Constrained Wyner-Ziv Source Coding

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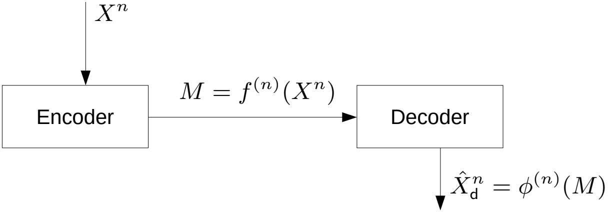
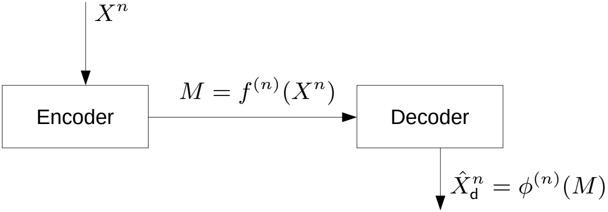
Lossless Source Coding (Perfect Reconstruction)



- Source sequence $X^n = (X_1, \ldots, X_n)$ IID $\sim P_X$ over $\mathcal{X}$
- Message $M \in \{1, \ldots, \lfloor 2^{nR} \rfloor\}$
- Decoder's source-reconstruction $\hat{X}_{\mathsf{d}}^n \in \mathcal{X}^n$
- Error: $\hat{X}_{\mathsf{d}}^n \neq X^n$

Lossless Source Coding (Perfect Reconstruction)

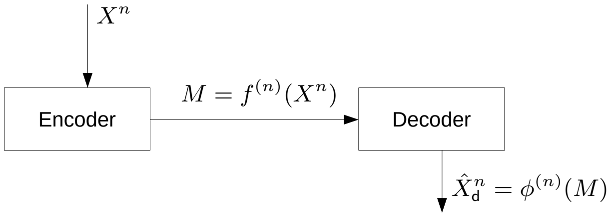


- Rate R achievable, if $\overline{\lim_{n\to\infty}}\, \mathsf{Pr}\left[X^n \neq \hat{X}_{\mathsf{d}}^n\right] = 0$

Lossless Source Coding Theorem (Shannon)

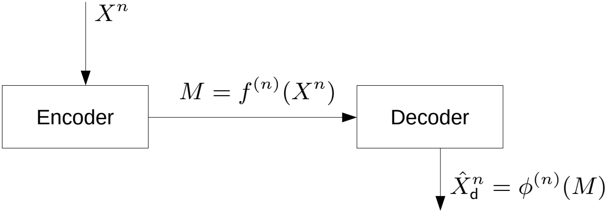
Infimum over achievable rates: $R^\star = H(X)$

Lossy Source Coding (Approximate Reconstruction)



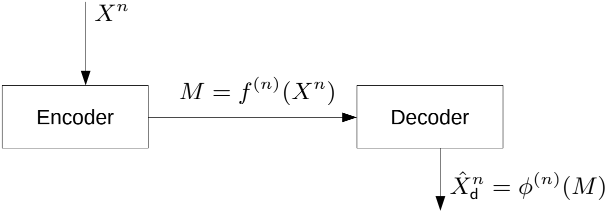
- Reconstruction alphabet $\hat{\mathcal{X}}$
- Distortion constraint $\mathsf{E}\left[d_{\mathsf{d}}^{(n)}(X^n, \hat{X}_{\mathsf{d}}^n)\right] \leq D_{\mathsf{d}}$
- Symbol-wise distortion-function $d_{\mathsf{d}} \colon \mathcal{X} \times \hat{\mathcal{X}} \to \mathbb{R}_0^+$
- Average per-symbol distortion: $d_{\mathsf{d}}^{(n)}(X^n, \hat{X}_{\mathsf{d}}^n) = \dfrac{1}{n} \sum_{i=1}^{n} d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})$

Lossy Source Coding (Approximate Reconstruction)



- (R, D_{d}) achievable if $\overline{\lim}_{n\to\infty} \mathsf{E}\left[\frac{1}{n}\sum_{i=1}^n d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \leq D_{\mathsf{d}}$
- $R(D_{\mathsf{d}})$: infimum $R \geq 0$ such that (R, D_{d}) achievable

Lossy Source Coding (Approximate Reconstruction)



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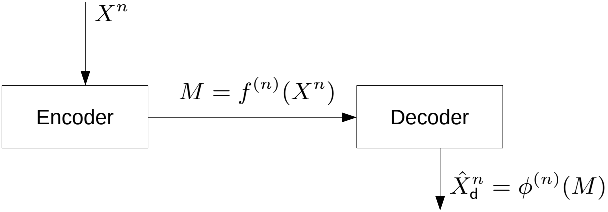
- $R(D_{\mathsf{d}})$: infimum $R \ge 0$ such that (R, D_{d}) achievable

- Lossless problem not a special case;
- If $d_{\mathsf{d}}(x, \hat{x}) = I\{x \ne \hat{x}\}$ and $D_{\mathsf{d}} = 0$ (average symbol-error vs. block-error prob.)

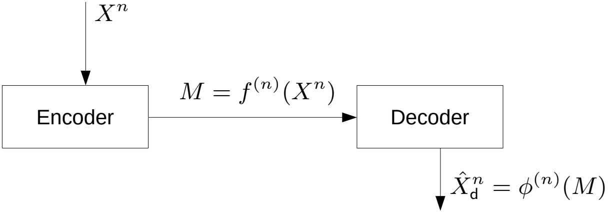
Rate-Distortion Function of Lossy Source Coding



Rate-Distortion Function

$$R(D_\mathsf{d}) = \min_{\substack{\hat{X}_\mathsf{d} \text{ s.t.} \\ \mathsf{E}\left[d_\mathsf{d}(X,\hat{X}_\mathsf{d})\right] \le D_\mathsf{d}}} I(X;\hat{X}_\mathsf{d})$$

Rate-Distortion Function of Lossy Source Coding

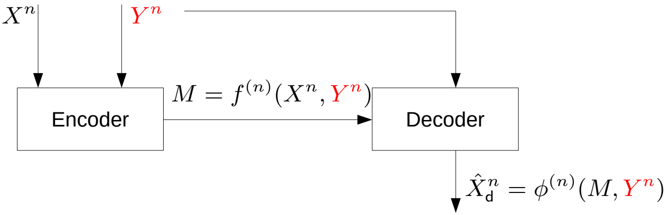


Rate-Distortion Function

$$R(D_{\mathsf{d}}) = \min_{\substack{\hat{X}_{\mathsf{d}} \text{ s.t.} \\ \mathsf{E}\left[d_{\mathsf{d}}(X,\hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} H(X) - H(X|\hat{X}_{\mathsf{d}})$$

- Corollary: When $d_{\mathsf{d}}(x, \hat{x}) = I\{x \neq \hat{x}\}$, then $R(0) = R^{\star} = H(X)$

Lossy Source Coding with Side-Information



- Source/side-information sequence $\{(X_i, Y_i)\}$ IID $\sim P_{XY}$ over $\mathcal{X} \times \mathcal{Y}$

Rate-Distortion Function with Side-Information

$$R_{\mathsf{SI}}(D_{\mathsf{d}}) = \min_{\substack{\hat{X}_{\mathsf{d}} \text{ s.t.} \\ \mathsf{E}\left[d_{\mathsf{d}}(X, \hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} I(X; \hat{X}_{\mathsf{d}} | Y)$$

So far: Encoder can compute $\hat{X}_{\mathsf{d}}^{n}$

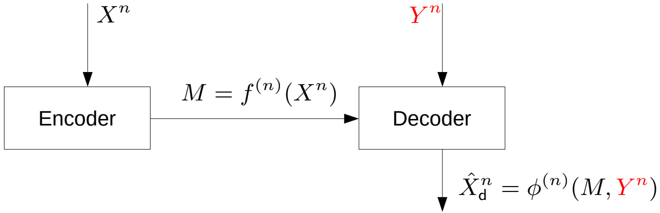
Encoder knows decoder's reconstruction $\hat{X}_{\mathsf{d}}^{n}$:

1. $\hat{X}_{\mathsf{d}}^{n}$ can serve as common reference

2. If $\hat{X}_{\mathsf{d}}^{n}$ not sufficient, encoder can re-describe X^{n}

   $\rightarrow$ strengthening of expected average distortion constraint

Lossy Source Coding with Side-Information @ Receiver only



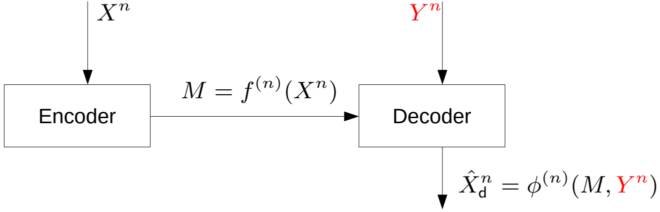
- Side-information known at decoder only!
- $R(D_{\mathsf{d}}) \geq R_{\mathsf{WZ}}(D_{\mathsf{d}}) \geq R_{\mathsf{SI}}(D_{\mathsf{d}})$

Wyner-Ziv Rate-Distortion Function

$$R_{\mathsf{WZ}}(D_{\mathsf{d}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y) \text{ s.t.} \\ Z \multimap\!- X \multimap\!- Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X, \hat{X}_{\mathsf{d}})\right] \leq D_{\mathsf{d}}}} I(X; Z|Y)$$

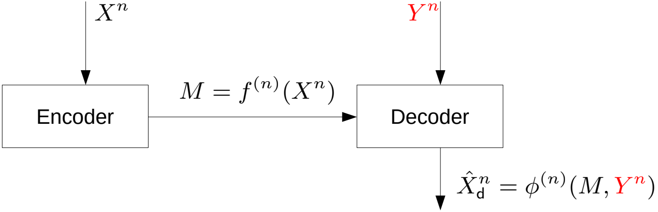
where $|\mathcal{Z}| = |\mathcal{X}| + 1$ suffices

Problem in Wyner-Ziv: Encoder cannot compute $\hat{X}_{\mathsf{d}}^n$!



Encoder ignorant of $Y^n \Rightarrow$ cannot compute $\hat{X}_{\mathsf{d}}^n(M, Y^n)$!

Problem in Wyner-Ziv: Encoder cannot compute $\hat{X}_{\mathsf{d}}^n$!

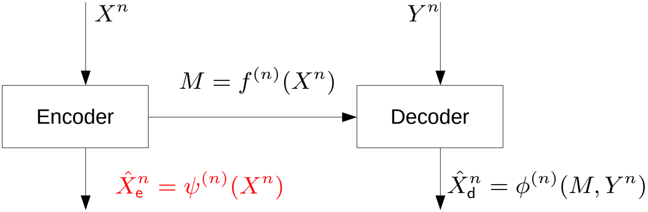


Encoder ignorant of $Y^n \Rightarrow$ cannot compute $\hat{X}_{\mathsf{d}}^n(M, Y^n)$!

Wish to constrain $\hat{X}_{\mathsf{d}}^n$:

- Encoder can reconstruct $\hat{X}_{\mathsf{d}}^n$ losslessly (Steinberg'09)
- Encoder can reconstruct $\hat{X}_{\mathsf{d}}^n$ lossily (Malär/Lapidoth/Wigger'11)

Wyner-Ziv with Common-Reconstruction (Steinberg'09)



- Same reconstruction alphabet: $\hat{X}_{\mathsf{e}}^n \in \hat{\mathcal{X}}^n$
- (R, D_{d}) achievable if:

1. $\overline{\lim}_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \leq D_{\mathsf{d}}$

2. $\overline{\lim}_{n\to\infty} \mathsf{Pr}\left[\hat{X}_{\mathsf{e}}^n \neq \hat{X}_{\mathsf{d}}^n\right] = 0$

Wyner-Ziv with Common-Reconstruction Constraint

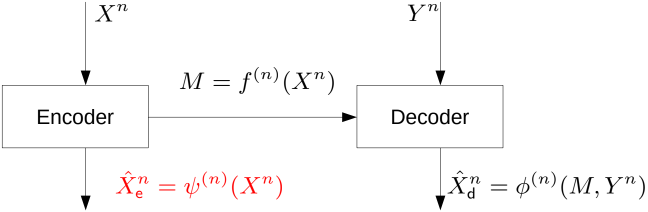
Rate-Distortion Function with Common Reconstruction (Steinberg'09)

$$R_{\mathsf{CR}}(D_{\mathsf{d}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z) \text{ s.t.} \\ Z - \circ - X - \circ - Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X, \hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} I(X; Z|Y)$$

Recall Wyner-Ziv:

$$R_{\mathsf{WZ}}(D_{\mathsf{d}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y) \text{ s.t.} \\ Z - \circ - X - \circ - Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X, \hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} I(X; Z|Y)$$

Wyner-Ziv with Lossy Common-Reconstruction Constraint



- Encoder-side distortion-function $d_{\mathsf{e}} \colon \hat{\mathcal{X}} \times \hat{\mathcal{X}} \to \mathbb{R}_0^+$
- $(R, D_{\mathsf{d}}, D_{\mathsf{e}})$ achievable if:

1. $\displaystyle \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \le D_{\mathsf{d}}$
2. $\displaystyle \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d},i}, \hat{X}_{\mathsf{e},i})\right] \le D_{\mathsf{e}}$

Comparison with Previous Setups

Wyner-Ziv:

- Special case of our setup, e.g., when $d_{\mathsf{d}} = d_{\mathsf{e}}$ and $D_{\mathsf{d}} \leq D_{\mathsf{e}}$
  - Then choose $\hat{X}_e^n = X^n$

Steinberg:

- Not a special case of our setup $\rightarrow$ 2. not a per-symbol average
- If $d_{\mathsf{e}}(\hat{x}_{\mathsf{d}}, \hat{x}_{\mathsf{e}}) = I\{\hat{x}_{\mathsf{d}} \neq \hat{x}_{\mathsf{e}}\}$ and $D_{\mathsf{e}} = 0$:

  block-error probability vs. average symbol-error probability

Main Result

Theorem (Rate-Distortions Function for Discrete Finite Alphabets)

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \text{ s.t.} \\ Z -\circ- X -\circ- Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X,\hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}} \\ \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d}},\hat{X}_{\mathsf{e}})\right] \le D_{\mathsf{e}}}} I(X;Z|Y)$$

where $|\mathcal{Z}| = |\mathcal{X}| + 3$ *suffices*

Recall Wyner-Ziv:

$$R_{\mathsf{WZ}}(D_{\mathsf{d}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y) \text{ s.t.} \\ Z -\circ- X -\circ- Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X,\hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} I(X;Z|Y)$$

Main Result

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where $|\mathcal{Z}| = |\mathcal{X}| + 3$ *suffices*

Recall Wyner-Ziv: $$R_{\mathsf{WZ}}(D_{\mathsf{d}}) = \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y) \text{ s.t.} \\ Z -\circ- X -\circ- Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X,\hat{X}_{\mathsf{d}})\right] \le D_{\mathsf{d}}}} I(X;Z|Y)$$

Corollary

When $d_{\mathsf{e}}(\hat{x}_{\mathsf{d}}, \hat{x}_{\mathsf{e}}) = I\{\hat{x}_{\mathsf{e}} \neq \hat{x}_{\mathsf{d}}\}$, *then* $R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, 0) = R_{\mathsf{CR}}(D_{\mathsf{d}})$

Gaussian Sources and Quadratic Distortions

- $X \sim \mathcal{N}\big(0, \sigma_X^2\big)$
- $Y = X + U$, where $U \sim \mathcal{N}\big(0, \sigma_U^2\big)$ independent of X
- $d_\mathsf{d}(x, \hat{x}_\mathsf{d}) = (x - \hat{x}_\mathsf{d})^2$ and $d_\mathsf{e}(\hat{x}_\mathsf{d}, \hat{x}_\mathsf{e}) = (\hat{x}_\mathsf{d} - \hat{x}_\mathsf{e})^2$

Rate-Distortions Function of Gaussian Setup

Theorem (Gaussian Sources and Quadratic Distortions)

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) = \begin{cases} \left[\frac{1}{2}\log\left(\frac{\sigma_X^2\sigma_U^2}{(\sigma_X^2+\sigma_U^2)D_{\mathrm{d}}}\right)\right]^+, & \text{if } \sqrt{D_{\mathsf{e}}\sigma_U^2} \geq \min\left\{D_{\mathsf{d}}, \frac{\sigma_X^2\sigma_U^2}{\sigma_X^2+\sigma_U^2}\right\} \\ \left[\frac{1}{2}\log\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_U^2}\frac{\sigma_U^2+D_{\mathrm{d}}-2\sqrt{\sigma_U^2 D_{\mathrm{e}}}}{D_{\mathrm{d}}-D_{\mathrm{e}}}\right)\right]^+, & \text{else.} \end{cases}$$

Corollary

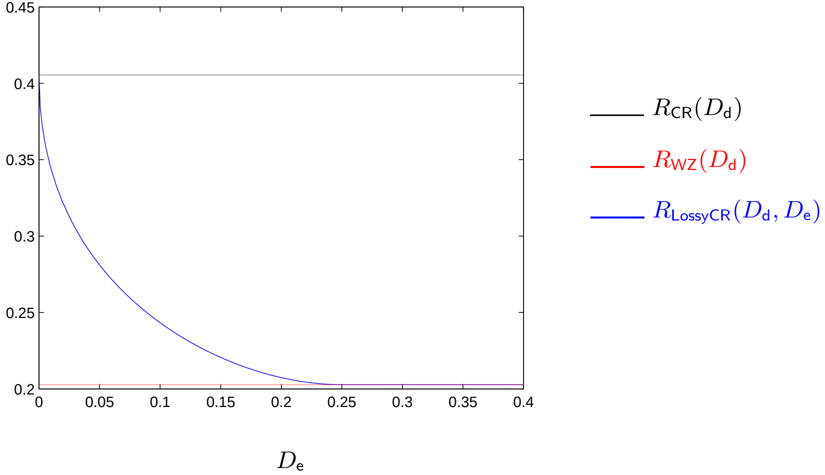
- *If* $\sqrt{D_{\mathsf{e}}\sigma_U^2} \geq \min\left\{D_{\mathsf{d}}, \frac{\sigma_X^2\sigma_U^2}{\sigma_X^2+\sigma_U^2}\right\}$ *or* $\left(1-\sqrt{\frac{D_{\mathsf{e}}}{\sigma_U^2}}\right)^2\sigma_X^2 \leq D_{\mathsf{d}} - D_{\mathsf{e}}$,

  then: $$R_{\mathsf{lossyCR}} = R_{\mathsf{WZ}} = R_{\mathsf{SI}}$$

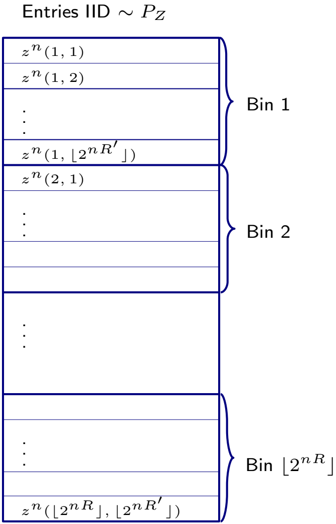
- *If* $D_{\mathsf{e}} = 0$, *then* $R_{\mathsf{lossyCR}} = R_{\mathsf{CR}}$

Plots for Quadratic-Gaussian Setup

$$\sigma_X^2 = 3; \quad \sigma_U^2 = 1; \quad D_{\mathsf{d}} = 0.5$$



Wyner-Ziv's Scheme



- Encoding:
  - Choose M, K s.t.

    $$(Z^n(M,K), X^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZX})$$

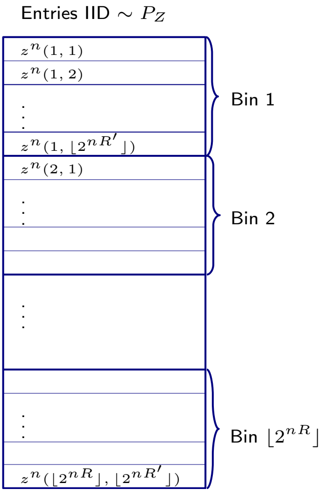
  - Message M is bin-index!
- Decoding:
  - Binning phase: Look for $\hat{K}$ s.t.

    $$(Z^n(M,\hat{K}), Y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZY})$$

  - Estimation phase:
    $\hat{X}_{\mathsf{d},i} = \phi(Z_i(M,\hat{K}), Y_i)$

With high prob: $Z^n(M,K) = Z^n(M,\hat{K})$

Steinberg's Scheme



- Encoding:
  - Choose M, K s.t.

    $$(Z^n(M,K), X^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZX})$$

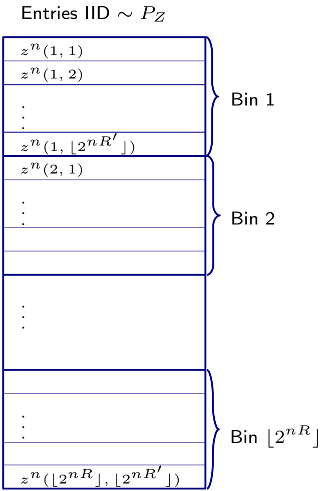
  - Message M is bin-index!
  - $\hat{X}_{\mathsf{e}}^n = Z^n(M,K)$
- Decoding:
  - Binning phase: Look for $\hat{K}$ s.t.

    $$(Z^n(M,\hat{K}), Y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZY})$$

  - "Estimation phase": $\hat{X}_{\mathsf{d}}^n = Z^n(M,\hat{K})$

Estimation phase independent of Y^n!

Our Scheme



- Encoding:
  - Choose M, K s.t.

    $$(Z^n(M,K), X^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZX})$$

  - Message M is bin-index!
  - $\hat{X}_{\mathsf{e}}^n = \psi(Z_i(M,K), X_i)$
- Decoding:
  - Binning phase: Look for $\hat{K}$ s.t.

    $$(Z^n(M,\hat{K}), Y^n) \in \mathcal{T}_\epsilon^{(n)}(P_{ZY})$$

  - Estimation phase: $\hat{X}_{\mathsf{d},i} = \phi(Z_i(M,\hat{K}), Y_i)$

Estimation phase can *moderately* depend on Y^n!

Achievability in Quadratic-Gaussian Case

- Previous achievability fails (strong typicality!)
- New achievability: similar, but with coding over spheres

Converse for Discrete Case

Converse:
$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq \bar{R}(D_{\mathsf{d}}, D_{\mathsf{e}}) \triangleq \min_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z -\!\circ\!- X -\!\circ\!- Y \\ \mathsf{E}\left[d_{\mathsf{d}}(X, \hat{X}_{\mathsf{d}})\right] \leq D_{\mathsf{d}} \\ \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d}}, \hat{X}_{\mathsf{e}})\right] \leq D_{\mathsf{e}}}} I(X; Z|Y)$$

a) Relax source coding problem, i.e., relax 2. distortion constraint

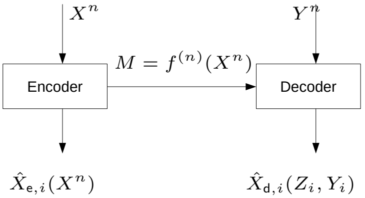
Then:
$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq R_{\mathsf{Relaxed}}(D_{\mathsf{d}}, D_{\mathsf{e}})$$

b) Converse to relaxed problem:

$$R_{\mathsf{Relaxed}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq \bar{R}(D_{\mathsf{d}}, D_{\mathsf{e}})$$

Converse Step a): Relax Source-Coding Problem

original problem

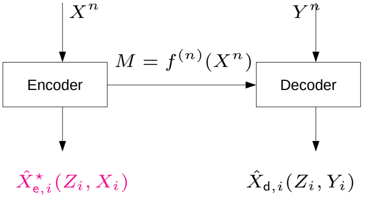


1. $\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \le D_{\mathsf{d}}$
2. $\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d},i}, \hat{X}_{\mathsf{e},i})\right] \le D_{\mathsf{e}}$

- Define $Z_i \triangleq (M, Y^{i-1}, Y_{i+1}^{n})$

Converse Step a): Relax Source-Coding Problem

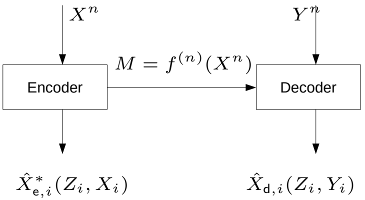
relaxed problem



1. $\varlimsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \leq D_{\mathsf{d}}$
2. $\varlimsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d},i}, \hat{X}^{\star}_{\mathsf{e},i})\right] \leq D_{\mathsf{e}}$

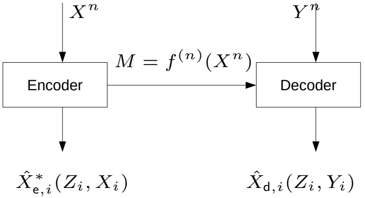
- Define $Z_i \triangleq (M, Y^{i-1}, Y_{i+1}^{n})$
- Because of $X^n -\!\circ\!- (Z_i, X_i) -\!\circ\!- (Z_i, Y_i)$: new constraint 2. weaker

Converse Step b): Converse to Relaxed Problem



1. $\overline{\lim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right] \leq D_{\mathsf{d}}$
2. $\overline{\lim}_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d},i}, \hat{X}^*_{\mathsf{e},i})\right] \leq D_{\mathsf{e}}$

Converse Step b): Converse to Relaxed Problem



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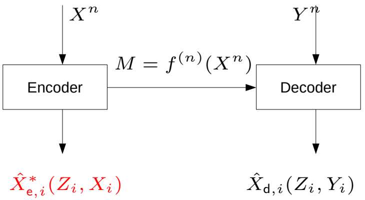
- By definition $Z_i \triangleq (M, Y^{i-1}, Y^n_{i+1})$: $\quad Z_i -\!\!\circ\!\!- X_i -\!\!\circ\!\!- Y_i$
- $R_{\text{Relaxed}} \ge \frac{1}{n} H(M) \ge \ldots \ge \frac{1}{n} \sum_{i=1}^{n} I(X_i; Z_i | Y_i)$

Converse Step b): Converse to Relaxed Problem



1. $$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \underbrace{\mathsf{E}\left[d_{\mathsf{d}}(X_i, \hat{X}_{\mathsf{d},i})\right]}_{D_{\mathsf{d},i}} \leq D_{\mathsf{d}}$$

2. $$\varlimsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \underbrace{\mathsf{E}\left[d_{\mathsf{e}}(\hat{X}_{\mathsf{d},i}, \hat{X}^*_{\mathsf{e},i})\right]}_{D_{\mathsf{e},i}} \leq D_{\mathsf{e}}$$

- By definition $Z_i \triangleq (M, Y^{i-1}, Y^n_{i+1})$: $\quad Z_i \multimap X_i \multimap Y_i$

- $$R_{\mathsf{Relaxed}} \geq \frac{1}{n} H(M) \geq \ldots \geq \frac{1}{n} \sum_{i=1}^{n} I(X_i; Z_i | Y_i) \geq \frac{1}{n} \sum_{i=1}^{n} \bar{R}(D_{\mathsf{d},i}, D_{\mathsf{e},i})$$
  $$\geq \bar{R}\left(\frac{1}{n} \sum_{i=1}^{n} D_{\mathsf{d},i}, \frac{1}{n} \sum_{i=1}^{n} D_{\mathsf{e},i}\right) \geq \bar{R}(D_{\mathsf{d}}, D_{\mathsf{e}})$$

Converse in Quadratic-Gaussian Case

- $X \sim \mathcal{N}\left(0, \sigma_X^2\right)$
- $Y = X + U$, where $U \sim \mathcal{N}\left(0, \sigma_U^2\right)$ independent of X
- $d_{\mathsf{d}}(x, \hat{x}_{\mathsf{d}}) = (x - \hat{x}_{\mathsf{d}})^2$ and $d_{\mathsf{e}}(\hat{x}_{\mathsf{d}}, \hat{x}_{\mathsf{e}}) = (\hat{x}_{\mathsf{d}} - \hat{x}_{\mathsf{e}})^2$

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq \begin{cases} \left[\frac{1}{2}\log\left(\frac{\sigma_X^2 \sigma_U^2}{(\sigma_X^2+\sigma_U^2)D_{\mathrm{d}}}\right)\right]^+, & \text{if } \sqrt{D_{\mathsf{e}}\sigma_U^2} \geq \min\left\{D_{\mathsf{d}}, \frac{\sigma_X^2\sigma_U^2}{\sigma_X^2+\sigma_U^2}\right\} \\ \left[\frac{1}{2}\log\left(\frac{\sigma_X^2}{\sigma_X^2+\sigma_U^2}\frac{\sigma_U^2+D_{\mathrm{d}}-2\sqrt{\sigma_U^2 D_{\mathrm{e}}}}{D_{\mathrm{d}}-D_{\mathrm{e}}}\right)\right]^+, & \text{else.} \end{cases}$$

Converse in Quadratic-Gaussian Case, First Step

Step 1: $$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq \inf_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z -\circ- X -\circ- Y \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right] \leq D_{\mathsf{d}} \\ \mathsf{E}\left[(\hat{X}_{\mathsf{d}}-\hat{X}_{\mathsf{e}})^2\right] \leq D_{\mathsf{e}}}} I(X;Z|Y) \qquad (1)$$

Step 2-: Evaluate RHS(1);

Converse in Quadratic-Gaussian Case, First Step

Step 1: $$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq h(X|Y) - \sup_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z -\circ- X -\circ- Y \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right] \leq D_{\mathsf{d}} \\ \mathsf{E}\left[(\hat{X}_{\mathsf{d}}-\hat{X}_{\mathsf{e}})^2\right] \leq D_{\mathsf{e}}}} h(X|YZ) \qquad (1)$$

Step 2-: Evaluate RHS(1); First Thoughts:

- Conditional Max-Entropy Theorem:
  Given $K_{XYZ\hat{X}_{\mathsf{d}}\hat{X}_{\mathsf{e}}}$ Gaussian tuple $(Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X))$ optimizes (1)
- Not $\forall$ $K_{XYZ\hat{X}_{\mathsf{d}}\hat{X}_{\mathsf{e}}}$ the Gaussian tuple is valid because $\hat{X}_{\mathsf{d}}(Z,Y)$ and $\hat{X}_{\mathsf{e}}(Z,X)$
- If we relax $\hat{X}_{\mathsf{d}}(Z,Y)$ and $\hat{X}_{\mathsf{e}}(Z,X)$ $\Rightarrow$ RHS(1)=0 (too low!)

Converse in Quadratic-Gaussian Case, Further Steps

Step 1:
$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq h(X|Y) - \sup_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z -\!\!\circ\!\!- X -\!\!\circ\!\!- Y \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right] \leq D_{\mathsf{d}} \\ \mathsf{E}\left[(\hat{X}_{\mathsf{d}}-\hat{X}_{\mathsf{e}})^2\right] \leq D_{\mathsf{e}}}} h(X|YZ) \quad (1)$$

Step 2: RHS(1) lower bounded by:

$$h(X|Y) - \sup_{\substack{\hat{X}_{\mathsf{d}} \text{ s.t.:} \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right] \leq D_{\mathsf{d}} \\ \left|\mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})U\right]\right| \leq \sqrt{\sigma_U^2 D_{\mathsf{e}}}}} h(X - \hat{X}_{\mathsf{d}} | Y - \hat{X}_{\mathsf{d}}, \hat{X}_{\mathsf{d}}) \quad (2)$$

Step 3: (2) maximized by jointly Gaussian $(\hat{X}_{\mathsf{d}}, X, U)$ (cond. max-entropy thm)

Step 4: Evaluate (2) for jointly Gaussian $(\hat{X}_{\mathsf{d}}, X, U)$

Step 2-I: Apply $\hat{X}_{\mathsf{d}}(Z,Y)$ to transform Objective Function

- Because $\hat{X}_{\mathsf{d}}(Z,Y)$:

$$h(X|Y,Z) = h(X|Y,Z,\hat{X}_{\mathsf{d}}) = h(X-\hat{X}_{\mathsf{d}}|Y-\hat{X}_{d},Z,\hat{X}_{\mathsf{d}})$$
$$\leq h(X-\hat{X}_{\mathsf{d}}|X-\hat{X}_{\mathsf{d}}+U,\hat{X}_{\mathsf{d}})$$

Step 2-I:

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}},D_{\mathsf{e}}) \geq h(X|Y) - \sup_{\substack{Z,\hat{X}_{\mathsf{d}}(Z,Y),\hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z -\circ- X -\circ- Y \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right]\leq D_{\mathsf{d}} \\ \mathsf{E}\left[(\hat{X}_{\mathsf{d}}-\hat{X}_{\mathsf{e}})^2\right]\leq D_{\mathsf{e}}}} h(X-\hat{X}_{\mathsf{d}}|X-\hat{X}_{\mathsf{d}}+U,\hat{X}_{\mathsf{d}})$$

Step 2-I: Apply $\hat{X}_{\mathsf{d}}(Z,Y)$ to transform Objective Function

- Because $\hat{X}_{\mathsf{d}}(Z,Y)$:

$$h(X|Y,Z) = h(X|Y,Z,\hat{X}_{\mathsf{d}}) = h(X-\hat{X}_{\mathsf{d}}|Y-\hat{X}_{d},Z,\hat{X}_{\mathsf{d}})$$
$$\leq h(X-\hat{X}_{\mathsf{d}}|X-\hat{X}_{\mathsf{d}}+U,\hat{X}_{\mathsf{d}})$$

Step 2-I:

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}},D_{\mathsf{e}}) \geq h(X|Y) - \sup_{\substack{Z,\hat{X}_{\mathsf{d}}(Z,Y),\hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } Z-\circ-X-\circ-Y \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right]\leq D_{\mathsf{d}} \\ \mathsf{E}\left[(\hat{X}_{\mathsf{d}}-\hat{X}_{\mathsf{e}})^2\right]\leq D_{\mathsf{e}}}} h(X-\hat{X}_{\mathsf{d}}|X-\hat{X}_{\mathsf{d}}+U,\hat{X}_{\mathsf{d}})$$

- Relax function-constraint now $\rightarrow$ Wyner-Ziv result (too loose)
- First need to use $\hat{X}_{\mathsf{e}}(Z,X)$ to limit dependence of $\hat{X}_{\mathsf{d}}$ on U

Step 2-II: Apply $\hat{X}_{\mathsf{e}}(Z,X)$ to transform Constraints

- $Z - \circ - X - \circ - Y = X + U \Rightarrow (X,Z)$ ind. of U
- $\hat{X}_{\mathsf{e}}(Z,X)$ & Constraint $\mathsf{E}\left[(\hat{X}_{\mathsf{d}} - \hat{X}_{\mathsf{e}})^2\right] \leq D_{\mathsf{e}}$:

$$\left|\mathsf{E}\left[\hat{X}_{\mathsf{d}} \cdot U\right]\right| = \left|\mathsf{E}\left[(\hat{X}_{\mathsf{d}} - \hat{X}_{\mathsf{e}})U\right]\right| \leq \sqrt{\sigma_U^2 D_{\mathsf{e}}} \tag{3}$$

Step 2-II: relax constraints

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \geq h(X|Y) - \sup_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } (Z,X) \text{ ind. of } U \\ \mathsf{E}\left[(X - \hat{X}_{\mathsf{d}})^2\right] \leq D_{\mathsf{d}} \\ \left|\mathsf{E}\left[\hat{X}_{\mathsf{d}} U\right]\right| \leq \sqrt{\sigma_U^2 D_{\mathsf{e}}}}} h(X - \hat{X}_{\mathsf{d}} | X - \hat{X}_{\mathsf{d}} + U, \hat{X}_{\mathsf{d}})$$

Step 2-II: Apply $\hat{X}_{\mathsf{e}}(Z,X)$ to transform Constraints

- $Z -\!\!\circ\!\!- X -\!\!\circ\!\!- Y = X + U \Rightarrow (X, Z)$ ind. of U
- $\hat{X}_{\mathsf{e}}(Z,X)$ & Constraint $\mathsf{E}\left[(\hat{X}_{\mathsf{d}} - \hat{X}_{\mathsf{e}})^2\right] \le D_{\mathsf{e}}$:

$$\left|\mathsf{E}\left[\hat{X}_{\mathsf{d}} \cdot U\right]\right| = \left|\mathsf{E}\left[(\hat{X}_{\mathsf{d}} - \hat{X}_{\mathsf{e}})U\right]\right| \le \sqrt{\sigma_U^2 D_{\mathsf{e}}} \quad (3)$$

Step 2-II: relax constraints

$$R_{\mathsf{lossyCR}}(D_{\mathsf{d}}, D_{\mathsf{e}}) \ge h(X|Y) - \sup_{\substack{Z, \hat{X}_{\mathsf{d}}(Z,Y), \hat{X}_{\mathsf{e}}(Z,X) \\ \text{s.t.: } (Z,X) \text{ ind. of } U \\ \mathsf{E}\left[(X-\hat{X}_{\mathsf{d}})^2\right] \le D_{\mathsf{d}} \\ \left|\mathsf{E}\left[\hat{X}_{\mathsf{d}}U\right]\right| \le \sqrt{\sigma_U^2 D_{\mathsf{e}}}}} h(X-\hat{X}_{\mathsf{d}} | X-\hat{X}_{\mathsf{d}}+U, \hat{X}_{\mathsf{d}})$$

- Relax function constraints now

Summary

- Wyner-Ziv source coding with 2. lossy reconstruction-constraint
- Rate-distortions function (single-letter) for discrete case
- Rate-distortions function for quadratic-Gaussian case