Joint Coding of URLLC and eMBB in Wyner's Soft-Handoff Network in the Finite Blocklength Regime

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***Abstract*—Wyner's soft-handoff network is considered where transmitters simultaneously send messages of enhanced mobile broadband (eMBB) and ultra-reliable low-latency communication (URLLC) services. Due to the low-latency requirements, the URLLC messages are transmitted over fewer channel uses compared to the eMBB messages. To improve the reliability of the URLLC transmissions, we propose a coding scheme with finite blocklength codewords that exploits dirty-paper coding (DPC) to precancel the interference from eMBB transmissions. Rigorous bounds are derived for the error probabilities of eMBB and URLLC transmissions achieved by our scheme. Numerical results illustrate that they are lower than for standard time-sharing.**

I. Introduction

The fifth and the forthcoming sixth generations of mobile communications have to accommodate both ultra-reliable and low-latency communication (URLLC) and enhanced mobile broadband (eMBB) services [1], [2]. URLLC services aim at guaranteeing high-reliability at a maximum end-to-end delay of 1ms and are used for delay-sensitive applications such as industrial control management as well as autonomous vehicle and remote surgery applications [2]. On the other hand, eMBB services aim to provide high data rates and are used for delay-tolerant applications such as video streaming, virtual and augmented reality applications [3].

The difference in the latency requirements of eMBB and URLLC services along with the fact that that they are scheduled in the same frequency band make their coexistence challenging. Networks with such mixed-delay constraints have been studied recently. See [4]–[8] for a comprehensive review on related works. The previous studies are mostly focused on the performance of such networks in the asymptotic regime where the number of channel uses goes to infinity. Since the URLLC delay constraint limits the number of available channel uses, the problem of joint coding of messages with heterogeneous blocklengths is of an increasing interest. Notably, for the Gaussian point-to-point channel with messages of heterogeneous decoding deadlines, the work in [9], proposes a coding scheme which exploits dirty-paper coding (DPC) [10], [11]. Accounting for finite decoding deadline constraints, rigorous bounds are derived on the achievable error probabilities of the messages. Their numerical results illustrate that their proposed scheme outperforms time sharing for a wide range of blocklengths. For the Gaussian broadcast channel with heterogeneous blocklength constraints, the work in [12], proposes a coding scheme which decodes the messages at time-instances that depend on the realizations of the random channel fading. The authors showed that significant improvements are possible over standard successive interference cancellation. In [13] achievable rates and latency of the early-decoding scheme in [12] are improved by introducing *concatenated shell codes*. Finally, [14] and [15] studied the uplink of the cloud radio access networks where URLLC messages are directly decoded at the base stations whereas decoding of eMBB messages can be delayed to the cloud center. In particular, [14] performs a hybrid analysis where URLLC transmissions are studied in the finite blocklength regime and eMBB transmissions in the asymptotic infinite blocklength regime.

In this paper, we consider Wyner's soft-handoff model with K interfering transmitters and receivers pairs. Each transmitter wishes to simultaneously transmit two messages of heterogeneous blocklengths: an URLLC message and an eMBB message. The URLLC message is transmitted over a shorter blocklength compared to the eMBB message. Txs can hold a conferencing communication that depends only on the eMBB messages but not on the URLLC messages. By exploiting the DPC principle in [10], [16], we propose a coding scheme to jointly transmit the URLLC and eMBB messages. Unlike [9], [12], [14], we consider that codebooks are generated randomly according to independent uniform distributions on the power-shell. Rigorous bounds are derived for achievable error probabilities of eMBB and URLLC transmissions. To this end, Gel'fand-Pinsker analysis techniques for finite blocklengths in [11] are combined with the multiple parallel channels approach in [17]. Numerical results illustrate that our proposed scheme significantly outperforms standard time-sharing.

II. Problem Setup

Consider Wyner's soft-handoff network with K transmitters (Txs) and K receivers (Rxs) that are aligned on two parallel lines so that each Tx k has two neighbours, Tx $k-1$ and Tx $k+1$, and each Rx k has two neighbours, Rx $k-1$ and Rx $k+1$. Define $\mathcal{K} := \{1, \ldots, K\}$. The signal transmitted by Tx $k \in \mathcal{K}$ is observed by Rx k and the neighboring Rx $k+1$. See Figure 1. Each Tx $k \in \mathcal{K}$ sends a so called *eMBB* type message

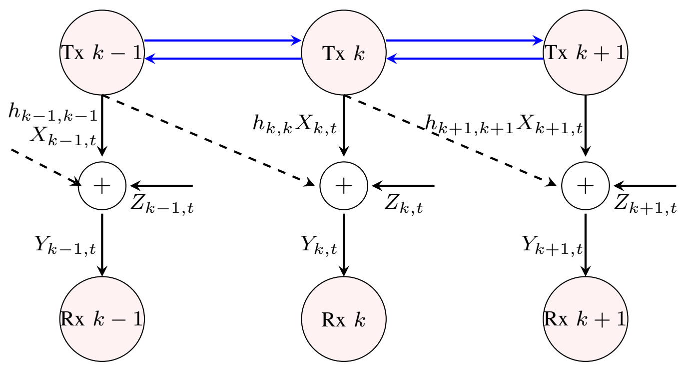

Fig. 1: System model.

$M_k^{(\mathsf{e})}$ to its corresponding Rx k, for $M_k^{(\mathsf{e})}$ uniformly distributed over $\mathcal{M}_k^{(\mathsf{e})} := \{1,\ldots,L_{\mathsf{e}}\}$. A subset of Txs $\mathcal{K}_{\mathsf{U}} \subset \mathcal{K}$ also sends additional *URLLC* messages $M_k^{(\mathsf{U})}$, for $k \in \mathcal{K}_{\mathsf{U}}$, for $M_k^{(\mathsf{U})}$ uniformly distributed over the set $\mathcal{M}^{(\mathsf{U})} := \{1,\ldots,L_{\mathsf{U}}\}$. We assume that

$$\mathcal{K}_{\mathsf{U}} := \{1,3,\ldots,K-1\}, \tag{1}$$

so that URLLC transmissions are only interfered by the eMBB transmissions but not by other URLLC transmissions. (The study of sets $\mathcal{K}_{\mathsf{U}}$ with interfering URLLC messages is left as a future research direction.)

Communication takes place in two phases.

Tx-cooperation phase:
The encoding starts with a first *Tx-cooperation phase* in which Txs share their eMBB message with their neighbouring Txs in $\mathcal{K}_{\mathsf{U}}$. (For example over high-rate optical fibers if the Txs are BSs.) The URLLC messages, which are subject to stringent delay constraints, are only generated after the Tx-cooperation phase, at the beginning of the subsequent *channel transmission phase*.

Channel transmission phase:
URLLC messages are transmitted over n_{U} channel uses and eMBB messages over $n_{\mathsf{e}} > n_{\mathsf{U}}$ channel uses. The blocklengths n_{U} and n_{e} are assumed to be fixed constants. Notice that while the transmission delay of URLLC messages is determined by the n_{U} channel uses, transmission delay of eMBB consists of both the delay of the Tx-cooperation phase as well as the delay induced by the n_{e} channel uses.

For each $k \in \mathcal{K}$, Tx k computes its time-t channel input $X_{k,t}$ with $t \in \{1,\ldots,n_e\}$ as

$$X_{k,t} = \begin{cases} f_k^{(b)}\left(M_k^{(\mathsf{U})}, M_k^{(\mathsf{e})}, T_{\ell\to k}\right), & k \in \mathcal{K}_{\mathsf{U}} \text{ and } t \le n_{\mathsf{U}} \\ f_k^{(e)}\left(M_k^{(\mathsf{e})}, T_{\ell\to k}\right), & k \notin \mathcal{K}_{\mathsf{U}} \text{ or } n_{\mathsf{U}} < t \le n_{\mathsf{e}}, \end{cases}$$

for each $\ell \in \{k-1,k+1\}$ and for some encoding functions $f_k^{(b)}$ and $f_k^{(e)}$ on appropriate domains satisfying the average block-power constraint

$$\frac{1}{n_{\mathsf{e}}}\sum_{t=1}^{n_{\mathsf{e}}} X_{k,t}^2 \le \mathsf{P}, \quad \forall\, k \in \mathcal{K}, \qquad \text{almost surely.} \tag{2}$$

The input-output relation of the network is described as

$$Y_{k,t} = h_{k,k}X_{k,t} + h_{k-1,k}X_{k-1,t} + Z_{k,t}, \tag{3}$$

where $\{Z_{k,t}\}$ are independent and identically distributed (i.i.d.) standard Gaussian for all k and t and independent of all messages; $h_{k,\ell} > 0$ is the fixed channel coefficient between Tx k and Rx ℓ; and we define $X_{0,t} = 0$ for all t.

After n_{U} channel uses, each Rx $k \in \mathcal{K}_{\mathsf{U}}$ decodes the URLLC message $M_k^{(\mathsf{U})}$ based on its own channel outputs $\boldsymbol{Y}_k^{n_{\mathsf{U}}} := \{Y_{k,1},\ldots,Y_{k,n_{\mathsf{U}}}\}$. So, it produces:

$$\hat{M}_k^{(\mathsf{U})} = g_k^{(n_{\mathsf{U}})}(\boldsymbol{Y}_k^{n_{\mathsf{U}}}), \tag{4}$$

for some decoding function $g_k^{(n_{\mathsf{U}})}$ on appropriate domains. The average error probability for each message $M_k^{(\mathsf{U})}$ is given by

$$\epsilon_{\mathsf{U},k} := \mathbb{P}\left\{\hat{M}_k^{(\mathsf{U})} \neq M_k^{(\mathsf{U})}\right\}, \quad \text{for} \quad k \in \mathcal{K}_{\mathsf{U}}. \tag{5}$$

After n_{e} channel uses, each Rx k decodes its desired eMBB messages as

$$\hat{M}_k^{(\mathsf{e})} = b_k^{(n_{\mathsf{e}})}\left(\boldsymbol{Y}_k^{n_{\mathsf{e}}}\right), \tag{6}$$

where $b_k^{(n_{\mathsf{e}})}$ is a decoding function on appropriate domains. The average error probability for message $M_k^{(\mathsf{e})}$ is given by

$$\epsilon_{\mathsf{e},k} := \mathbb{P}\left\{\hat{M}_k^{(\mathsf{e})} \neq M_k^{(\mathsf{e})}\right\}, \quad \text{for} \quad k \in \mathcal{K}. \tag{7}$$

We will be interested in the average URLLC and eMBB error probabilities

$$\epsilon_{\mathsf{U}} := \frac{1}{K}\sum_{k\in\mathcal{K}_{\mathsf{U}}} \epsilon_{\mathsf{U},k} \tag{8}$$

$$\epsilon_{\mathsf{e}} := \frac{1}{K}\sum_{k\in\mathcal{K}} \epsilon_{\mathsf{e},k}. \tag{9}$$

III. Coding Scheme

Txs in $\mathcal{K}_{\mathsf{U}}$ use DPC to precancel the interference of eMBB transmissions from their neighbouring transmissions and from their own eMBB transmissions on their URLLC transmissions. (Recall that during the Tx-cooperation rounds Txs in $\mathcal{K}_{\mathsf{U}}$ learn the eMBB messages of their neighbouring Txs.)

A. Encoding at Txs in $\mathcal{K}\backslash\mathcal{K}_U$

Each Tx $k \in \mathcal{K}\backslash\mathcal{K}_{\mathsf{U}}$ transmits only the eMBB message $M_k^{(\mathsf{e})}$ over the entire block of n_{e} channel uses. Over the first n_{U} channel uses, it transmits a codeword $\boldsymbol{X}_k^{(\mathsf{e},1)}(M_k^{(\mathsf{e})})$ that is uniformly distributed on the centered n_{U}-dimensional sphere of radius $\sqrt{n_{\mathsf{U}}\beta_{\mathsf{e}}\mathsf{P}}$, for some $\beta_{\mathsf{e}} \in [0,1]$, independently of all other codewords. Tx k also describes its message $M_k^{(\mathsf{e})}$, and thus its input signal $\boldsymbol{X}_k^{(\mathsf{e},1)}$, to the neigbouring Tx to its right during the only Tx-cooperation round.

To encode $M_k^{(\mathsf{e})}$ over the following $(n_{\mathsf{e}} - n_{\mathsf{U}})$ channel uses, Tx k employs a second codeword $\boldsymbol{X}_k^{(\mathsf{e},2)}(M_k^{(\mathsf{e})})$ that is uniformly distributed on the centered $(n_{\mathsf{e}} - n_{\mathsf{U}})$-dimensional sphere of radius $\sqrt{(n_{\mathsf{e}} - n_{\mathsf{U}})(1-\beta_{\mathsf{e}})\mathsf{P}}$, independently of all other codewords.

B. Encoding at Txs in $\mathcal{K}_U$

Each Tx $k \in \mathcal{K}_{\mathsf{U}}$ has both eMBB and URLLC messages to transmit. To transmit its URLLC message $M_k^{(\mathsf{U})}$, Tx k employs DPC encoding to precancel the interference of the eMBB transmission of the Tx to its left and its own eMBB transmission. Tx k transmits its URLLC message over only n_{U} channel uses whereas it sends its eMBB message over the entire block of n_{e} channel uses. To transmit both messages while satisfying (2), we divide the total transmit power P into three parts $\beta_{\mathsf{U}}\mathsf{P}$, $\beta_{\mathsf{e},1}\mathsf{P}$, $\beta_{\mathsf{e},2}\mathsf{P}$, where power $\beta_{\mathsf{U}}\mathsf{P}$ is used for URLLC transmission, power $\beta_{\mathsf{e},1}\mathsf{P}$ for eMBB transmission during the first n_{U} channel uses, and power $\beta_{\mathsf{e},2}\mathsf{P}$ for eMBB transmission during the last $n_{\mathsf{e}}-n_{\mathsf{U}}$ channel uses. The coefficients β_{U}, $\beta_{\mathsf{e},1}$, $\beta_{\mathsf{e},2} \in [0,1]$ are chosen such that

$$\beta_{\mathsf{U}} + \beta_{\mathsf{e},1} + \beta_{\mathsf{e},2} = 1. \tag{10}$$

Transmitting $M_k^{(\mathsf{e})}$ and $M_k^{(\mathsf{U})}$: Over the first n_{U} channel uses, Tx k sends its eMBB message $M_k^{(\mathsf{e})}$ jointly with its URLLC message $M_k^{(\mathsf{U})}$. To this end, it encodes $M_k^{(\mathsf{e})}$ using a codeword $\boldsymbol{X}_k^{(\mathsf{e},1)}(M_k^{(\mathsf{e})})$ that is uniformly distributed on the centered n_{U}-dimensional sphere of radius $\sqrt{n_{\mathsf{U}}\beta_{\mathsf{e},1}\mathsf{P}}$. To encode $M_k^{(\mathsf{U})}$, for each realization m of message $M_k^{(\mathsf{U})}$, $\lfloor 2^{n_{\mathsf{U}}R_{\mathsf{U}}}\rfloor$ codewords $\boldsymbol{V}_k(m,i)$, $i=1,\ldots,\lfloor 2^{n_{\mathsf{U}}R_{\mathsf{U}}}\rfloor$, are drawn uniformly from a centered n_{U}-dimensional sphere of radius $\sqrt{r_k n_{\mathsf{U}}\mathsf{P}}$ independently of each other and of all other codewords, where

$$r_k := \beta_{\mathsf{U}} + \alpha_{k,1}^2\beta_{\mathsf{e},1} + \alpha_{k,2}^2\beta_{\mathsf{e}}. \tag{11}$$

Tx k then chooses a codeword $\boldsymbol{V}_k(M_k^{(\mathsf{U})}, i)$ such that the sequence

$$\boldsymbol{X}_k^{(\mathsf{U})} := \boldsymbol{V}_k(M_k^{(\mathsf{U})}, i) - \alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} - \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)} \tag{12}$$

lies in the set

$$\mathcal{D}_k := \left\{ \boldsymbol{x}_k^{(\mathsf{U})} : n_{\mathsf{U}}\beta_{\mathsf{U}}\mathsf{P} - \delta_k \le \left\| \boldsymbol{x}_k^{(\mathsf{U})} \right\|^2 \le n_{\mathsf{U}}\beta_{\mathsf{U}}\mathsf{P} \right\} \tag{13}$$

for a given $\delta_k > 0$. If multiple such codewords exist, one of them is chosen at random, and if no appropriate codeword exists, an error is declared.

Over the first n_{U} channel uses, Tx k transmits

$$\boldsymbol{X}_k^{(\mathsf{U})} + \boldsymbol{X}_k^{(\mathsf{e},1)}. \tag{14}$$

Over the last $(n_{\mathsf{e}} - n_{\mathsf{U}})$ channel uses, Tx k simply encodes $M_k^{(\mathsf{e})}$ using a codeword $\boldsymbol{X}_k^{(\mathsf{e},2)}(M_k^{(\mathsf{e})})$ that is uniformly distributed on the centered $(n_{\mathsf{e}}-n_{\mathsf{U}})$-dimensional sphere of radius $\sqrt{(n_{\mathsf{e}}-n_{\mathsf{U}})\beta_{\mathsf{e},2}\mathsf{P}}$.

C. Decoding at Rxs in $\mathcal{K}\backslash\mathcal{K}_U$

Each Rx k in $\mathcal{K}\backslash\mathcal{K}_{\mathsf{U}}$ only has an eMBB message to decode. Rx $k \in \mathcal{K}\backslash\mathcal{K}_{\mathsf{U}}$ decomposes its channel outputs into two output blocks consisting of the first n_{U} and the last $(n_{\mathsf{e}}-n_{\mathsf{U}})$ channel uses, respectively. These blocks are of the form:

$$\boldsymbol{Y}_{k,1} = h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)} + h_{k-1,k}(\boldsymbol{X}_{k-1}^{(\mathsf{U})} + \boldsymbol{X}_{k-1}^{(\mathsf{e},1)}) + \boldsymbol{Z}_{k,1}, \tag{15a}$$

$$\boldsymbol{Y}_{k,2} = h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)} + h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},2)} + \boldsymbol{Z}_{k,2}, \tag{15b}$$

where $\boldsymbol{Z}_{k,1}$ and $\boldsymbol{Z}_{k,2}$ are independent i.i.d. standard Gaussian noise sequences. For $\boldsymbol{Y}_{k,1} = \boldsymbol{y}_{k,1}$ and $\boldsymbol{Y}_{k,2} = \boldsymbol{y}_{k,2}$, Rx k estimates $M_k^{(\mathsf{e})}$ as an index m for which the corresponding codewords $\boldsymbol{x}_k^{(\mathsf{e},1)}(m)$ and $\boldsymbol{x}_k^{(\mathsf{e},2)}(m)$ maximize the information density

$$i_1(\boldsymbol{x}_k^{(\mathsf{e},1)}, \boldsymbol{x}_k^{(\mathsf{e},2)}; \boldsymbol{y}_{k,1}, \boldsymbol{y}_{k,2}) := \ln \frac{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{X}_k^{(\mathsf{e},1)}}(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)}) f_{\boldsymbol{Y}_{k,2}|\boldsymbol{X}_k^{(\mathsf{e},2)}}(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})}{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1}) f_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})}, \tag{16}$$

among all codeword pairs $\boldsymbol{x}_k^{(\mathsf{e},1)} = \boldsymbol{x}_k^{(\mathsf{e},1)}(m')$ and $\boldsymbol{x}_k^{(\mathsf{e},2)} = \boldsymbol{x}_k^{(\mathsf{e},2)}(m')$.

D. Decoding at Rxs in $\mathcal{K}_U$

Similarly to the previous subsection, also Rxs in $\mathcal{K}_{\mathsf{U}}$ decompose their channel outputs into two output blocks consisting of the first n_{U} and the last $(n_{\mathsf{e}} - n_{\mathsf{U}})$ channel uses, respectively. For a Rx $k \in \mathcal{K}_{\mathsf{U}}$, these blocks are of the form:

$$\boldsymbol{Y}_{k,1} = h_{k,k}(\boldsymbol{X}_k^{(\mathsf{U})} + \boldsymbol{X}_k^{(\mathsf{e},1)}) + h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)} + \boldsymbol{Z}_{k,1}, \tag{17a}$$

$$\boldsymbol{Y}_{k,2} = h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)} + h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},2)} + \boldsymbol{Z}_{k,2}. \tag{17b}$$

where $\boldsymbol{Z}_{k,1}$ and $\boldsymbol{Z}_{k,2}$ are independent i.i.d. standard Gaussian noise sequences.

1) Decoding $M_k^{(\mathsf{U})}$: Rx k decodes $M_k^{(\mathsf{U})}$ based on the outputs of the first channel inputs $\boldsymbol{Y}_{k,1}$ defined in (17a). Rx k estimates $M_k^{(\mathsf{U})}$ as an index m for which the corresponding codeword $\boldsymbol{v}_k(m,i)$ maximizes the information density

$$i(\boldsymbol{v}_k; \boldsymbol{y}_{k,1}) := \ln \frac{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k}(\boldsymbol{y}_{k,1}|\boldsymbol{v}_k)}{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})}, \tag{18}$$

among all codewords $\boldsymbol{v}_k = \boldsymbol{v}_k(m', j)$.

2) Decoding $M_k^{(\mathsf{e})}$: Rx k decodes $M_k^{(\mathsf{e})}$ based on the channel outputs of the first and second channels $\boldsymbol{Y}_{k,1}$ and $\boldsymbol{Y}_{k,2}$ by looking for the index m for which the corresponding codewords $\boldsymbol{x}_k^{(\mathsf{e},1)}(m)$ and $\boldsymbol{x}_k^{(\mathsf{e},2)}(m)$ maximize the information density

$$i_2(\boldsymbol{x}_k^{(\mathsf{e},1)}, \boldsymbol{x}_k^{(\mathsf{e},2)}; \boldsymbol{y}_{k,1}, \boldsymbol{y}_{k,2}) := \ln \frac{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{X}_k^{(\mathsf{e},1)}}(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)}) f_{\boldsymbol{Y}_{k,2}|\boldsymbol{X}_k^{(\mathsf{e},2)}}(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})}{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1}) f_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})} \tag{19}$$

among all codeword pairs $\boldsymbol{x}_k^{(\mathsf{e},1)}(m')$ and $\boldsymbol{x}_k^{(\mathsf{e},2)}(m')$.

IV. Main Result

Fix $\beta_{\mathsf{e}}, \beta_{\mathsf{e},1}, \beta_{\mathsf{e},2}, \beta_{\mathsf{U}} \in [0,1]$ such that (10) is satisfied. Define

$$\sigma_1^2 := h_{k,k}^2\left(r_k + (1-\alpha_{k,1})^2\beta_{\mathsf{e},1}\right)\mathsf{P} + (h_{k-1,k} - h_{k,k}\alpha_{k,2})^2\beta_{\mathsf{e}}\mathsf{P} + 1, \tag{24a}$$

$$\sigma_2^2 := h_{k-1,k}^2\left(r_k + (1-\alpha_{k-1})^2\beta_{\mathsf{e},1}\right)\mathsf{P} + h_{k-1,k}^2\alpha_{k-1,2}^2\beta_{\mathsf{e}}\mathsf{P} + h_{k,k}^2\beta_{\mathsf{e}}\mathsf{P} + 1, \tag{24b}$$

$$\sigma_3^2 := \left(h_{k,k}^2(1-\beta_{\mathsf{e}}) + h_{k-1,k}^2\beta_{\mathsf{e},2}\right)\mathsf{P} + 1, \tag{24c}$$

$$\sigma_4^2 := \left(h_{k,k}^2\beta_{\mathsf{e},2} + h_{k-1,k}^2(1-\beta_{\mathsf{e}})\right)\mathsf{P} + 1, \tag{24d}$$

$$c_1 := \left(h_{k,k}\sqrt{\beta_{\mathsf{e}}} + h_{k-1,k}(\sqrt{\beta_{\mathsf{U}}} + \sqrt{\beta_{\mathsf{e},1}})\right), \tag{24e}$$

$$c_2 := \left(h_{k,k}(\sqrt{\beta_{\mathsf{U}}} + \sqrt{\beta_{\mathsf{e},1}}) + h_{k-1,k}\sqrt{\beta_{\mathsf{e}}}\right), \tag{24f}$$

$$c_3 := \left(h_{k,k}\sqrt{1-\beta_{\mathsf{e}}} + h_{k-1,k}\sqrt{\beta_{\mathsf{e},2}}\right), \tag{24g}$$

$$c_4 := \left(h_{k,k}\sqrt{\beta_{\mathsf{e},2}} + h_{k-1,k}\sqrt{(1-\beta_{\mathsf{e}})}\right). \tag{24h}$$

By employing the scheme proposed in Section III, we have the following theorem on the upper bounds on the average URLLC and eMBB error probabilities ϵ_{U} and ϵ_{e}.

Theorem 1: For fixed message set sizes L_{U} and L_{e}, the average error probabilities ϵ_{U} and ϵ_{e} are bounded by

$$\epsilon_{\mathsf{U}} \le \frac{1}{K}\sum_{k\in\mathcal{K}_{\mathsf{U}}}\Big(1 - F(u_{k,2}-u_{k,1}) + F(-u_{k,2}-u_{k,1}) + (1-\max\{\mathcal{L}_{k,1},\mathcal{L}_{k,2}\})^{\lfloor 2^{n_{\mathsf{U}}R_v}\rfloor}\Big) + L_{\mathsf{U}}\lfloor 2^{n_{\mathsf{U}}R_v}\rfloor e^{-\gamma_{\mathsf{U}}} \tag{25}$$

$$\epsilon_{\mathsf{e}} \le \frac{1}{K}\sum_{k\in\mathcal{K}\setminus\mathcal{K}_{\mathsf{U}}}\frac{1}{\bar{\gamma}_{\mathsf{e},1}}(\zeta_1 l_{k,1} + \zeta_2 l_{k,2} + n_{\mathsf{U}} l_3 + (n_{\mathsf{e}}-n_{\mathsf{U}})l_4) + \frac{1}{K}\sum_{k\in\mathcal{K}_{\mathsf{U}}}\frac{1}{\bar{\gamma}_{\mathsf{e},2}}(\zeta_1 d_{k,1} + \zeta_2 d_{k,2} + n_{\mathsf{U}} d_3 + (n_{\mathsf{e}}-n_{\mathsf{U}})d_4) + L_{\mathsf{e}}(e^{-\gamma_{\mathsf{e},1}} + e^{-\gamma_{\mathsf{e},2}}) \tag{26}$$

for any γ_{U}, $\gamma_{\mathsf{e},1}$ and $\gamma_{\mathsf{e},2}$, and where

$$u_{k,1} := \sqrt{n_{\mathsf{U}}\mathsf{P}}\left(c_2 + \frac{h_{k,k}\sqrt{r_k}\sigma_1^2}{\sigma_1^2-1}\right), \tag{27}$$

$$u_{k,2} := \sqrt{\frac{\sigma_1^2}{\sigma_1^2-1}\left(n_{\mathsf{U}}\ln(\sigma_1^2) - \frac{2\gamma_1}{J_{\mathsf{U}}} + \frac{n_{\mathsf{U}}\mathsf{P}r_k h_{k,k}^2}{\sigma_1^2}\right)}, \tag{28}$$

$$l_{k,1} := 2\sqrt{n_{\mathsf{U}}\mathsf{P}}\left(c_1 + h_{k,k}\sqrt{\beta_{\mathsf{e}}} - \frac{c_1}{\sigma_2^2}\right), \tag{29}$$

$$l_{k,2} := 2\sqrt{(n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}}\left(c_3 + h_{k,k}\sqrt{1-\beta_{\mathsf{e}}} - \frac{c_3}{\sigma_3^2}\right), \tag{30}$$

$$d_{k,1} := 2\sqrt{n_{\mathsf{U}}\mathsf{P}}\left(c_2 + h_{k,k}\sqrt{\beta_{\mathsf{e},1}} - \frac{c_2}{\sigma_1^2}\right), \tag{31}$$

$$d_{k,2} := 2\sqrt{(n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}}\left(c_4 + h_{k,k}\sqrt{\beta_{\mathsf{e},2}} - \frac{c_4}{\sigma_4^2}\right), \tag{32}$$

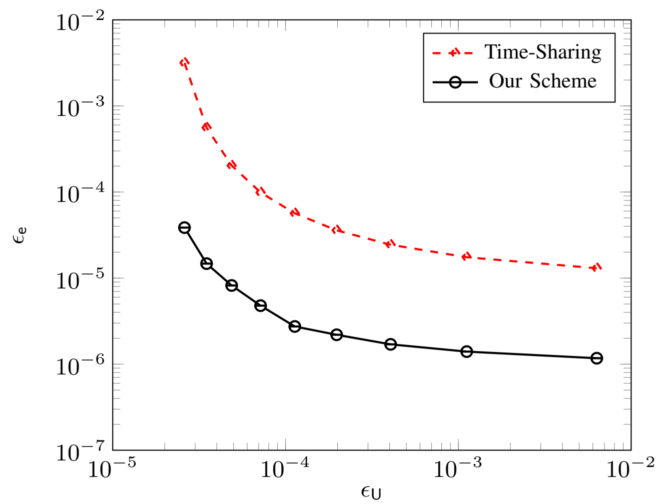

Fig. 2: ϵ_{e} vs ϵ_{U} for $\mathsf{P}=10, n_{\mathsf{e}}=100$ and n_{U} decreases from 90 to 10 with steps of 10.

and $l_3 := (\sigma_2^2-1)/\sigma_2^2$, $l_4 := (\sigma_3^2-1)/\sigma_3^2$, $d_3 := (\sigma_1^2-1)/\sigma_1^2$, $d_4 := (\sigma_4^2-1)/\sigma_4^2$, $\zeta_1 := \sqrt{2}\Gamma(\frac{n_{\mathsf{U}}+1}{2})/\Gamma(\frac{n_{\mathsf{U}}}{2})$, $\zeta_2 := \sqrt{2}\Gamma(\frac{n_{\mathsf{e}}-n_{\mathsf{U}}+1}{2})/\Gamma(\frac{n_{\mathsf{e}}-n_{\mathsf{U}}}{2})$, $\mathcal{L}_{k,1}$, $\mathcal{L}_{k,2}$, $\bar{\gamma}_{\mathsf{e},1}$, and $\bar{\gamma}_{\mathsf{e},2}$ are defined in (20) to (23), J_{U}, $J_{\mathsf{e},1}$ and $J_{\mathsf{e},2}$ are defined in (58), (78) and (96), and $F(\cdot)$ represents the cumulative distribution function (CDF) of a chi distribution of degree n_{U}.

Proof: See Appendix A. ■

In Figure 2, we numerically compare the bounds in Theorem 1 with the time-sharing scheme where only Txs in $\mathcal{K}_{\mathsf{U}}$ send URLLC messages over n_{U} channel uses whereas all the Txs in $\mathcal{K}$ send eMBB messages but over only the remaining $n_{\mathsf{e}}-n_{\mathsf{U}}$ channel uses. In this plot, the value of n_{U} varies from 90 to 10 with step size 10, while the value of n_{e} is fixed at 100. In our simulations, the values of the parameters β_{e}, β_{U}, $\beta_{\mathsf{e},1}$, $\beta_{\mathsf{e},2}$, $\alpha_{k,1}$ and $\alpha_{k,2}$ are optimized to minimize ϵ_{e} for a given ϵ_{U}. As can be seem from this figure, our scheme outperforms the time-sharing scheme.

V. Conclusions

We considered Wyner's soft-handoff model where transmitters simultaneously send eMBB and URLLC messages of heterogeneous blocklengths. We proposed a coding scheme to jointly transmit URLLC and eMBB messages in such

$$\mathcal{L}_{k,1} := \frac{\delta_k}{2n_{\mathsf{U}}\mathsf{P}\sqrt{\pi r_k}}\frac{\Gamma(\frac{n_{\mathsf{U}}}{2})}{\Gamma(\frac{n_{\mathsf{U}}-1}{2})}\frac{1}{\alpha_{k,1}\sqrt{\beta_{\mathsf{e},1}}}\left(1-\left(r_k-\beta_{\mathsf{U}}+\alpha_{k,2}\sqrt{\beta_{\mathsf{e}}}(\alpha_{k,1}\sqrt{\beta_{\mathsf{e},1}}+\sqrt{r_k})+\frac{\delta_k}{2n_{\mathsf{U}}\mathsf{P}}\right)^2/(\alpha_{k,1}^2 r_k\beta_{\mathsf{e},1})\right)^{\frac{n_{\mathsf{U}}-3}{2}}, \tag{20}$$

$$\mathcal{L}_{k,2} := \frac{\delta_k}{2n_{\mathsf{U}}\mathsf{P}\sqrt{\pi r_k}}\frac{\Gamma(\frac{n_{\mathsf{U}}}{2})}{\Gamma(\frac{n_{\mathsf{U}}-1}{2})}\frac{1}{\alpha_{k,2}\sqrt{\beta_{\mathsf{e}}}}\left(1-\left(r_k-\beta_{\mathsf{U}}+\alpha_{k,1}\sqrt{\beta_{\mathsf{e},1}}(\alpha_{k,2}\sqrt{\beta_{\mathsf{e}}}+\sqrt{r_k})+\frac{\delta_k}{2n_{\mathsf{U}}\mathsf{P}}\right)^2/(\alpha_{k,2}^2 r_k\beta_{\mathsf{e}})\right)^{\frac{n_{\mathsf{U}}-3}{2}}, \tag{21}$$

$$\bar{\gamma}_{\mathsf{e},1} := n_{\mathsf{U}}\left(\ln(\sigma_2^2)-\mathsf{P}\left(\left(c_1+h_{k,k}\sqrt{\beta_{\mathsf{e}}}\right)^2-\frac{c_1^2}{\sigma_2^2}\right)\right)+(n_{\mathsf{e}}-n_{\mathsf{U}})\left(\ln(\sigma_3^2)-\mathsf{P}\left(\left(c_3+h_{k,k}\sqrt{1-\beta_{\mathsf{e}}}\right)^2-\frac{c_3^2}{\sigma_3^2}\right)\right)-\frac{2\gamma_{\mathsf{e},1}}{J_{\mathsf{e},1}}, \tag{22}$$

$$\bar{\gamma}_{\mathsf{e},2} := n_{\mathsf{U}}\left(\ln(\sigma_1^2)-\mathsf{P}\left(\left(c_2+h_{k,k}\sqrt{\beta_{\mathsf{e},1}}\right)^2-\frac{c_2^2}{\sigma_1^2}\right)\right)+(n_{\mathsf{e}}-n_{\mathsf{U}})\left(\ln(\sigma_4^2)-\mathsf{P}\left(\left(c_4+h_{k,k}\sqrt{\beta_{\mathsf{e},2}}\right)^2-\frac{c_4^2}{\sigma_4^2}\right)\right)-\frac{2\gamma_{\mathsf{e},2}}{J_{\mathsf{e},2}}. \tag{23}$$

a network. We derived rigorous upper bounds on the error probability of eMBB and URLLC transmissions. Our numerical analysis showed that the proposed scheme significantly improves over the standard time-sharing.

An interesting future line of work is to study this network under the assumption that n_{e} is much larger than n_{U}. This assumption allows the eMBB transmissions to benefit from their delay-tolerance feature. Another interesting scenario is to let all the Txs send URLLC messages which requires dealing with the interference from the URLLC messages on the URLLC transmissions as well.

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Appendix A
Proof of Theorem 1

A. Bounding $\epsilon_{\mathsf{U},k}$

We start by bounding the decoding error probability of a URLLC message at a given Rx k in $\mathcal{K}_{\mathsf{U}}$. Define the decoding error event $\mathcal{E}_k^{(\mathsf{U})} := \{\hat{M}_k^{(\mathsf{U})} \neq M_k^{(\mathsf{U})}\}$ and let $\mathcal{E}_{k,v}$ be the encoding error event that no appropriate codeword $\boldsymbol{V}_k(M_k^{(\mathsf{U})}, i)$ can be found so that $\boldsymbol{X}_k^{(\mathsf{U})}(M_k^{(\mathsf{U})}) \in \mathcal{D}_k$. We have:

$$\epsilon_{U,k} \leq \mathbb{P}[\mathcal{E}_{k,v}] + \mathbb{P}[\mathcal{E}_k^{(\mathsf{U})}|\mathcal{E}_{k,v}]. \tag{33}$$

1) Analyzing $\mathbb{P}[\mathcal{E}_{k,v}]$*:* To calculate this probability, we follow a similar argument as in [11, Appendix E]. From (13) we notice that $(\boldsymbol{V}_k - \alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} - \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}) \in \mathcal{D}_k$ if and only if

$$n_{\mathsf{U}}\beta_{\mathsf{U}}\mathsf{P} - \delta_k \leq ||\boldsymbol{V}_k - \alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} - \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||^2 \leq n_{\mathsf{U}}\beta_{\mathsf{U}}\mathsf{P}.$$

Recall that $||\boldsymbol{V}_k||^2 = n_{\mathsf{U}} r_k \mathsf{P}$ almost surely. Thus event $\mathcal{E}_{k,v}$ holds whenever the following condition is violated:

$$\begin{aligned} & n_{\mathsf{U}}(r_k - \beta_{\mathsf{U}})\mathsf{P} + ||\alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} + \alpha_{k,2}\hat{\boldsymbol{X}}_{k-1}^{(\mathsf{e},1)}||^2 \\ & \leq 2\alpha_{k,1}\langle \boldsymbol{V}_k, \boldsymbol{X}_k^{(\mathsf{e},1)}\rangle + 2\alpha_{k,2}\langle \boldsymbol{V}_k, \boldsymbol{X}_{k-1}^{(\mathsf{e},1)}\rangle \\ & \leq n_{\mathsf{U}}(r_k - \beta_{\mathsf{U}})\mathsf{P} + ||\alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} + \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||^2 + \delta_k. \end{aligned} \tag{34}$$

Define

$$\begin{aligned} C_k := & \frac{n_{\mathsf{U}}(r_k - \beta_{\mathsf{U}})\mathsf{P}}{2\alpha_{k,1}} \\ & + \frac{||\alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} + \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||^2}{2\alpha_{k,1}} - \frac{\alpha_{k,2}}{\alpha_{k,1}}\langle \boldsymbol{V}_k, \boldsymbol{X}_{k-1}^{(\mathsf{e},1)}\rangle. \end{aligned} \tag{35}$$

Equation (34) then is equivalent to

$$C_k \leq \langle \boldsymbol{V}_k, \boldsymbol{X}_k^{(\mathsf{e},1)}\rangle \leq C_k + \frac{\delta_k}{2\alpha_{k,1}}. \tag{36}$$

Since $\boldsymbol{X}_k^{(\mathsf{e},1)}$ is drawn uniformly from the sphere, the distribution of $\langle \boldsymbol{V}_k, \boldsymbol{X}_k^{(\mathsf{e},1)}\rangle$ depends on $\boldsymbol{V}_k$ only through its magnitude, this is seen by noting that the inner product of two vectors is unchanged when an orthogonal transformation is applied to both arguments, and the distribution of $\boldsymbol{X}_k^{(\mathsf{e},1)}$ is unchanged under any orthogonal transformation. In the following we therefore assume that $\boldsymbol{V}_k = (||\boldsymbol{V}_k||, 0, \ldots, 0)$, in which case (36) is equivalent to:

$$\frac{C_k}{||\boldsymbol{V}_k||} \leq X_{k,1}^{(\mathsf{e},1)} \leq \frac{C_k}{||\boldsymbol{V}_k||} + \frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||} \tag{37}$$

where $X_{k,1}^{(\mathsf{e},1)}$ is the first entry of the vector $\boldsymbol{X}_k^{(\mathsf{e},1)}$. We conclude that $\mathbb{P}[\boldsymbol{V}_k - \alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} - \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)} \in \mathcal{D}_k]$ is lower bounded by the probability of the first entry of $\boldsymbol{X}_k^{(\mathsf{e},1)}$ falling into the interval of length $\frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||}$ that starts at $\frac{C_k}{||\boldsymbol{V}_k||}$. (Notice that the length of the interval is deterministic because $\|\boldsymbol{V}_k\| = \sqrt{n_{\mathsf{U}} r_k \mathsf{P}}$ is a constant, but its starting point is random because C_k is a random variable.

The distribution of a given symbol in a length-n_{U} random sequence distributed uniformly on the sphere is [19]

$$\begin{aligned} f_{X_{k,1}^{(\mathsf{e},1)}}(x_{k,1}^{(\mathsf{e},1)}) = & \frac{1}{\sqrt{\pi n_{\mathsf{U}}\beta_{\mathsf{e},1}\mathsf{P}}} \frac{\Gamma(\frac{n_{\mathsf{U}}}{2})}{\Gamma(\frac{n_{\mathsf{U}}-1}{2})} \left(1 - \frac{(x_{k,1}^{(\mathsf{e},1)})^2}{n_{\mathsf{U}}\beta_{\mathsf{e},1}\mathsf{P}}\right)^{\frac{n_{\mathsf{U}}-3}{2}} \\ & \times \mathbb{1}\{(x_{k,1}^{(\mathsf{e},1)})^2 \leq n_{\mathsf{U}}\beta_{\mathsf{e},1}\mathsf{P}\}. \end{aligned} \tag{38}$$

This density function is decreasing in $(x_{k,1}^{(\mathsf{e},1)})^2$, which implies that

$$\begin{aligned} & \mathbb{P}[\boldsymbol{V}_k - \alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)} - \alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)} \in \mathcal{D}_k] \\ & \geq \frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||} f_{X_{k,1}^{(\mathsf{e},1)}}\left(\frac{C_k}{||\boldsymbol{V}_k||} + \frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||}\right). \end{aligned} \tag{39}$$

Furthermore by the Cauchy-Schwartz inequality:

$$\begin{aligned} C_k \leq & \frac{n_{\mathsf{U}}(\alpha_{k,1}^2\beta_{\mathsf{e},1} + \alpha_{k,2}^2\beta_{\mathsf{e}})\mathsf{P}}{2\alpha_{k,1}} \\ & + \frac{\alpha_{k,1}^2||\boldsymbol{X}_k^{(\mathsf{e},1)}||^2 + \alpha_{k,2}^2||\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||^2}{2\alpha_{k,1}} \\ & + \frac{2\alpha_{k,1}\alpha_{k,2}||\boldsymbol{X}_k^{(\mathsf{e},1)}|| \cdot ||\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||}{2\alpha_{k,1}} \\ & + \frac{\alpha_{k,2}}{\alpha_{k,1}}||\boldsymbol{V}_k|| \cdot ||\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||. \end{aligned} \tag{40}$$

Thus

$$\frac{C_k}{||\boldsymbol{V}_k||} \leq A_k, \tag{41}$$

where

$$A_k := \sqrt{n_{\mathsf{U}}\mathsf{P}} \left(\frac{r_k - \beta_{\mathsf{U}} + (\alpha_{k,2}\sqrt{\beta_{\mathsf{e}}} + \alpha_{k,1}\sqrt{\beta_{\mathsf{e},1}})^2}{2\alpha_{k,1}\sqrt{r_k}}\right)$$

$$+\frac{\alpha_{k,2}\sqrt{n_{\mathsf{U}}\mathsf{P}\beta_{\mathsf{e}}}}{\alpha_{k,1}}. \tag{42}$$

Therefore

$$\mathbb{P}[\boldsymbol{V}_k-\alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)}-\alpha_{k,2}\hat{\boldsymbol{X}}_{k-1}^{(\mathsf{e},1)}\in\mathcal{D}_k] \geq \frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||}f_{X_{k,1}^{(\mathsf{e},1)}}\left(A_k+\frac{\delta_k}{2\alpha_{k,1}||\boldsymbol{V}_k||}\right). \tag{43}$$

In a similar way, in (34), one can move the term $2\alpha_{k,1}\langle\boldsymbol{V}_k,\boldsymbol{X}_k^{(\mathsf{e},1)}\rangle$ to both sides and bound the above probability by the probability of the first entry of $\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}$ falling within a given interval. This leads to an equivalent bound, which combined with (43) yields:

$$\mathbb{P}[\boldsymbol{V}_k-\alpha_{k,1}\boldsymbol{X}_k^{(\mathsf{e},1)}-\alpha_{k,2}\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}\in\mathcal{D}_k]\geq\max\{\mathcal{L}_1,\mathcal{L}_2\}, \tag{44}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined in (20) and (21), respectively. Since the $\lfloor 2^{n_{\mathsf{U}}R_v}\rfloor$ codewords are generated independently, thus

$$\mathbb{P}[\mathcal{E}_{k,v}]\leq(1-\max\{\mathcal{L}_1,\mathcal{L}_2\})^{\lfloor 2^{n_{\mathsf{U}}R_v}\rfloor}. \tag{45}$$

2) Analyzing $\mathbb{P}[\mathcal{E}_k^{(\mathsf{U})}|\mathcal{E}_{k,v}]$*:* To evaluate this error event, we use the threshold bound for maximum-metric decoding. I.e.,

$$\mathbb{P}[\mathcal{E}_k^{(\mathsf{U})}|\mathcal{E}_{k,v}]\leq\mathbb{P}[i(\boldsymbol{V}_k;\boldsymbol{Y}_{k,1})\leq\gamma_{\mathsf{U}}] +M_{\mathsf{U}}\lfloor 2^{n_{\mathsf{U}}R_v}\rfloor\cdot\mathbb{P}[i(\bar{\boldsymbol{V}}_k;\boldsymbol{Y}_{k,1})>\gamma_{\mathsf{U}}] \tag{46}$$

for any γ_{U}, where $\bar{\boldsymbol{V}}_k\sim f_{\boldsymbol{V}_k}$ and is independent of $(\boldsymbol{V}_k,\boldsymbol{Y}_{k,1})$. We start by calculating $\mathbb{P}[i(\bar{\boldsymbol{V}}_k,\boldsymbol{Y}_{k,1})>\gamma_{\mathsf{U}}]$. By Bayes rule we have

$$f_{\boldsymbol{V}_k}(\bar{\boldsymbol{v}}_k)=\frac{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})}{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k}(\boldsymbol{y}_{k,1}|\bar{\boldsymbol{v}}_k)} \tag{47}$$

$$=f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})\exp\left(-i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})\right). \tag{48}$$

By multiplying both sides of the above equation by $\mathbb{1}\{i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}\}$ and integrating over all $\bar{\boldsymbol{v}}_k$, we have

$$\int_{\bar{\boldsymbol{v}}_k}\mathbb{1}\{i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}\}f_{\boldsymbol{V}_k}(\bar{\boldsymbol{v}}_k)d\bar{\boldsymbol{v}}_k= \int_{\bar{\boldsymbol{v}}_k}\mathbb{1}\{i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}\}e^{-i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})}f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})d\bar{\boldsymbol{v}}_k \tag{49}$$

Note that the left-hand side of (49) is equivalent to $\mathbb{P}[i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}|\boldsymbol{Y}_{k,1}=\boldsymbol{y}_{k,1}]$. Thus

$$\mathbb{P}[i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}|\boldsymbol{Y}_{k,1}=\boldsymbol{y}_{k,1}] \tag{50}$$

$$=\int_{\bar{\boldsymbol{v}}_k}\mathbb{1}\{i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})>\gamma_{\mathsf{U}}\} \times\exp\left(-i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})\right)f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})d\bar{\boldsymbol{v}}_k \tag{51}$$

$$=\int_{\bar{\boldsymbol{v}}_k}\mathbb{1}\{\frac{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k}(\boldsymbol{y}_{k,1}|\bar{\boldsymbol{v}}_k)}{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})}e^{-\gamma_{\mathsf{U}}}>1\} \times\exp\left(-i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})\right)f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})d\bar{\boldsymbol{v}}_k \tag{52}$$

$$\leq\int_{\bar{\boldsymbol{v}}_k}\frac{f_{\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k}(\boldsymbol{y}_{k,1}|\bar{\boldsymbol{v}}_k)}{f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})}e^{-\gamma_{\mathsf{U}}} \times\exp\left(-i(\bar{\boldsymbol{v}}_k,\boldsymbol{y}_{k,1})\right)f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})d\bar{\boldsymbol{v}}_k \tag{53}$$

$$=\int_{\bar{\boldsymbol{v}}_k}e^{-\gamma_{\mathsf{U}}}f_{\boldsymbol{V}_k|\boldsymbol{Y}_{k,1}}(\bar{\boldsymbol{v}}_k|\boldsymbol{y}_{k,1})d\bar{\boldsymbol{v}}_k \tag{54}$$

$$\leq e^{-\gamma_{\mathsf{U}}}. \tag{55}$$

Now we calculate $\mathbb{P}[i(\boldsymbol{V}_k,\boldsymbol{Y}_{k,1})\leq\gamma_{\mathsf{U}}]$. Note that $\boldsymbol{Y}_{k,1}$ and $\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k$ do not follow a Gaussian distribution. Now define $Q^{(\mathsf{U})}(\boldsymbol{y}_{k,1})=\mathcal{N}(\boldsymbol{y}_{k,1};\boldsymbol{0},I_n\sigma_1^2)$ and $W^{(\mathsf{U})}(\boldsymbol{y}_{k,1}|\boldsymbol{v}_k)=\mathcal{N}(\boldsymbol{y}_{k,1};h_{k,k}\boldsymbol{V}_k,I_n\sigma_{y|v}^2)$ where σ_1^2 is defined in (24a) and $\sigma_{y|v}^2=1$.

Introduce

$$\tilde{i}(\boldsymbol{v}_k;\boldsymbol{y}_{k,1}):=\ln\frac{W^{(\mathsf{U})}(\boldsymbol{y}_{k,1}|\boldsymbol{v}_k)}{Q^{(\mathsf{U})}(\boldsymbol{y}_{k,1})}. \tag{56}$$

Lemma 1: We can prove that

$$\frac{i(\boldsymbol{v}_k;\boldsymbol{y}_{k,1})}{\tilde{i}(\boldsymbol{v}_k;\boldsymbol{y}_{k,1})}\geq J_{\mathsf{U}}, \tag{57}$$

where

$$J_{\mathsf{U}}:=(n_{\mathsf{U}}-2)\ln(2a_1a_2) -2n_{\mathsf{U}}\mathsf{P}(a_1^2\beta_{\mathsf{e},1}+a_2^2\beta_{\mathsf{e}})-\frac{e^{c_\Gamma}a_2^2\beta_{\mathsf{e}}\mathsf{P}}{\sqrt{2\pi a_1^2\beta_{\mathsf{e},1}\mathsf{P}}}-\kappa \tag{58}$$

and $a_1:=h_{k,k}(1-\alpha_{k,1})$, $a_2:=h_{k-1,k}-h_{k,k}\alpha_{k,2}$, $\kappa:=\ln(\frac{1}{2})+c_\Gamma+\ln(\sqrt{\frac{\pi}{8}})-2\ln(h_{k,k})$ with $c_\Gamma\leq 2$.

Proof: We use Lemma 7 of Appendix B to upper bound $f_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})/Q^{(\mathsf{U})}(\boldsymbol{y}_{k,1})$ and Lemma 6 of Appendix B to lower bound $f_{\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k}(\boldsymbol{y}_{k,1}|\boldsymbol{v}_k)/W^{(\mathsf{U})}(\boldsymbol{y}_{k,1}|\boldsymbol{v}_k)$. ■

As a result, we have

$$\mathbb{P}[i(\boldsymbol{V}_k;\boldsymbol{Y}_{k,1})\leq\gamma_{\mathsf{U}})] \tag{59}$$

$$\leq\mathbb{P}[\tilde{i}(\boldsymbol{V}_k;\boldsymbol{Y}_{k,1})\leq\frac{\gamma_{\mathsf{U}}}{J_{\mathsf{U}}}] \tag{60}$$

$$=\mathbb{P}\left[\ln\frac{W^{(\mathsf{U})}(\boldsymbol{Y}_{k,1}|\boldsymbol{V}_k)}{Q^{(\mathsf{U})}(\boldsymbol{Y}_{k,1})}\leq\frac{\gamma_{\mathsf{U}}}{J_{\mathsf{U}}}\right] \tag{61}$$

$$=\mathbb{P}\left[\ln\frac{\frac{1}{(\sqrt{2\sigma_{y|v}^2\pi})^{n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,1}-h_{k,k}\boldsymbol{V}_k||^2}{2\sigma_{y|v}^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_1^2})^{n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,1}||^2}{2\sigma_1^2}\right)}\leq\frac{\gamma_{\mathsf{U}}}{J_{\mathsf{U}}}\right] \tag{62}$$

$$=\mathbb{P}\Bigg[h_{k,k}^2||\boldsymbol{X}_k^{(\mathsf{U})}||^2+h_{k,k}^2||\boldsymbol{X}_k^{(\mathsf{e},1)}||^2+h_{k-1,k}^2||\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}||^2 +||\boldsymbol{Z}_{k,1}||^2+2h_{k,k}^2\langle\boldsymbol{X}_k^{(\mathsf{U})},\boldsymbol{X}_k^{(\mathsf{e},1)}\rangle +2h_{k,k}h_{k-1,k}\langle\boldsymbol{X}_k^{(\mathsf{U})},\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}\rangle+2h_{k,k}\langle\boldsymbol{X}_k^{(\mathsf{U})},\boldsymbol{Z}_{k,1}\rangle +2h_{k,k}h_{k-1,k}\langle\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}\rangle+2h_{k,k}\langle\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{Z}_{k,1}\rangle +2h_{k-1,k}\langle\boldsymbol{X}_{k-1}^{(\mathsf{e},1)},\boldsymbol{Z}_{k,1}\rangle-\frac{2h_{k,k}^2\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}\langle\boldsymbol{X}_k^{(\mathsf{U})},\boldsymbol{V}_k\rangle -\frac{2h_{k,k}^2\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}\langle\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{V}_k\rangle-\frac{2h_{k,k}\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}\langle\boldsymbol{Z}_{k,1},\boldsymbol{V}_k\rangle -\frac{2h_{k-1,k}h_{k,k}\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}\langle\boldsymbol{X}_{k-1}^{(\mathsf{e},1)},\boldsymbol{V}_k\rangle\geq\tilde{\gamma}_{\mathsf{U}}\Bigg] \tag{63}$$

$$\leq\mathbb{P}\Bigg[h_{k,k}^2n_{\mathsf{U}}\beta_{\mathsf{U}}\mathsf{P}+||\boldsymbol{Z}_{k,1}||^2 +2h_{k,k}^2n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{U}}\beta_{\mathsf{e},1}}+2h_{k,k}h_{k-1,k}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{U}}\beta_{\mathsf{e}}}$$

$$+2h_{k,k}\sqrt{n_{\mathsf{U}}\mathsf{P}\beta_{\mathsf{U}}}\cdot||\boldsymbol{Z}_{k,1}||+2h_{k,k}h_{k-1,k}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{e},1}\beta_{\mathsf{e}}}$$
$$+2h_{k,k}\sqrt{n_{\mathsf{U}}\mathsf{P}\beta_{\mathsf{e},1}}||\boldsymbol{Z}_{k,1}||+2h_{k-1,k}\sqrt{n_{\mathsf{U}}\mathsf{P}\beta_{\mathsf{e}}}||\boldsymbol{Z}_{k,1}||$$
$$+\frac{2h_{k,k}^2\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{U}}r_k}+\frac{2h_{k,k}^2\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{e},1}r_k}$$
$$+\frac{2h_{k,k}\sigma_1^2}{\sigma_1^2-\sigma_{y|v}^2}\left(h_{k-1,k}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{e}}r_k}+\sqrt{n_{\mathsf{U}}\mathsf{P}r_k}||\boldsymbol{Z}_{k,1}||\right)$$
$$\left.\geq\tilde{\gamma}_{\mathsf{U}}-n_{\mathsf{U}}\mathsf{P}(h_{k,k}^2\beta_{\mathsf{e},1}+h_{k-1,k}^2\beta_{\mathsf{e}})\right] \tag{64}$$
$$=\mathbb{P}\left[||\boldsymbol{Z}_{k,1}||^2+b_1||\boldsymbol{Z}_{k,1}||\geq\bar{\gamma}_{\mathsf{U}}\right] \tag{65}$$
$$=\mathbb{P}\left[\left(||\boldsymbol{Z}_{k,1}||+\frac{b_1}{2}\right)^2\geq\bar{\gamma}_{\mathsf{U}}+\frac{b_1^2}{4}\right] \tag{66}$$
$$=1-F\left(\sqrt{\bar{\gamma}_{\mathsf{U}}+\frac{b_1^2}{4}}-\frac{b_1}{2}\right)+F\left(-\sqrt{\bar{\gamma}_{\mathsf{U}}+\frac{b_1^2}{4}}-\frac{b_1}{2}\right) \tag{67}$$

where

$$\tilde{\gamma}_{\mathsf{U}}:=\frac{\sigma_1^2}{\sigma_1^2-1}\left(n_{\mathsf{U}}\ln(\sigma_1^2)-\frac{2\gamma_1}{J_{\mathsf{U}}}-h_{k,k}^2r_kn_{\mathsf{U}}\mathsf{P}\right), \tag{68a}$$
$$\bar{\gamma}_{\mathsf{U}}:=\tilde{\gamma}_{\mathsf{U}}-n_{\mathsf{U}}\mathsf{P}c_2^2-\frac{2h_{k,k}\sigma_1^2n_{\mathsf{U}}\mathsf{P}\sqrt{r_k}}{\sigma_1^2-1}c_2, \tag{68b}$$
$$b_1:=2\sqrt{n_{\mathsf{U}}\mathsf{P}}\left(c_2+\frac{h_{k,k}\sqrt{r_k}\sigma_1^2}{\sigma_1^2-1}\right), \tag{68c}$$

where c_2 is defined in (24f). Note that $||\boldsymbol{Z}_{k,1}||$ follows a chi distribution of degree n_{U} and $F(\cdot)$ is its corresponding CDF. By defining $u_1:=\frac{b_1}{2}$ and $u_2:=\sqrt{\bar{\gamma}_{\mathsf{U}}+\frac{b_1^2}{4}}$ and combining this bound with the bound in (55), we prove the upper bound in (25).

B. Bounding ϵ_{e}

Define the decoding error event $\mathcal{E}_{k,1}^{(\mathsf{e})}:=\{\hat{M}_k^{(\mathsf{e})}\neq M_k^{(\mathsf{e})}\}$ for $k\in\mathcal{K}\backslash\mathcal{K}_{\text{U}}$ and the decoding error event $\mathcal{E}_{k,2}^{(\mathsf{e})}:=\{\hat{M}_k^{(\mathsf{e})}\neq M_k^{(\mathsf{e})}\}$ for $k\in\mathcal{K}_{\text{U}}$. The average error probability of decoding eMBB messages over the K Tx/Rx pairs is given by

$$\epsilon_{\mathsf{e}}=\frac{1}{K}\left(\sum_{k\in\mathcal{K}\backslash\mathcal{K}_{\mathsf{U}}}\mathbb{P}[\mathcal{E}_{k,1}^{(\mathsf{e})}]+\sum_{k\in\mathcal{K}_{\mathsf{U}}}\mathbb{P}[\mathcal{E}_{k,2}^{(\mathsf{e})}]\right) \tag{69}$$

1) Analyzing $\mathbb{P}[\mathcal{E}_{k,1}^{(\mathsf{e})}]$: To evaluate this error event, we use the threshold bound for maximum-metric decoding. I.e.

$$\mathbb{P}[\mathcal{E}_{k,1}^{(\mathsf{e})}]\leq\mathbb{P}[i_1(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},1}]$$
$$+M_{\mathsf{e}}\mathbb{P}[i_1(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},1}] \tag{70}$$

for any $\gamma_{\mathsf{e},1}$, where $\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)}\sim f_{\boldsymbol{X}_k^{(\mathsf{e},1)}}$ and $\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)}\sim f_{\boldsymbol{X}_k^{(\mathsf{e},2)}}$ and are independent of $(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)},\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})$. To calculate $\mathbb{P}[i_1(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},1}]$ we follow similar steps as in (47)-(55) and show that

$$\mathbb{P}[i_1(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},1}]\leq e^{-\gamma_{\mathsf{e},1}}. \tag{71}$$

Now to calculate $\mathbb{P}[i_1(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},1}]$ we first define the following distributions:

$$Q_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})\sim\mathcal{N}(\boldsymbol{y}_{k,1};0,\sigma_2^2I_{n_{\mathsf{U}}}) \tag{72}$$
$$Q_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})\sim\mathcal{N}(\boldsymbol{y}_{k,2};0,\sigma_3^2I_{n_{\mathsf{U}}}) \tag{73}$$
$$W(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)})\sim\mathcal{N}(\boldsymbol{y}_{k,1};h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)},\tilde{\sigma}_2^2I_{n_{\mathsf{e}}-n_{\mathsf{U}}}) \tag{74}$$
$$W(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})\sim\mathcal{N}(\boldsymbol{y}_{k,2};h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)},\tilde{\sigma}_3^2I_{n_{\mathsf{e}}-n_{\mathsf{U}}}), \tag{75}$$

where σ_2^2 and σ_3^2 are defined in (24b), (24c) and $\tilde{\sigma}_2^2=1$ and $\tilde{\sigma}_3^2=1$. Introduce

$$\tilde{i}_1(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2}):=\ln\frac{W(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)})W(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})}{Q_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})Q_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})}. \tag{76}$$

Lemma 2: It can be shown that

$$\frac{i_1(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2})}{\tilde{i}_1(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2})}\geq J_{\mathsf{e},1} \tag{77}$$

where

$$J_{\mathsf{e},1}:=\frac{3(n_{\mathsf{U}}-2)}{2}\ln(2)+(n_{\mathsf{U}}-2)\ln(h_{k-1,k}a_1a_2)$$
$$-\ \frac{3n_{\mathsf{U}}\mathsf{P}}{2}(h_{k-1,k}^2r_k+a_1^2\beta_{\mathsf{e},1}+a_2^2\beta_{\mathsf{e}})$$
$$+\ (n_{\mathsf{e}}-n_{\mathsf{U}}-2)\ln(2h_{k,k}h_{k-1,k})$$
$$-\ (n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}(h_{k,k}^2(1-\beta_{\mathsf{e}})+h_{k-1,k}^2\beta_{\mathsf{e},2})$$
$$-\ \frac{e^{c_\Gamma}\sqrt{\mathsf{P}}}{\sqrt{2\pi}}\left(\frac{a_1^2\beta_{\mathsf{e},1}}{\sqrt{h_{k-1,k}^2r_k}}+\frac{h_{k,k}^2\beta_{\mathsf{e}}}{\sqrt{a_2^2\beta_{\mathsf{e}}}}+\frac{h_{k-1,k}^2\beta_{\mathsf{e},2}}{\sqrt{h_{k,k}^2(1-\beta_{\mathsf{e}})}}\right) \tag{78}$$

and $a_1:=h_{k-1,k}(1-\alpha_{k-1,1})$, $a_2:=-h_{k-1,k}\alpha_{k-1,2}$ with $c_\Gamma\leq 2$.

Proof: Similar to the proof of Lemma 1. ■

As a result of the above lemma, we have

$$\mathbb{P}\left[i_1(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},1})\right] \tag{79}$$
$$\leq\mathbb{P}[\tilde{i}_1(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\frac{\gamma_{\mathsf{e},1}}{J_{\mathsf{e},1}}] \tag{80}$$
$$=\mathbb{P}\left[\ln\frac{W(\boldsymbol{Y}_{k,1}|\boldsymbol{X}_k^{(\mathsf{e},1)})W(\boldsymbol{Y}_{k,2}|\boldsymbol{X}_k^{(\mathsf{e},2)})}{Q_{\boldsymbol{Y}_{k,1}}(\boldsymbol{Y}_{k,1})Q_{\boldsymbol{Y}_{k,2}}(\boldsymbol{Y}_{k,2})}\leq\frac{\gamma_{\mathsf{e},1}}{J_{\mathsf{e},1}}\right] \tag{81}$$
$$=\mathbb{P}\left[\ln\frac{\frac{1}{(\sqrt{2\tilde{\sigma}_2^2\pi})^{n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,1}-h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)}||^2}{2\tilde{\sigma}_2^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_2^2})^{n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,1}||^2}{2\sigma_2^2}\right)}\right.$$
$$\left.+\ln\frac{\frac{1}{(\sqrt{2\tilde{\sigma}_3^2\pi})^{n_{\mathsf{e}}-n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,2}-h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)}||^2}{2\tilde{\sigma}_3^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_3^2})^{n_{\mathsf{e}}-n_{\mathsf{U}}}}\exp\left(-\frac{||\boldsymbol{Y}_{k,2}||^2}{2\sigma_3^2}\right)}\leq\frac{\gamma_{\mathsf{e},1}}{J_{\mathsf{e},1}}\right]$$
$$=\mathbb{P}\left[\frac{\sigma_2^2-\tilde{\sigma}_2^2}{\sigma_2^2\tilde{\sigma}_2^2}||h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)}+h_{k-1,k}(\boldsymbol{X}_{k-1}^{(\mathsf{U})}+\boldsymbol{X}_{k-1}^{(\mathsf{e},1)})||^2\right.$$
$$+\frac{\sigma_2^2-\tilde{\sigma}_2^2}{\sigma_2^2\tilde{\sigma}_2^2}||\boldsymbol{Z}_{k,1}||^2+\frac{h_{k,k}^2}{\tilde{\sigma}_2^2}||\boldsymbol{X}_k^{(\mathsf{e},1)}||^2$$

$$+2\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}\left\langle \boldsymbol{Z}_{k,1}, h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)}\right\rangle$$
$$+2\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}\left\langle \boldsymbol{Z}_{k,1}, h_{k-1,k}(\boldsymbol{X}_{k-1}^{(\mathsf{U})}+\boldsymbol{X}_{k-1}^{(\mathsf{e},1)})\right\rangle$$
$$-\frac{2h_{k,k}}{\tilde\sigma_2^2}\left\langle h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)}+\boldsymbol{Z}_{k,1}, \boldsymbol{X}_k^{(\mathsf{e},1)}\right\rangle$$
$$-\frac{2h_{k,k}}{\tilde\sigma_2^2}\left\langle h_{k-1,k}(\boldsymbol{X}_{k-1}^{(\mathsf{U})}+\boldsymbol{X}_{k-1}^{(\mathsf{e},1)}), \boldsymbol{X}_k^{(\mathsf{e},1)}\right\rangle$$
$$+\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}\left(||h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)}+h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},2)}||^2\right)$$
$$+2\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}\left\langle \boldsymbol{Z}_{k,2}, h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)}+h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},2)}\right\rangle$$
$$-\frac{2h_{k,k}}{\tilde\sigma_3^2}\left\langle h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)}+h_{k-1,k}\boldsymbol{X}_{k-1}^{(\mathsf{e},2)}+\boldsymbol{Z}_{k,2}, \boldsymbol{X}_k^{(\mathsf{e},2)}\right\rangle$$
$$\left.+\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}||\boldsymbol{Z}_{k,2}||^2+\frac{h_{k,k}^2}{\tilde\sigma_3^2}||\boldsymbol{X}_k^{(\mathsf{e},2)}||^2\geq\tilde\gamma_{\mathsf{e},1}\right] \quad (82)$$
$$\leq\mathbb{P}\left[\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}n_{\mathsf{U}}\mathsf{P}\left(h_{k,k}^2\beta_{\mathsf{e}}+h_{k-1,k}^2(\beta_{\mathsf{U}}+\beta_{\mathsf{e},1})\right)\right.$$
$$+2\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}n_{\mathsf{U}}\mathsf{P}h_{k,k}h_{k-1,k}\left(\sqrt{\beta_{\mathsf{e}}\beta_{\mathsf{U}}}+\sqrt{\beta_{\mathsf{e}}\beta_{\mathsf{e},1}}\right)$$
$$+2\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}n_{\mathsf{U}}\mathsf{P}h_{k-1,k}^2\sqrt{\beta_{\mathsf{U}}\beta_{\mathsf{e},1}}+\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}||\boldsymbol{Z}_{k,1}||^2$$
$$+||\boldsymbol{Z}_{k,1}||\sqrt{n_{\mathsf{U}}\mathsf{P}}\left(2\frac{\sigma_2^2-\tilde\sigma_2^2}{\sigma_2^2\tilde\sigma_2^2}c_1+\frac{2h_{k,k}}{\tilde\sigma_2^2}\sqrt{\beta_{\mathsf{e}}}\right)$$
$$+\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}(n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}c_3^2+\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}||\boldsymbol{Z}_{k,2}||^2$$
$$+2||\boldsymbol{Z}_{k,2}||\sqrt{(n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}}\left(\frac{\sigma_3^2-\tilde\sigma_3^2}{\sigma_3^2\tilde\sigma_3^2}c_3+\frac{h_{k,k}}{\tilde\sigma_3^2}\sqrt{1-\beta_{\mathsf{e}}}\right)$$
$$+\frac{h_{k,k}}{\tilde\sigma_3^2}\sqrt{1-\beta_{\mathsf{e}}}(n_{\mathsf{e}}-n_{\mathsf{U}})\mathsf{P}\left(h_{k,k}\sqrt{1-\beta_{\mathsf{e}}}+2c_3\right)$$
$$\left.+\frac{h_{k,k}^2}{\tilde\sigma_2^2}n_{\mathsf{U}}\mathsf{P}\beta_{\mathsf{e}}+\frac{2h_{k,k}}{\tilde\sigma_2^2}n_{\mathsf{U}}\mathsf{P}\sqrt{\beta_{\mathsf{e}}}c_1\geq\tilde\gamma_{\mathsf{e},1}\right] \quad (83)$$
$$=\mathbb{P}[l_{k,1}||\boldsymbol{Z}_{k,1}||+l_{k,2}||\boldsymbol{Z}_{k,2}|| +l_{k,3}||\boldsymbol{Z}_{k,1}||^2+l_{k,4}||\boldsymbol{Z}_{k,2}||^2\geq\bar\gamma_{\mathsf{e},1}] \quad (84)$$
$$\leq\frac{1}{\bar\gamma_{\mathsf{e},1}}\mathbb{E}[l_{k,1}||\boldsymbol{Z}_{k,1}||+l_{k,2}||\boldsymbol{Z}_{k,2}|| +l_{k,3}||\boldsymbol{Z}_{k,1}||^2+l_{k,4}||\boldsymbol{Z}_{k,2}||^2] \quad (85)$$
$$=\frac{1}{\bar\gamma_{\mathsf{e},1}}\left(l_{k,1}\sqrt{2}\frac{\Gamma(\frac{n_{\mathsf{U}}+1}{2})}{\Gamma(\frac{n_{\mathsf{U}}}{2})}+l_{k,2}\sqrt{2}\frac{\Gamma(\frac{n_{\mathsf{e}}-n_{\mathsf{U}}+1}{2})}{\Gamma(\frac{n_{\mathsf{e}}-n_{\mathsf{U}}}{2})}\right) +\frac{1}{\bar\gamma_{\mathsf{e},1}}\left(l_{k,3}n_{\mathsf{U}}+l_{k,4}(n_{\mathsf{e}}-n_{\mathsf{U}})\right) \quad (86)$$

where

$$l_{k,3}:=\frac{\sigma_2^2-1}{\sigma_2^2} \quad (87a)$$
$$l_{k,4}:=\frac{\sigma_3^2-1}{\sigma_3^2} \quad (87b)$$
$$\tilde\gamma_{\mathsf{e},1}:=-\frac{2\gamma_{\mathsf{e},1}}{J_{\mathsf{e},1}}+n_{\mathsf{U}}\ln\frac{\sigma_2^2}{\tilde\sigma_2^2}+(n_{\mathsf{e}}-n_{\mathsf{U}})\ln\frac{\sigma_3^2}{\tilde\sigma_3^2}, \quad (87c)$$

and $l_{k,1}$, $l_{k,2}$, $\bar\gamma_{\mathsf{e},1}$ are defined in (29), (30), and (22), respectively, c_1 and c_3 are defined in (24e) and (24g). Note that in (85) we employ Markov's inequality. In (86), we use the fact that $||\boldsymbol{Z}_{k,1}||$ follows a chi distribution of degree n_{U}, $||\boldsymbol{Z}_{k,1}||^2$ follows a chi-squared distribution of degree n_{U}, $||\boldsymbol{Z}_{k,2}||$ follows a chi distribution of degree $n_{\mathsf{e}}-n_{\mathsf{U}}$, and $||\boldsymbol{Z}_{k,2}||^2$ follows a chi-squared distribution of degree $n_{\mathsf{e}}-n_{\mathsf{U}}$ to calculate their corresponding expectations.

2) Analyzing $\mathbb{P}[\mathcal{E}_{k,2}^{(\mathsf{e})}]$ *:* To evaluate this error event, we use the threshold bound for maximum-metric decoding. I.e.

$$\mathbb{P}[\mathcal{E}_{k,2}^{(\mathsf{e})}]\leq\mathbb{P}[i_2(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},2}] + M_{\mathsf{e}}\mathbb{P}[i_2(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},2}] \quad (88)$$

for any $\gamma_{\mathsf{e},2}$, where $\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)}\sim f_{\boldsymbol{X}_k^{(\mathsf{e},1)}}$ and $\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)}\sim f_{\boldsymbol{X}_k^{(\mathsf{e},2)}}$ and are independent of $(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)},\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})$. To calculate $\mathbb{P}[i_2(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},2}]$ we follow similar steps as in (47)-(55) and show that

$$\mathbb{P}[i_2(\bar{\boldsymbol{X}}_k^{(\mathsf{e},1)},\bar{\boldsymbol{X}}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})>\gamma_{\mathsf{e},2}]\leq e^{-\gamma_{\mathsf{e},2}}. \quad (89)$$

To calculate $\mathbb{P}[i_2(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},2}]$, we first define the following distributions:

$$\tilde Q_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})\sim\mathcal{N}(\boldsymbol{y}_{k,1};\boldsymbol{0},I_n\sigma_1^2) \quad (90)$$
$$\tilde Q_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})\sim\mathcal{N}(\boldsymbol{y}_{k,2};\boldsymbol{0},\sigma_4^2 I_{n_{\mathsf{U}}}) \quad (91)$$
$$\tilde W(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)})\sim\mathcal{N}(\boldsymbol{y}_{k,1};h_{k,k}(1-\alpha_{k,1})\boldsymbol{X}_k^{(\mathsf{e},1)},\tilde\sigma_1^2 I_{n_{\mathsf{e}}-n_{\mathsf{U}}}) \quad (92)$$
$$\tilde W(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})\sim\mathcal{N}(\boldsymbol{y}_{k,2};h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)},\tilde\sigma_4^2 I_{n_{\mathsf{e}}-n_{\mathsf{U}}}), \quad (93)$$

where σ_1^2 and σ_4^2 are defined in (24a) and (24d), respectively and $\tilde\sigma_1^2=1$ and $\tilde\sigma_4^2=1$. Introduce

$$\tilde i_2(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2}) :=\ln\frac{\tilde W(\boldsymbol{y}_{k,1}|\boldsymbol{x}_k^{(\mathsf{e},1)})\tilde W(\boldsymbol{y}_{k,2}|\boldsymbol{x}_k^{(\mathsf{e},2)})}{\tilde Q_{\boldsymbol{Y}_{k,1}}(\boldsymbol{y}_{k,1})\tilde Q_{\boldsymbol{Y}_{k,2}}(\boldsymbol{y}_{k,2})}. \quad (94)$$

Lemma 3: It can be shown that

$$\frac{i_2(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2})}{\tilde i_2(\boldsymbol{x}_k^{(\mathsf{e},1)},\boldsymbol{x}_k^{(\mathsf{e},2)};\boldsymbol{y}_{k,1},\boldsymbol{y}_{k,2})}\geq J_{\mathsf{e},2} \quad (95)$$

where

$$J_{\mathsf{e},2}:=(n_{\mathsf{U}}-2)\ln(2h_{k,k}a_1)+(n_{\mathsf{e}}-n_{\mathsf{U}}-2)\ln(\sqrt{2}h_{k-1,k}) - n_{\mathsf{U}}\mathsf{P}(h_{k,k}^2r_k+a_1^2\beta_{\mathsf{e}})-\frac{(n_{\mathsf{e}}-n_{\mathsf{U}})h_{k-1,k}^2(1-\beta_{\mathsf{e}})\mathsf{P}}{2} - \frac{e^{c_\Gamma}a_1^2\beta_{\mathsf{e}}\mathsf{P}}{\sqrt{2\pi h_{k,k}^2r_k\mathsf{P}}}-\frac{e^{c_\Gamma}h_{k,k}^2\beta_{\mathsf{e},2}\mathsf{P}}{\sqrt{2\pi h_{k-1,k}^2(1-\beta_{\mathsf{e}})\mathsf{P}}}-\kappa_2 \quad (96)$$

and $a_1:=h_{k-1,k}-h_{k,k}\alpha_{k,2}$ and $\kappa_2:=\ln(\frac{1}{2})+c_\Gamma+\ln(\sqrt{\frac{\pi}{8}})-2\ln(h_{k,k}(1-\alpha_{k,1}))$ with $c_\Gamma\leq 2$.

Proof: Similar to the proof of Lemma 1. ■

As a result of the above lemma, we have

$$\mathbb{P}[i_2(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\gamma_{\mathsf{e},2})] \quad (97)$$
$$\leq\mathbb{P}[\tilde i_2(\boldsymbol{X}_k^{(\mathsf{e},1)},\boldsymbol{X}_k^{(\mathsf{e},2)};\boldsymbol{Y}_{k,1},\boldsymbol{Y}_{k,2})\leq\frac{\gamma_{\mathsf{e},2}}{J_{\mathsf{e},2}}] \quad (98)$$

$$= \mathbb{P}\left[\ln \frac{\tilde{W}(\boldsymbol{Y}_{k,1}|\boldsymbol{X}_k^{(\mathsf{e},1)})\tilde{W}(\boldsymbol{Y}_{k,2}|\boldsymbol{X}_k^{(\mathsf{e},2)})}{\tilde{Q}_{\boldsymbol{Y}_{k,1}}(\boldsymbol{Y}_{k,1})\tilde{Q}_{\boldsymbol{Y}_{k,2}}(\boldsymbol{Y}_{k,2})} \leq \frac{\gamma_{\mathsf{e},2}}{J_{\mathsf{e},2}}\right] \tag{99}$$

$$= \mathbb{P}\Bigg[\ln \frac{\frac{1}{(\sqrt{2\tilde{\sigma}_1^2\pi})^{n_\mathsf{U}}} \exp\left(-\frac{||\boldsymbol{Y}_{k,1}-h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},1)}||^2}{2\tilde{\sigma}_1^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_1^2})^{n_\mathsf{U}}} \exp\left(-\frac{||\boldsymbol{Y}_{k,1}||^2}{2\sigma_1^2}\right)}$$

$$+ \ln \frac{\frac{1}{(\sqrt{2\tilde{\sigma}_4^2\pi})^{n_\mathsf{e}-n_\mathsf{U}}} \exp\left(-\frac{||\boldsymbol{Y}_{k,2}-h_{k,k}\boldsymbol{X}_k^{(\mathsf{e},2)}||^2}{2\tilde{\sigma}_4^2}\right)}{\frac{1}{(\sqrt{2\pi\sigma_4^2})^{n_\mathsf{e}-n_\mathsf{U}}} \exp\left(-\frac{||\boldsymbol{Y}_{k,2}||^2}{2\sigma_4^2}\right)} \leq \frac{\gamma_{\mathsf{e},2}}{J_{\mathsf{e},2}}\Bigg]$$

$$\leq \mathbb{P}\big[d_{k,1}||\boldsymbol{Z}_{k,1}|| + d_{k,2}||\boldsymbol{Z}_{k,2}|| + d_{k,3}||\boldsymbol{Z}_{k,2}||^2 + d_{k,4}||\boldsymbol{Z}_{k,2}||^2 \geq \bar{\gamma}_{\mathsf{e},2}\big] \tag{100}$$

$$\leq \frac{1}{\bar{\gamma}_{\mathsf{e},2}}\mathbb{E}\big[d_{k,1}||\boldsymbol{Z}_{k,1}|| + d_{k,2}||\boldsymbol{Z}_{k,2}|| + d_{k,3}||\boldsymbol{Z}_{k,1}||^2 + d_{k,4}||\boldsymbol{Z}_{k,2}||^2\big] \tag{101}$$

$$= \frac{1}{\bar{\gamma}_{\mathsf{e},2}}\left(d_{k,1}\sqrt{2}\frac{\Gamma(\frac{n_\mathsf{U}+1}{2})}{\Gamma(\frac{n_\mathsf{U}}{2})} + d_{k,2}\sqrt{2}\frac{\Gamma(\frac{n_\mathsf{e}-n_\mathsf{U}+1}{2})}{\Gamma(\frac{n_\mathsf{e}-n_\mathsf{U}}{2})}\right) + \frac{1}{\bar{\gamma}_{\mathsf{e},2}}\left(d_{k,3}n_\mathsf{U} + d_{k,4}(n_\mathsf{e} - n_\mathsf{U})\right) \tag{102}$$

where

$$d_{k,3} := \frac{\sigma_1^2 - 1}{\sigma_1^2}, \quad d_{k,4} := \frac{\sigma_4^2 - 1}{\sigma_4^2} \tag{103a}$$

and $d_{k,1}$, $d_{k,2}$, and $\bar{\gamma}_{\mathsf{e},1}$ are defined in (31), (32), and (23), respectively.

Appendix B
Lemmas on Bounding Distributions

Lemma 4: Consider the vector $\boldsymbol{S} = b\boldsymbol{X}+\boldsymbol{Z}$ where $||\boldsymbol{X}||^2 = nP$ and $\boldsymbol{Z} \sim \mathcal{N}(0, \sigma_z^2 I_n)$ and b is a constant. Let $f_{\boldsymbol{S}}(\boldsymbol{s})$ be the pdf of $\boldsymbol{S}$ and is given by

$$f_{\boldsymbol{S}}(\boldsymbol{s}) = \frac{1}{2(\sqrt{\sigma_z^2\pi})^n}\Gamma(\frac{n}{2})b^{n-2}\exp(-\frac{b^2nP}{2\sigma_z^2}) \times \exp\left(-\frac{||\boldsymbol{s}||^2}{2\sigma_z^2}\right)\frac{\mathcal{I}_{\frac{n}{2}-1}\left(||\boldsymbol{s}||b\sqrt{nP}/\sigma_z^2\right)}{\left(||\boldsymbol{s}||b\sqrt{nP}/\sigma_z^2\right)^{\frac{n}{2}-1}}, \tag{104}$$

where $\mathcal{I}_{\frac{n}{2}-1}$ is the modified Bessel function of the first kind and $(\frac{n}{2}-1)$-th order.

Proof: [18, equation 52]. ■

Lemma 5: Consider the vector $\boldsymbol{S} = b\boldsymbol{X}+\boldsymbol{Z}$ where $||\boldsymbol{X}||^2 = nP$ and $\boldsymbol{Z} \sim \mathcal{N}(0, \sigma_z^2 I_n)$ and b is a constant. Let $f_{\boldsymbol{S}}(\boldsymbol{s})$ be the pdf of $\boldsymbol{S}$ and

$$Q_{\boldsymbol{S}}(\boldsymbol{s}) \sim \mathcal{N}(\boldsymbol{s}; 0, (b^2P+1)I_n), \tag{105}$$

$$\tilde{Q}_{\boldsymbol{S}}(\boldsymbol{s}) \sim \mathcal{N}(\boldsymbol{s}; 0, I_n), \tag{106}$$

Thus

$$\frac{f_{\boldsymbol{S}}(\boldsymbol{s})}{Q_{\boldsymbol{S}}(\boldsymbol{s})} \leq T_u \tag{107}$$

$$\frac{f_{\boldsymbol{S}}(\boldsymbol{s})}{\tilde{Q}_{\boldsymbol{S}}(\boldsymbol{s})} \geq T_l \tag{108}$$

where

$$T_l := 2^{\frac{n-2}{2}}b^{n-2}(e^{-b^2P/\sigma_z^2})^{\frac{n}{2}} \tag{109}$$

$$T_u := e^{\kappa} \tag{110}$$

with $\kappa := \big(\ln(\frac{1}{2}) + c_\Gamma + \ln(\sqrt{\frac{\pi}{8}}) - 2\ln(b)\big)$ and $c_\Gamma \leq 2$.

Proof: Define

$$T_{fq} := \frac{f_{\boldsymbol{S}}(\boldsymbol{s})}{\tilde{Q}_{\boldsymbol{S}}(\boldsymbol{s})} \tag{111}$$

By Lemma 4, we have

$$T_{fq} = \frac{1}{2}\Gamma(\frac{n}{2})b^{n-2}(2e^{-b^2P/\sigma_z^2})^{\frac{n}{2}}\frac{\mathcal{I}_{\frac{n}{2}-1}\left(||\boldsymbol{s}||b\sqrt{nP}/\sigma_z^2\right)}{\left(||\boldsymbol{s}||b\sqrt{nP}/\sigma_z^2\right)^{\frac{n}{2}-1}}. \tag{112}$$

We start by lower bounding T_{fq}. To this end, we use the following lower bound on the Bessel function:

$$\mathcal{I}_n(x) > \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n. \tag{113}$$

that is valid for $x > 0$ and $n > -\frac{1}{2}$. Therefore,

$$T_{fq} > 2^{\frac{n-2}{2}}b^{n-2}(e^{-b^2P/\sigma_z^2})^{\frac{n}{2}} \tag{114}$$

which proves the bound (109). The upper bound (110) follows the argument provided in [18, Appendix B]. ■

Lemma 6: Consider the vector $\boldsymbol{U} = a_1\boldsymbol{X}_1 + a_2\boldsymbol{X}_2 + \boldsymbol{Z}$ where $||\boldsymbol{X}_1||^2 = nP_1$, $||\boldsymbol{X}_2||^2 = nP_2$, $\boldsymbol{Z} \sim \mathcal{N}(0, \sigma_z^2 I_n)$, and a_1 and a_2 are constants. Let $f_{\boldsymbol{U}}(\boldsymbol{u})$ be the pdf of $\boldsymbol{U}$ and

$$\tilde{Q}_{\boldsymbol{U}}(\boldsymbol{u}) \sim \mathcal{N}(\boldsymbol{u}; 0, \sigma_z^2 I_n), \tag{115}$$

$$Q_{\boldsymbol{U}}(\boldsymbol{u}) \sim \mathcal{N}(\boldsymbol{u}; 0, (a_1^2P_1 + a_2^2P_2 + \sigma_z^2)I_n). \tag{116}$$

Thus

$$\frac{f_{\boldsymbol{U}}(\boldsymbol{u})}{\tilde{Q}_{\boldsymbol{U}}(\boldsymbol{u})} \geq (2a_1a_2)^{(n-2)}e^{-\frac{n}{2}\left(\frac{a_1^2P_1}{\sigma_{z_1}^2} + \frac{a_2^2P_2}{\sigma_{z_2}^2}\right)} \tag{117}$$

$$\frac{f_{\boldsymbol{U}}(\boldsymbol{u})}{Q_{\boldsymbol{U}}(\boldsymbol{u})} \leq e^{\frac{e^{c_\Gamma}a_2^2P_2}{\sqrt{2\pi a_1^2P_1}}} \tag{118}$$

with $c_\Gamma \leq 2$.

Proof: Define $\boldsymbol{U}_1 := a_1\boldsymbol{X} + \boldsymbol{Z}_1$ and $\boldsymbol{U}_2 := a_2\boldsymbol{X}_2 + \boldsymbol{Z}_2$ where $\boldsymbol{Z}_1 \sim \mathcal{N}(0, \sigma_{z_1}^2 I_n)$ and $\boldsymbol{Z}_2 \sim \mathcal{N}(0, \sigma_{z_2}^2 I_n)$ with $\sigma_{z_1}^2 + \sigma_{z_2}^2 = \sigma_z^2$. Let $f_{\boldsymbol{U}_1}(\boldsymbol{u}_1)$ be the pdf of $\boldsymbol{U}_1$, $f_{\boldsymbol{U}_2}(\boldsymbol{u}_2)$ be the pdf of $\boldsymbol{U}_2$, $Q_{\boldsymbol{U}_1}(\boldsymbol{u}_1) \sim \mathcal{N}(\boldsymbol{u}_1; 0, \sigma_{z_1}^2 I_n)$, and $Q_{\boldsymbol{U}_2}(\boldsymbol{u}_2) \sim \mathcal{N}(\boldsymbol{u}_2; 0, \sigma_{z_2}^2 I_n)$. Thus

$$f_{\boldsymbol{U}}(\boldsymbol{u}) = \int_{\mathbb{R}^n} f_{\boldsymbol{U}_1}(\boldsymbol{u}_1)f_{\boldsymbol{U}_2}(\boldsymbol{u}-\boldsymbol{u}_1)d\boldsymbol{u}_1 \tag{119}$$

$$\geq 2^{\frac{n-2}{2}}a_1^{n-2}(e^{-a_1^2P_1/\sigma_{z_1}^2})^{\frac{n}{2}} \times 2^{\frac{n-2}{2}}a_2^{n-2}(e^{-a_2^2P_2/\sigma_{z_2}^2})^{\frac{n}{2}} \times \int_{\mathbb{R}^n} Q_{\boldsymbol{U}_1}(\boldsymbol{u}_1)Q_{\boldsymbol{U}_2}(\boldsymbol{u}-\boldsymbol{u}_1)d\boldsymbol{u}_1 \tag{120}$$

$$= (2a_1a_2)^{(n-2)}e^{-\frac{n}{2}\left(\frac{a_1^2P_1}{\sigma_{z_1}^2} + \frac{a_2^2P_2}{\sigma_{z_2}^2}\right)}\tilde{Q}_{\boldsymbol{U}}(\boldsymbol{u}), \tag{121}$$

where the inequality (120) is based on Lemma 5. This proves the lower bound in (117). The upper bound (118) follows the argument provided in [18, Appendix C]. ■

Lemma 7: Consider the vector $\boldsymbol{Y} = a_1\boldsymbol{X}_1 + a_2\boldsymbol{X}_2 + a_3\boldsymbol{X}_3 + \boldsymbol{Z}$ where $||\boldsymbol{X}_i||^2 = nP_i$ for $i \in \{1,2,3\}$, $\boldsymbol{Z} \sim \mathcal{N}(0, \sigma_z^2 I_n)$, and a_is with $i \in \{1,2,3\}$ are constants. Let $f_{\boldsymbol{Y}}(\boldsymbol{Y})$ be the pdf of $\boldsymbol{Y}$ and

$$\tilde{Q}_{\boldsymbol{Y}}(\boldsymbol{y}) \sim \mathcal{N}(\boldsymbol{y}; 0, \sigma_z^2 I_n), \tag{122}$$

$$Q_{\boldsymbol{Y}}(\boldsymbol{y}) \sim \mathcal{N}(\boldsymbol{y}; 0, (a_1^2 P_1 + a_2^2 P_2 + a_3 P_3 + \sigma_z^2) I_n). \tag{123}$$

One can prove that

$$\frac{f_{\boldsymbol{Y}}(\boldsymbol{Y})}{\tilde{Q}_{\boldsymbol{Y}}(\boldsymbol{y})} \geq 2^{\frac{3(n-2)}{2}} (a_1 a_2 a_3)^{(n-2)} \times e^{-\frac{n}{2}\left(\frac{a_1^2 P_1}{\sigma_{z_1}^2} + \frac{a_2^2 P_2}{\sigma_{z_2}^2} + \frac{a_3^2 P_3}{\sigma_{z_3}^2}\right)}, \tag{124}$$

$$\frac{f_{\boldsymbol{Y}}(\boldsymbol{Y})}{Q_{\boldsymbol{Y}}(\boldsymbol{y})} \leq e^{\kappa} e^{\frac{e^{c_\Gamma} a_2^2 P_2}{\sqrt{2\pi a_1^2 P_1}}}, \tag{125}$$

where $\kappa := \left(\ln(\frac{1}{2}) + c_\Gamma + \ln(\sqrt{\frac{\pi}{8}}) - 2\ln(a_3)\right)$ with $c_\Gamma \leq 2$, and $\sigma_{z_1}^2 + \sigma_{z_2}^2 + \sigma_{z_3}^2 = \sigma_z^2$.

Proof: The proof is based on the argument provided in the proof of Lemma 6. ■

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