M} \leq nI(X;Y) - \sqrt{n\mathsf{V}}\mathbb{Q}^{-1}(\epsilon - \beta_u) - \mathsf{K}\log(n), \tag{56}$$

establishing the bound in (17).

F. Expected Distortion

The expected distortion can be written as

$$\begin{aligned} \Delta^{(n)} &= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[d(S_i, \hat{S}_i)] && (57)\\ &= \sum_{x\in\mathcal{X}} \sum_{s\in\mathcal{S}} \sum_{z\in\mathcal{Z}} d(s, \hat{s}^*(x,z)) P_X(x) P_S(s) P_{Z|XS}(z|x,s). && (58) \end{aligned}$$

This proves the inequality (18) and consequently Theorem 1.

VII. Conclusions

We have studied the rate-distortion-error tradeoff of a point-to-point ISAC system where a transmitter conveys a message to a receiver over a discrete memoryless state-dependent channel and simultaneously estimates the state of the channel. We have derived achievability and converse bounds on the rate-distortion-error tradeoff in the finite blocklength regime. We also have characterized the second-order rate-distortion-error region of the proposed setup. Our numerical analysis shows that our joint design scheme significantly outperforms the resource-sharing baseline schemes where the available resources are split between the sensing and communication tasks. In our model the receiver has no state-information. The generality of our model allows however to obtain results for perfect or partial state-information as special cases from our Theorems 1 and 2, simply by including the state-information as part of the receiver's output. An interesting line of future work is to study the ISAC problem with general state and channel distribution in the finite blocklength regime [25].

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Appendix A
Proof of Theorem 2

The proof of the bound in (20) follows similar steps as the proof of [22, Lemma 58, Theorem 28]. In the following, we sketch the proof of the converse bound (20).

Consider a random variable Y on $\mathcal{Y}$ which can take probability measures $P_{Y|X}$ and P_Y. Define by $P_{Z|Y}: \mathcal{Y} \to \{0,1\}$ a randomized test between those two distributions where 0 indicates that the test chooses P_Y. The best performance achievable among such randomized tests is given by

$$\beta_\alpha(P_{Y|X}, P_Y)$$

$$= \min_{P_{Z|Y}:\sum_{y\in\mathcal{Y}} P_Y(y)P_{Z|Y}(1|y)\geq\alpha} \sum_{y\in\mathcal{Y}} P_Y(y)P_{Z|Y}(1|y). \quad (59)$$

Thus $\beta_\alpha(P_{Y|X}, P_Y)$ gives the minimum probability of error under hypothesis P_Y if the probability of error under hypothesis $P_{Y|X}$ is below $1-\alpha$. It is easy to show that for any $\tilde{\gamma} > 0$,

$$\alpha \leq \mathbb{P}[\frac{dP_{Y|X}}{dP_Y} \geq \tilde{\gamma}] + \tilde{\gamma}\beta_\alpha(P_{Y|X}, P_Y). \quad (60)$$

Equivalently, for any $\gamma_n > 0$

$$\beta_\alpha(P_{Y^n|X^n}, P_{Y^n}) \geq \frac{1}{\gamma_n}\left(\alpha - \mathbb{P}[\log\frac{dP_{Y^n|X^n}}{dP_{Y^n}} \geq \log\gamma_n]\right). \quad (61)$$

Set

$$\log\gamma_n = nI(X;Y) + \sqrt{n\mathsf{V}}Q^{-1}(\alpha_n) \quad (62)$$

with

$$\alpha_n = \alpha - \frac{6T}{\sqrt{nV^3}} - \frac{\delta}{\sqrt{n}} \quad (63)$$

for some $\delta > 0$. By employing the Berry-Esseen theorem, we have

$$\left|\mathbb{P}\left[\log\frac{dP_{Y^n|X^n}}{dP_{Y^n}} \geq \log\gamma_n\right] - \alpha_n\right| \leq \frac{6T}{\sqrt{nV^3}}. \quad (64)$$

Consequently

$$\mathbb{P}\left[\log\frac{dP_{Y^n|X^n}}{dP_{Y^n}} \geq \log\gamma_n\right] \leq \alpha - \frac{\delta}{\sqrt{n}}. \quad (65)$$

Substituting (65) into (61), we have

$$\beta_\alpha(P_{Y^n|X^n}, P_{Y^n}) \geq \frac{\delta}{\gamma_n\sqrt{n}}, \quad (66)$$

and by (62)

$$\log\big(\beta_\alpha(P_{Y^n|X^n}, P_{Y^n})\big)$$
$$\geq \log(\delta) - nI(X:Y) - \sqrt{nV}\mathbb{Q}^{-1}(\alpha_n) - \frac{1}{2}\log(n). \quad (67)$$

Using [22, Theorem 27], every (M, ϵ)-code satisfies

$$\log\mathsf{M} \leq -\log\beta_{1-\epsilon}(P_{Y^n|X^n}, P_{Y^n})). \quad (68)$$

By (67) and $\alpha = 1-\epsilon$ and the fact that $\mathbb{Q}^{-1}(1-x) = -\mathbb{Q}^{-1}(x)$,

$$\log\mathsf{M} \leq nI(X:Y) - \sqrt{nV}\mathbb{Q}^{-1}(\epsilon + \frac{6T}{\sqrt{nV^3}} + \frac{\delta}{\sqrt{n}})$$
$$+\frac{1}{2}\log(n) - \log(\delta) \quad (69)$$

which proves the inequality (20).

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