MIMO Optical Communication in Free Space

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joint work with Longguang Li, Stefan Moser, and Ligong Wang

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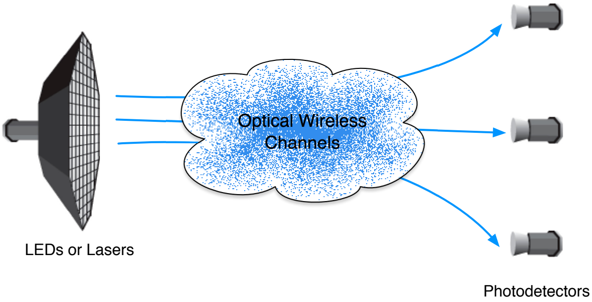
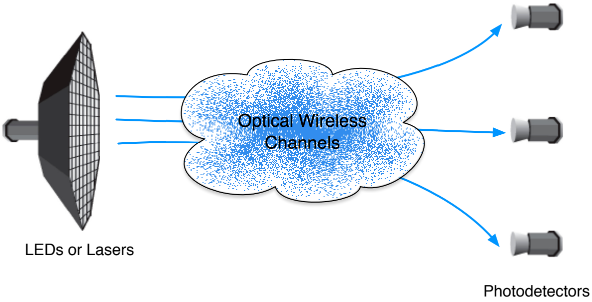
Free-Space Optical Wireless Communication



Intensity Modulation-Direct Detection Communication

Signal proportional to intensity/power

Free-Space Optical Wireless Communication



Multiple-Input Multiple-Output (MIMO) Channel

Assume non-coherent superpositioning

Free-Space Optical Intensity Channel

- Transmitter equipped with n_T LEDs
- Receiver with n_R photodetectors
- Channel output

$$\mathbf{Y} = \mathsf{H}\mathbf{x} + \mathbf{Z}$$

  - H channel matrix with nonnegative entries

$$\mathsf{H} := [\mathbf{h}_1,\ \mathbf{h}_2,\ \ldots,\ \mathbf{h}_{n_\mathsf{T}}] \in \mathbb{R}^{n_\mathsf{R} \times n_\mathsf{T}}$$

  - **Z** Gaussian noise

$$\mathbf{Z} \sim \mathcal{N}(0, \mathsf{I})$$

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Free-Space Optical Intensity Channel

Constraints on **X**

- Nonnegativity and peak-power constraint:

$$X_k \in [0, A], \quad \forall\, k \in \{1, \ldots, n_T\}$$

- Total average-power constraint:

$$\mathsf{E}\left[\|\mathbf{X}\|_1\right] \leq \alpha A, \qquad \alpha \in \left(0, \frac{n_T}{2}\right]$$

Channel Capacity

- Channel: $\mathbf{Y} = \mathsf{H}\mathbf{X} + \mathbf{Z}$
- Capacity:

$$\mathsf{C}_{\mathsf{H}}(A, \alpha A) = \max_{P_{\mathbf{X}} \text{ s.t. (C1) \& (C2)}} \mathrm{I}(\mathbf{X}; \mathbf{Y})$$

$$\text{(C1)} \qquad \mathbf{X} \in [0, A]^{n_{\mathsf{T}}}$$

$$\text{(C2)} \qquad \mathsf{E}\left[\|\mathbf{X}\|_1\right] \leq \alpha A$$

- Capacity-achieving input distribution discrete [Smith'69], [Fahs&Abou-Faycal'18]

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$$R_Y(y) = \begin{cases} \beta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} & \text{if } y < 0, \\ (1-\beta) \cdot \frac{1}{\mathrm{A}} \cdot \frac{\mu}{1-e^{-\mu}} \cdot e^{-\frac{\mu y}{\mathrm{A}}}, & \text{if } y \in [0, \mathrm{A}], \\ \beta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\mathrm{A})^2}{2}} & \text{if } y > \mathrm{A}. \end{cases}$$

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- Only peak-power: [McKellips'04],[Thangaraj-Kramer-Böcherer'17], [Rassouli-Clerckx'17]

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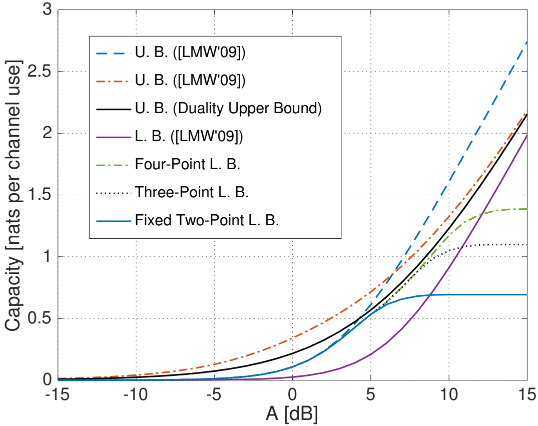
Numerical Result for a SISO Channel



- Average-to-peak power ratio $\alpha = 0.4$

High and Low-SNR Asymptotics

- Low-SNR capacity [Lapidoth-Moser-W'09]

$$C(A, \alpha A) = \frac{A^2(1-\alpha)}{2} + o(A^2)$$

- High-SNR capacity [Lapidoth-Moser-W'02]

$$C(A, \alpha A) = \frac{1}{2}\log\frac{A^2}{2\pi e} - (1-\alpha)\mu^\star + \log(1-\alpha\mu^\star) + o(1)$$

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Previous Results on MIMO channels

- n-by-n full-rank channel matrix:
  - Rx inverts channel matrix and Tx independently codes over parallel channels
  - Achieves high-SNR capacity
  - [Chaaban-Rezki-Alouini'17],[Dytso-Goldenbaum-Shamai-Poor'17],[Moser-Mylonakis-Wang-W'17]
- $n_R \geq n_T$: Rx performs SVD decomposition and ignores some outputs
- $n_T > n_R \geq 1$: Linear precoding matrix like for beamforming? [Chaaban-Rezki-Alouini'18]
- There are better solutions for $n_T > n_R$! [Moser-Wang-W'17]

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MISO Channel ($n_\mathsf{T} > n_\mathsf{R} = 1$)

- Channel: $Y = \mathbf{h}^\mathsf{T}\mathbf{X} + Z$
- Assume $\mathbf{h} = (h_1, \ldots, h_{n_\mathsf{T}})$ with $h_1 \geq h_2 \geq h_3 \geq \cdots \geq h_{n_\mathsf{T}}$

- Capacity: $\mathsf{C}(A, \alpha A) = \max_{P_\mathbf{X}} \mathrm{h}(\mathbf{h}^\mathsf{T}\mathbf{X} + Z) - \mathrm{h}(Z)$

  s.t.

$$(c1) \qquad X_i \in [0, A], \quad i \in \{1, \ldots, n_\mathsf{T}\},$$

$$(c2) \qquad \sum_{i=1}^{n_\mathsf{T}} \mathsf{E}[X_i] \leq \alpha A$$

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$$(c1\bar{X}) \qquad \bar{X} \in \left[0, \sum_{k=1}^{n_\mathsf{T}} h_i A\right],$$

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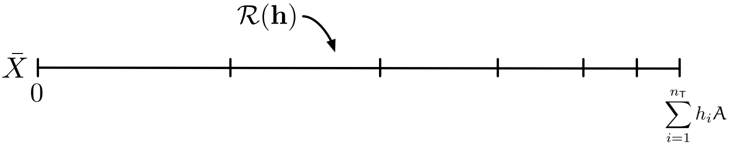
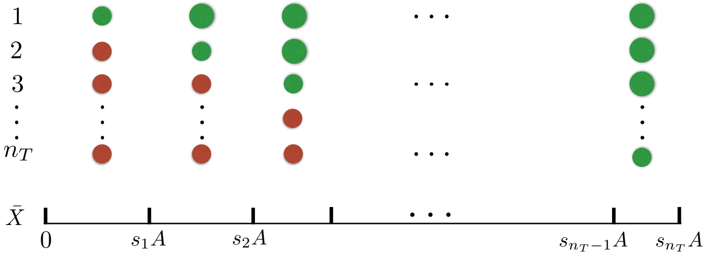
Restrict to inputs that consume minimum energy!



MISO Minimum-Energy Signaling

- Assume $h_1 \geq h_2 \geq h_3 \geq \cdots \geq h_{n_T}$
- Define $s_k := \sum_{i=1}^{k} h_i$ and $\quad U = k \iff \bar{X} \in [s_{k-1}, s_k)$
- Minimum-energy signaling (reduces $\sum \mathsf{E}[X_i]$):



- Average power:

$$\mathbb{E}[\bar{X}] = \sum_{i=1}^{n_T} \Pr[U = i] \left(\frac{\mathbb{E}[\bar{X} | U = i] - A s_{i-1}}{h_i} + (i-1)A \right)$$

Equivalent MISO Capacity Expression in Terms of $\bar{X}$

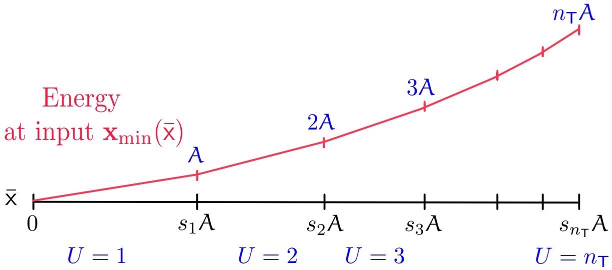
Theorem

$$\mathrm{C}_{\mathbf{h}}(A, \alpha A) = \max_{P_{\bar{X}}} I(\bar{X}; Y)$$

subject to:

$$(C1) \quad \bar{X} \in [0, s_{n_\mathsf{T}} A]$$

$$(C2) \quad \sum_{i=1}^{n_\mathsf{T}} \Pr[U = i]\left(\frac{\mathbb{E}[\bar{X}|U = i] - A s_{i-1}}{h_i} + (i-1)A\right) \le \alpha A$$



MISO Capacity Lower Bound with EPI

- Entropy-Power Inequality (EPI)

$$
\begin{aligned}
\mathrm{C} &\geq \frac{1}{2} \max_{P_{\bar{X}} \text{ s.t.} (c1\bar{X}) \& (c2\bar{X})} \log\left(1 + \frac{e^{2\,\mathrm{h}(\bar{X})}}{2\pi e}\right) \\
&= \frac{1}{2} \max_{P_{\bar{X}} \text{ s.t.} (c1\bar{X}) \& (c2\bar{X})} \log\left(1 + \frac{e^{2(\mathrm{H}(U) + \mathrm{h}(\bar{X}|U))}}{2\pi e}\right)
\end{aligned}
$$

- For each $U = i$, maximize $\mathrm{h}(\bar{X}|U = i)$ s.t. $\mathrm{E}\left[\bar{X}|U = i\right] \leq \mu$
  $\rightarrow$ uniform or truncated exponential distribution

- Optimize over P_U, μ

MISO Capacity Upper Bound

- $\bar{X}^\star, Y^\star$ the capacity-achieving input and output and

$$(U = i) \implies (\bar{X}^\star \in (s_{i-1}, s_i])$$

Then: $\mathrm{C} \leq \mathrm{I}(\bar{X}^\star; Y^\star, U) = \mathrm{H}(U) + \mathrm{I}(\bar{X}^\star; Y^\star | U)$

$$= \mathrm{H}(P_U) + \sum_{i=1}^{n_\mathrm{T}} P_U(i) \cdot \mathrm{I}(\bar{X}^\star; Y^\star | U = i)$$

- Given $U = i$, $\bar{X}^\star \mapsto Y^\star$ is a single-antenna channel with a linear power constraint $\Rightarrow$
apply previous UBs to each $\mathrm{I}(\bar{X}^\star; Y^\star | U = i)$

- Choose $\mathsf{E}\left[\bar{\mathbf{X}}^\star | U = i\right] = \mu,\ \forall i$ and optimize over P_U, μ

MIMO Channel Capacity $(n_\mathsf{T} > n_\mathsf{R})$

- Channel: $\mathbf{Y} = \mathsf{H}\mathbf{X} + \mathbf{Z}$
- Capacity: $\mathrm{C} = \max_{P_\mathbf{X}} \mathrm{h}(\mathsf{H}\mathbf{X} + \mathbf{Z}) - \mathrm{h}(\mathbf{Z})$
  s.t.

$$\begin{aligned} &(\mathrm{c}1) \qquad \mathbf{X} \in [0, A]^{n_\mathsf{T}} \\ &(\mathrm{c}2) \qquad \mathsf{E}[\|\mathbf{X}\|_1] \le \alpha A \end{aligned}$$

Recall $\mathsf{H} = [\mathbf{h}_1,\ \mathbf{h}_2,\ \ldots,\ \mathbf{h}_{n_\mathsf{T}}]$

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$$(\mathrm{c1}\bar{X}) \qquad \bar{\mathbf{X}} \in \mathcal{R}(\mathsf{H}) := \left\{ \sum_{k=1}^{n_\mathrm{T}} x_k \mathbf{h}_k : x_1, \ldots, x_{n_\mathrm{T}} \in [0, A] \right\}$$

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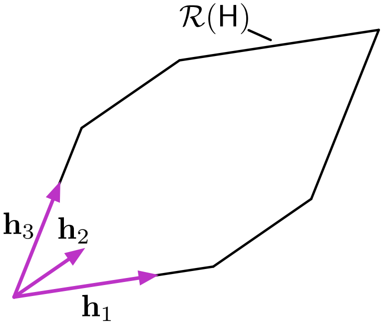
Restrict to minimum-energy signaling: $\mathbf{x}_{\min}(\bar{\mathbf{x}}) := \underset{\mathbf{x}:\, \mathsf{H}\mathbf{x} = \bar{\mathbf{x}}}{\operatorname{argmin}} \|\mathbf{x}\|_1$

Recall $\mathsf{H} = [\mathbf{h}_1,\ \mathbf{h}_2,\ \ldots,\ \mathbf{h}_{n_\mathrm{T}}]$

Example of 2 × 3 MIMO Minimum-Energy Signaling

For each $\bar{\mathbf{x}} \in \mathcal{R}(\mathsf{H})$: $\quad \mathbf{x}_{\min}(\bar{\mathbf{x}}) := \operatorname{argmin}_{\mathbf{x}:\mathsf{H}\mathbf{x}=\bar{\mathbf{x}}} \|\mathbf{x}\|_1$



$$\mathsf{H} = [\mathbf{h}_1 \ \mathbf{h}_2 \ \mathbf{h}_3] = \begin{pmatrix} 3 & 1.75 & 1 \\ 0.5 & 1 & 3 \end{pmatrix}$$

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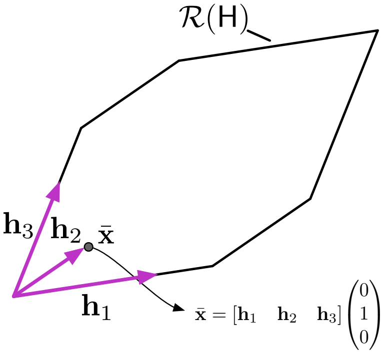

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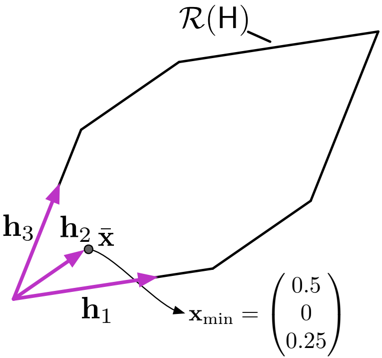

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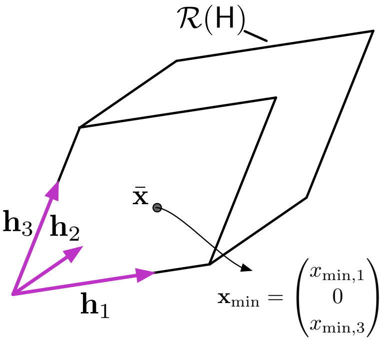

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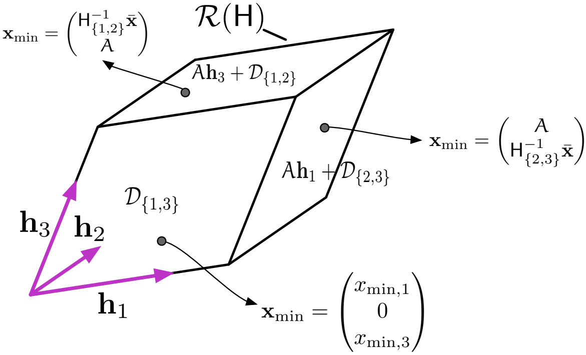

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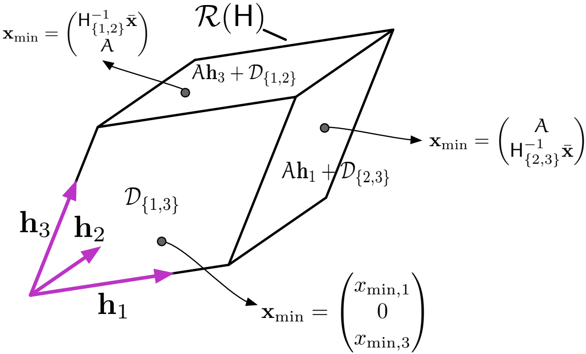

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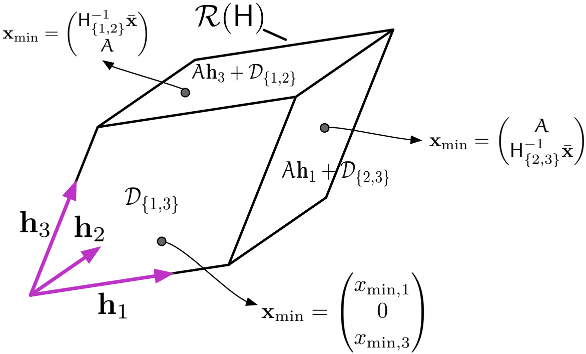

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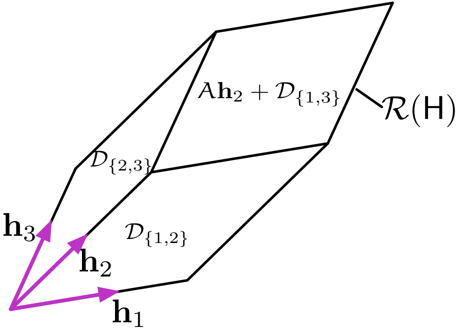
In each parallelepiped only two antennas are used for signaling $\rightarrow$ full-rank 2×2 MIMO channel

Example of 2 × 3 MIMO Minimum-Energy Signaling

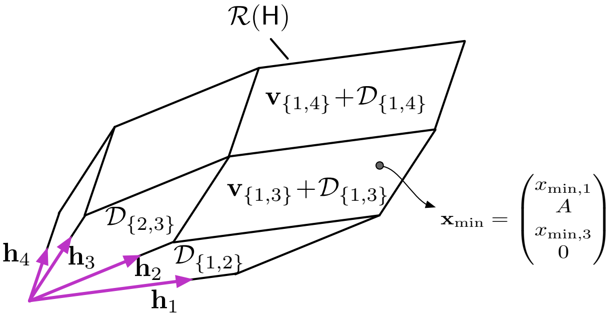
Another 2 × 3 MIMO channel:



$$\mathsf{H} = [\mathbf{h}_1\ \mathbf{h}_2\ \mathbf{h}_3] = \begin{pmatrix} 2.5 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Example of 2×4 MIMO Minimum-Energy Signaling

A 2×4 MIMO channel:



$$\mathsf{H} = [\mathbf{h}_1 \; \mathbf{h}_2 \; \mathbf{h}_3 \; \mathbf{h}_4] = \begin{pmatrix} 7 & 5 & 2 & 1 \\ 1 & 2 & 2.9 & 3 \end{pmatrix}$$

Minimum-Energy Signaling for $n_{\mathsf{R}} \times n_{\mathsf{T}}$ MIMO Channels

- Minimum-energy signaling partitions $\mathcal{R}(\mathsf{H})$ into $\binom{n_{\mathsf{T}}}{n_{\mathsf{R}}}$ parallelepipeds, each one spanned by one subset of n_{R} channel vectors
- In each parallelepiped $\mathcal{D}_{\mathcal{I}}$ only antennas in $\mathcal{I}$ are used for signaling, the others are set 0 or A $\Rightarrow$ an $n_{\mathsf{R}} \times n_{\mathsf{R}}$ full-rank MIMO channel
- Arrangement of parallelepipeds $\{\mathcal{D}_{\mathcal{I}}\}$ depends on the channel matrix and is determined by $\{\mathbf{v}_{\mathcal{I}}\}$:

$$\mathbf{v}_{\mathcal{I}} := A \sum_{j \in \mathcal{I}^{\mathsf{c}}} \mathbb{I}\left(\mathbf{1}_{n_{\mathsf{R}}}^{\mathsf{T}} \mathsf{H}_{\mathcal{I}}^{-1} \mathbf{h}_j > 1\right) \mathbf{h}_j$$

with

$$\mathsf{H}_{\mathcal{I}} := [\mathbf{h}_i : i \in \mathcal{I}]$$

Minimum-Energy Signaling: Proof based on KKT

- Lagrangean: $\mathcal{L} = \|\mathbf{x}\|_1 - \boldsymbol{\mu}^\mathsf{T}\mathbf{x} + \boldsymbol{\nu}^\mathsf{T}(\mathbf{x} - A) - \boldsymbol{\lambda}^\mathsf{T}(\mathsf{H}\mathbf{x} - \bar{\mathbf{x}})$

- KKT conditions:

$$\begin{aligned} \mathbf{1}_{n_\mathsf{T}} - \boldsymbol{\mu}^\star + \boldsymbol{\nu}^\star - \mathsf{H}^\mathsf{T}\boldsymbol{\lambda}^\star &= \mathbf{0} \\ \mathsf{H}\mathbf{x}^\star - \bar{\mathbf{x}} &= \mathbf{0} \\ \mu_i^\star x_i^\star &= 0, \quad i \in \{1, \ldots, n_\mathsf{T}\}, \\ \nu_i^\star (x_i^\star - A) &= 0, \quad i \in \{1, \ldots, n_\mathsf{T}\}, \end{aligned}$$

Minimum-Energy Signaling: Proof based on KKT

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- Set $i \in \mathcal{I} \iff x_i^\star \in (0, A);$:

$$\mu_\mathcal{I} = \nu_\mathcal{I} = \mathbf{0} \quad \Rightarrow \quad \mathsf{H}_\mathcal{I}^\mathsf{T}\boldsymbol{\lambda}^\star = \mathbf{1}_{|\mathcal{I}|} \quad \Rightarrow \quad \boldsymbol{\lambda}^\star = \mathsf{H}_\mathcal{I}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|}.$$

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Minimum-Energy Signaling: Proof based on KKT

- KKT conditions:

$$\begin{aligned} \mathbf{1}_{n_{\mathsf{T}}} - \boldsymbol{\mu}^\star + \boldsymbol{\nu}^\star - \mathsf{H}^{\mathsf{T}}\boldsymbol{\lambda}^\star &= \mathbf{0} \\ \mathsf{H}\mathbf{x}^\star - \bar{\mathbf{x}} &= \mathbf{0} \\ \mu_i^\star x_i^\star &= 0, \quad i \in \{1, \ldots, n_{\mathsf{T}}\}, \\ \nu_i^\star (x_i^\star - A) &= 0, \quad i \in \{1, \ldots, n_{\mathsf{T}}\}, \end{aligned}$$

- Set $i \in \mathcal{I} \quad \Longleftrightarrow \quad x_i^\star \in (0, A)$; restrict to $\operatorname{rank}(\mathsf{H}_{\mathcal{I}}) = n_{\mathsf{R}}$!:

$$\mu_{\mathcal{I}} = \nu_{\mathcal{I}} = \mathbf{0} \quad \Rightarrow \quad \mathsf{H}_{\mathcal{I}}^{\mathsf{T}}\boldsymbol{\lambda}^\star = \mathbf{1}_{|\mathcal{I}|} \quad \Rightarrow \quad \boldsymbol{\lambda}^\star = \mathsf{H}_{\mathcal{I}}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|}.$$

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Minimum-Energy Signaling: Proof based on KKT

- KKT conditions:

$$\begin{aligned} \mathbf{1}_{n_\mathsf{T}} - \boldsymbol{\mu}^\star + \boldsymbol{\nu}^\star - \mathsf{H}^\mathsf{T}\boldsymbol{\lambda}^\star &= \mathbf{0} \\ \mathsf{H}\mathbf{x}^\star - \bar{\mathbf{x}} &= \mathbf{0} \\ \mu_i^\star x_i^\star &= 0, \quad i \in \{1,\dots,n_\mathsf{T}\}, \\ \nu_i^\star (x_i^\star - \mathrm{A}) &= 0, \quad i \in \{1,\dots,n_\mathsf{T}\}, \end{aligned}$$

- Set $i \in \mathcal{I} \iff x_i^\star \in (0, \mathrm{A})$; restrict to $\operatorname{rank}(\mathsf{H}_\mathcal{I}) = n_\mathsf{R}$!:

$$\mu_\mathcal{I} = \nu_\mathcal{I} = \mathbf{0} \quad \Rightarrow \quad \mathsf{H}_\mathcal{I}^\mathsf{T}\boldsymbol{\lambda}^\star = \mathbf{1}_{|\mathcal{I}|} \quad \Rightarrow \quad \boldsymbol{\lambda}^\star = \mathsf{H}_\mathcal{I}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|}.$$

- Optimal inputs $\mathbf{x}_\mathcal{I}^\star = \mathsf{H}_\mathcal{I}^{-1}\Bigg(\underbrace{\bar{\mathbf{x}} - \sum_{j\in\mathcal{I}^\mathsf{c}} x_j^\star \mathbf{h}_j}_{\mathbf{v}_\mathcal{I}}\Bigg)$

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Minimum-Energy Signaling: Proof based on KKT

- KKT conditions:

$$\mathbf{1}_{n_\mathsf{T}} - \boldsymbol{\mu}^\star + \boldsymbol{\nu}^\star - \mathsf{H}^\mathsf{T}\boldsymbol{\lambda}^\star = \mathbf{0}$$
$$\mathsf{H}\mathbf{x}^\star - \bar{\mathbf{x}} = \mathbf{0}$$
$$\mu_i^\star x_i^\star = 0, \quad i \in \{1, \dots, n_\mathsf{T}\},$$
$$\nu_i^\star (x_i^\star - \mathrm{A}) = 0, \quad i \in \{1, \dots, n_\mathsf{T}\},$$

- Set $i \in \mathcal{I} \iff x_i^\star \in (0, \mathrm{A})$; restrict to $\mathrm{rank}(\mathsf{H}_\mathcal{I}) = n_\mathsf{R}$!:

$$\mu_\mathcal{I} = \nu_\mathcal{I} = \mathbf{0} \quad \Rightarrow \quad \mathsf{H}_\mathcal{I}^\mathsf{T}\boldsymbol{\lambda}^\star = \mathbf{1}_{|\mathcal{I}|} \quad \Rightarrow \quad \boldsymbol{\lambda}^\star = \mathsf{H}_\mathcal{I}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|}.$$

- Optimal inputs $x_j^\star$ for $j \in \mathcal{I}^\mathsf{c}$:

$$1 - \mathbf{h}_j^\mathsf{T}\mathsf{H}_\mathcal{I}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|} = \mu_i^\star \cdot \mathbb{I}\{x_j^\star = 0\} - \nu_i^\star \cdot \mathbb{I}\{x_j^\star = \mathrm{A}\}$$

$\mu_j^\star, \nu_j^\star \geq 0 \to$ value of $x_j^\star$ determined by $\mathrm{sign}(1 - \mathbf{h}_j^\mathsf{T}\mathsf{H}_\mathcal{I}^{-\mathsf{T}}\mathbf{1}_{|\mathcal{I}|})$

Equivalent Capacity Expression in terms of $\bar{\mathbf{X}}$

Theorem (Equivalent Capacity Expression)

The MIMO capacity satisfies

$$\mathrm{C} = \max_{P_{\bar{\mathbf{X}}}} \mathrm{I}(\bar{\mathbf{X}}; \mathbf{Y}),$$

where $P_{\bar{\mathbf{X}}}$ over $\mathcal{R}(\mathsf{H})$ and subject to the power constraint:

$$\sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,n_{\mathsf{T}}\} \\ |\mathcal{I}| = n_{\mathsf{R}}}} p_{\mathcal{I}} \cdot \left(\left\| \mathsf{H}_{\mathcal{I}}^{-1} \left(\mathsf{E}[\bar{\mathbf{X}} \,|\, \mathcal{I}] - \mathbf{v}_{\mathcal{I}} \right) \right\|_1 + \| \mathbf{v}_{\mathcal{I}} \|_1 \right) \leq \alpha \mathrm{A}$$

with

$$p_{\mathcal{I}} = \Pr \left(\bar{\mathbf{X}} \in (\mathbf{v}_{\mathcal{I}} + \mathcal{D}_{\mathcal{I}}) \right)$$

$$\mathbf{v}_{\mathcal{I}} = \mathrm{A} \cdot \sum_{j \in \mathcal{I}^{\mathsf{c}}} \mathbb{I}(\mathbf{1}_{n_{\mathsf{R}}}^{\mathsf{T}} \mathsf{H}_{\mathcal{I}}^{-1} \mathbf{h}_j > 1) \cdot \mathbf{h}_j$$

MIMO Capacity Upper Bound

- $\bar{\mathbf{X}}^\star, \mathbf{Y}^\star$ the capacity-achieving input and output and

$$(U = \mathcal{I}) \implies (\bar{\mathbf{X}}^\star \in (\mathbf{v}_\mathcal{I} + \mathcal{D}_\mathcal{I}))$$

Then: $\mathrm{C} \leq \mathrm{I}(\bar{\mathbf{X}}^\star; \mathbf{Y}^\star, U^\star) = \mathrm{H}(U^\star) + \mathrm{I}(\bar{\mathbf{X}}^\star; \mathbf{Y}^\star \big| U^\star)$

$$= H(\{p_\mathcal{I}^\star\}) + \sum_{\substack{\mathcal{I} \subseteq \{1,2,\ldots,n_\mathsf{T}\} \\ |\mathcal{I}| = n_\mathsf{R}}} p_\mathcal{I} \cdot \mathrm{I}(\bar{\mathbf{X}}^\star; \mathbf{Y}^\star \big| U^\star = \mathcal{I})$$

- Given $U = \mathcal{I}$, $\mathbf{X}^\star \mapsto \mathbf{Y}^\star$ is an $n_\mathsf{R} \times n_\mathsf{R}$ MIMO with invertible channel matrix $\to$ use previous UBs

- For duality bound: uniform or rotated truncated exponential distribution with Gaussian tails

MISO Capacity Lower Bound with EPI

- Entropy-Power Inequality (EPI)

$$\begin{aligned} \mathrm{C} &\geq \max_{P_{\bar{\mathbf{x}}} \text{ s.t.}(c1\bar{\mathbf{X}})\&(c2\bar{\mathbf{X}})} \frac{1}{2} \log\left(1 + \frac{e^{2\,\mathrm{h}(\bar{\mathbf{X}})}}{2\pi e}\right) \\ &= \max_{P_{\bar{\mathbf{x}}} \text{ s.t.}(c1\bar{\mathbf{X}})\&(c2\bar{\mathbf{X}})} \frac{1}{2} \log\left(1 + \frac{e^{2(\mathrm{H}(U)+\mathrm{h}(\bar{\mathbf{X}}|U))}}{2\pi e}\right) \end{aligned}$$

- For each $U = i$ choose $P_{\bar{\mathbf{X}}|U=i}$ to maximize $\mathrm{h}(\bar{\mathbf{X}}|U = i)$
  $\rightarrow$ uniform or rotated truncated exponential distribution

- Numerically optimize over P_U

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High-SNR Asymptotic Capacity

Theorem (High-SNR Asymptotics)

For $\alpha \geq \alpha_{\text{th}}(\mathrm{H})$,

$$\lim_{A\to\infty}\{C - n_R \log A\} = \frac{1}{2}\log\left(\frac{V_H^2}{(2\pi e)^{n_R}}\right)$$

For $\alpha < \alpha_{\text{th}}(\mathrm{H})$,

$$\lim_{A\to\infty}\{C - n_R \log A\} = \frac{1}{2}\log\left(\frac{V_H^2}{(2\pi e)^{n_R}}\right) + \nu$$

with

$$V_H := \mathrm{vol}(\mathcal{R}(\mathrm{H}))$$

$$\nu := \sup_{\lambda\in\left(\max\{0,\frac{n_R}{2}+\alpha-\alpha_{\text{th}}\},\min\{\frac{n_R}{2},\alpha\}\right)} \left\{ n_R\left(1 - \log\frac{\mu}{1-e^{-\mu}} - \frac{\mu\, e^{-\mu}}{1-e^{-\mu}}\right) - \inf_{\mathbf{p}} \mathrm{D}(\mathbf{p}\|\mathbf{q}) \right\}$$

- At high SNR, average-power constraint is not active for $\alpha > \alpha_{\text{th}}$

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Maximum-Variance Upper Bound

- The covariance matrix of $\bar{\mathbf{X}} = \mathsf{H}\mathbf{X}$:

$$\mathsf{K}_{\bar{\mathbf{X}}\bar{\mathbf{X}}} \triangleq \mathsf{E}\left[(\bar{\mathbf{X}} - \mathsf{E}[\bar{\mathbf{X}}])(\bar{\mathbf{X}} - \mathsf{E}[\bar{\mathbf{X}}])^{\mathsf{T}}\right]$$

Lemma

For an arbitrary α,

$$\mathsf{C} \leq \frac{n_{\mathrm{R}}}{2} \log\left(1 + \frac{1}{n_{\mathrm{R}}} \max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{X}}\bar{\mathbf{X}}}\right)\right)$$

MIMO Low-SNR Capacity Slope

- Maximum-Variance Upper Bound

$$C \leq \frac{n_R}{2} \log\left(1 + \frac{1}{n_R} \max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right)\right)$$

- Maximum-Variance Lower Bound [Prelov-van der Meulen'04]

$$C \geq \frac{1}{2} \log \max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right) + o(A^2)$$

- Low-SNR Capacity Slope [Chaaban-Rezki-Alouini'18]

$$\lim_{A \downarrow 0} \frac{C}{A^2} = \frac{1}{2} \log \max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right)$$

Known Properties of $\max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{X}}\bar{\mathbf{X}}}\right)$ [Chaaban-Rezki-Alouini'18]

- An optimal input to the maximum trace uses for each component of **X** only the values 0 and A

$$X_i \in \{0, A\} \quad \text{with probability } 1, \quad i = 1, \ldots, n_{\mathsf{T}}$$

- An optimal input is over a set $\{\mathbf{x}_1^*, \mathbf{x}_2^*, \ldots\}$ satisfying

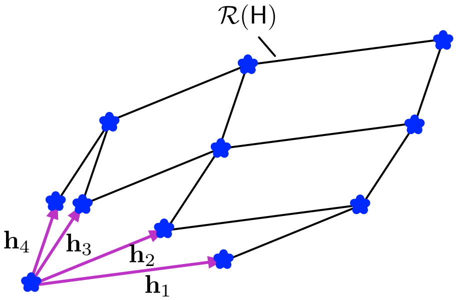
$$x_{k,\ell}^* \leq x_{k',\ell}^* \quad \text{for all } k < k',\ \ell = 1, \ldots, n_{\mathsf{T}}$$

e.g.

$$(0,0,0,0,0); \quad (0,A,0,0,0); \quad (0,A,0,A,0); \quad (0,A,0,A,A)$$

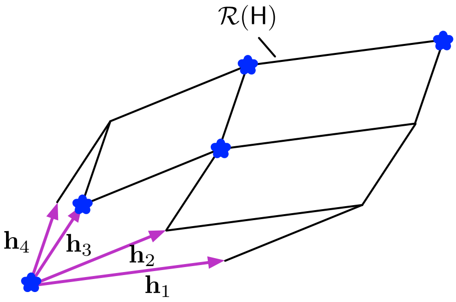
New Properties of $\max_{P_{\bar{\mathsf{x}}}} \mathrm{tr}\left(\mathsf{K}_{\bar{\mathsf{x}}\bar{\mathsf{x}}}\right)$

- By the optimality of minimum-energy signaling, we can restrict to corner points of parallelepipeds



New Properties of $\max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right)$

- By the optimality of minimum-energy signaling, we can restrict to corner points of parallelepipeds
- Examining the KKT conditions: under mild technical conditions $\rightarrow$ $n_{\mathsf{R}} + 2$ mass points suffice



New Properties of $\max_{P_{\bar{\mathbf{x}}}} \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right)$

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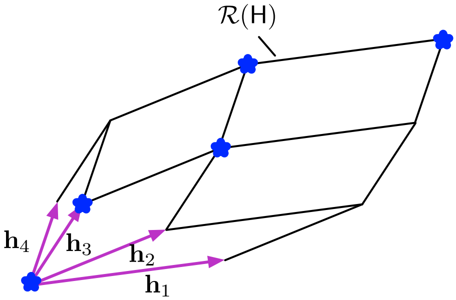

Significantly reduce numerical computation of $\max \operatorname{tr}\left(\mathsf{K}_{\bar{\mathbf{x}}\bar{\mathbf{x}}}\right)$

Numerical Examples on Maximum-Variance Signaling

channel matrix	α	$\max_{P_{\mathbf{X}}} \mathrm{tr}(\mathsf{K}_{\bar{\mathbf{X}}\bar{\mathbf{X}}})$	$P_{\mathbf{X}}: \max_{P_{\mathbf{X}}} \mathrm{tr}(\mathsf{K}_{\bar{\mathbf{X}}\bar{\mathbf{X}}})$
$\mathsf{H} = \begin{pmatrix} 1.3 & 0.6 & 1 & 0.1 \\ 2.1 & 4.5 & 0.7 & 0.5 \end{pmatrix}$	1.5	$16.3687A^2$	$P_{\mathbf{X}}(0,0,0,0) = 0.625$, $P_{\mathbf{X}}(A,A,A,A) = 0.375$
	0.9	$12.957A^2$	$P_{\mathbf{X}}(0,0,0,0) = 0.7$, $P_{\mathbf{X}}(A,A,A,0) = 0.3$
	0.6	$9.9575A^2$	$P_{\mathbf{X}}(0,0,0,0) = 0.7438$, $P_{\mathbf{X}}(A,A,0,0) = 0.1687$, $P_{\mathbf{X}}(A,A,A,0) = 0.0875$
	0.3	$6.0142A^2$	$P_{\mathbf{X}}(0,0,0,0) = 0.85$, $P_{\mathbf{X}}(A,A,0,0) = 0.15$
$\mathsf{H} = \begin{pmatrix} 0.9 & 3.2 & 1 & 2.1 \\ 0.5 & 3.5 & 1.7 & 2.5 \\ 0.7 & 1.1 & 1.1 & 1.3 \end{pmatrix}$	0.9	$23.8405A^2$	$P_{\mathbf{X}}(0,0,0,0) = 0.7755$, $P_{\mathbf{X}}(A,A,A,A) = 0.2245$

Summary I

- Minimum-energy signaling: decompose the original channel into a set of almost parallel channels. Each one an $n_R \times n_R$ MIMO channel with an invertible channel matrix
- New upper bounds by using duality capacity expression, and lower bounds by using the EPI. They match at high-SNR, establishing the high-SNR asymptotic capacity
- At low-SNR, a computationally easier way to evaluate the low-SNR capacity slope

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Block Fading FSO Channel

Channel Model

$$\mathbf{Y}_t[n] = \mathbb{H}_t \mathbf{x}_t[n] + \mathbf{Z}_t[n]$$

- $t \in \{1, 2, \ldots, \mathrm{B}\}$ the block index, B the number of blocks
- $n \in \{1, 2, \ldots, \mathrm{N}\}$ the symbol index, N the block length
- Channel input vector $\mathbf{x}_t[n]$

$$\text{(C1)} \qquad \mathbf{X_t[n]} \in [0, A]^{n_\mathrm{T}} \quad \forall\, n \in \{1, 2, \ldots, \mathrm{N}\}$$

$$\text{(C2)} \quad \frac{1}{\mathrm{N}} \mathsf{E}\left[\sum_{n=1}^{\mathrm{N}} \left\|\mathbf{X}_t[n]\right\|_1\right] \leq \alpha A \quad \forall\, t \in \{1, 2, \ldots, \mathrm{B}\}$$

- $\mathbb{H}_t$ a random $n_\mathrm{R} \times n_\mathrm{T}$ matrix, fixed in one block, i.i.d across blocks

Channel State Information (CSI)

- Ergodic Capacity
  - The highest achievable rate when $\mathrm{B} \to \infty$
- Availability of CSI
  - Receiver always has perfect CSI
  - Transmitter has no CSI, perfect CSI, and limited CSI

NO CSI at the Transmitter

- With no CSI at the transmitter, the choice of $P_{\mathbf{X}}$ is independent of H

Theorem

If the transmitter has no CSI, then

$$\mathrm{C}_{\mathbb{H}} = \max_{P_{\mathbf{X}}} \mathrm{E}_{\mathbb{H}} \left[\mathrm{I}(\mathbf{X}; \mathbb{H}\mathbf{X} + \mathbf{Z} | \mathbb{H} = \mathrm{H}) \right]$$

- A precoding strategy:

$$\mathbf{X} = \mathrm{G}\mathbf{V}$$

G is an $n_{\mathrm{T}} \times n_{\mathrm{R}}$ matrix, and $\mathbf{V}$ the n_{R}-dimensional random vector exponentially distributed with density

$$f(\mathbf{v}) = \frac{1}{A^{n_{\mathrm{R}}}} \left(\frac{\mu}{1 - e^{-\mu}} \right)^{n_{\mathrm{R}}} e^{-\frac{\mu \|\mathbf{v}\|_1}{A}}, \quad \mathbf{v} \in [0, A]^{n_{\mathrm{R}}}$$

NO CSI at the Transmitter

Theorem

If the transmitter has no CSI, then

$$\mathsf{C}_{\mathbb{H}} \geq \frac{1}{2} \sup_{\lambda \in (0, \frac{n_\mathsf{R}}{2})} \sup_{\mathsf{G}} \mathsf{E}_{\mathbb{H}} \left[\log_2 \left(1 + \frac{A^{2n_\mathsf{R}} (\det \mathbb{H}\mathsf{G})^2 \, e^{2\nu}}{(2\pi e)^{n_\mathsf{R}}} \right) \right]$$

the sup. over all $n_\mathsf{T} \times n_\mathsf{R}$ precoding matrices G*:*

- *nonnegative entries*
- $\mathrm{rank}(\mathsf{G}) = n_\mathsf{R}$
- $\|\mathsf{G}\|_1 \leq \alpha/\lambda$, *and with row vectors satisfying* $\|\mathbf{g}_i\|_1 \leq 1$, $\forall\, i \in \{1, \ldots, n_\mathsf{T}\}$

and ν *is defined as*

$$\nu \triangleq n_\mathsf{R} \left(1 - \log_2 \frac{\mu}{1 - e^{-\mu}} - \frac{\mu\, e^{-\mu}}{1 - e^{-\mu}} \right)$$

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Perfect CSI at the Transmitter

- With perfect CSI at the transmitter, the choice of $P_{\mathbf{X}}$ depends on the CSI

Theorem

If the transmitter has perfect CSI, then

$$\mathrm{C}_{\mathbb{H}} = \mathrm{E}_{\mathbb{H}}\left[\max_{P_{\mathbf{X}|\mathbb{H}=\mathrm{H}}} \mathrm{I}(\mathbf{X}; \bar{\mathbf{X}} + \mathbf{Z} | \mathbb{H} = \mathrm{H})\right]$$

- For each channel realization $\mathbb{H} = \mathrm{H}$, we use bounds from MIMO case
- Signaling exploits minimum-energy signaling

Limited CSI at the Transmitter

- By a rate-limited feedback link, at the beginning of each block, the receiver sends a function of $\mathbb{H}$, $\mathcal{F}(\mathbb{H})$, back to the transmitter
- The choice of $P_{\mathbf{X}}$ depends on $\mathcal{F}(\mathbb{H})$

Theorem

The capacity $\mathrm{C}_{\mathbb{H},\mathcal{F}}$ of a channel with limited CSI $\mathcal{F}(\mathbb{H})$ at the transmitter is:

$$\mathrm{C}_{\mathbb{H},\mathcal{F}} = \mathsf{E}_{\mathbb{H}}\left[\max_{P_{\mathbf{X}|\mathcal{F}(\mathsf{H})}} \mathrm{I}(\mathbf{X};\mathbf{Y}|\mathbb{H}=\mathsf{H})\right]$$

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Limited CSI at the Transmitter

- Receiver feeds back

$$\mathrm{R}_{\mathsf{FB}} = (n_{\mathsf{T}} - n_{\mathsf{R}}) \cdot \binom{n_{\mathsf{T}}}{n_{\mathsf{R}}}$$

  bits per channel block

- The R_{FB} bits describes the binary values of $n_{\mathsf{T}} - n_{\mathsf{R}}$ antennas not used for signaling in each parallelepiped in the minimum-energy signaling strategy
- The input distribution in each parallelepiped are chosen to be a truncated exponential distribution satisfying the power constraint

Limited CSI at the Transmitter

Theorem

For the limited CSI function $\mathcal{F}$:

$$C_{\mathbb{H},\mathcal{F}} \geq E_{\mathcal{F}(\mathbb{H})}\left[\sup_{\lambda\in(0,\frac{n_R}{2})} \sup_{\mathbf{p}} E_{\mathbb{H}|\mathcal{F}}\left[\frac{1}{2}\log\left(1+\frac{A^{2n_R}{V_H}^2\, e^{2\nu^\star}}{(2\pi e)^{n_R}}\right)\right]\right]$$

where

$$\nu^\star \triangleq n_R\left(1-\log_2\frac{\mu}{1-e^{-\mu}}-\frac{\mu\, e^{-\mu}}{1-e^{-\mu}}\right)-\mathrm{D}(\mathbf{p}\|\mathbf{q})$$

where the supremum over $\mathbf{p}$ *is over all PMFs satisfying*

$$\sum_{\mathcal{I}\in\mathcal{U}} p_{\mathcal{I}} s_{\mathcal{I}} = \alpha - \lambda.$$

Numerical Result for a 1 × 3 Channel

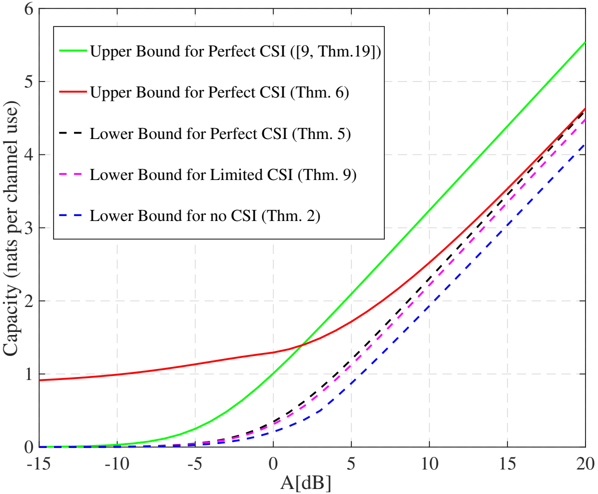

Figure 1: A 1 × 3 MISO channel, where the entries in $\mathbb{H}$ follow a Gamma-Gamma distribution.

Summary II

- Lower bounds presented for the capacities without CSI, with perfect CSI, and with limited CSI
- For perfect CSI, the lower bound asymptotically tight at high SNR
- For limited CSI, the lower bound is close to the one with perfect CSI, but with only $(n_{\mathrm{T}} - n_{\mathrm{R}})\binom{n_{\mathrm{T}}}{n_{\mathrm{R}}}$ bits of feedback in each block

Conclusion

- Minimum-energy signaling: decompose $\mathcal{R}(\mathsf{H})$ into $\binom{n_\mathsf{T}}{n_\mathsf{R}}$ n_R-dim. parallelepipeds, and use a different subset of n_R antennas for signaling in each parallelepiped

- Several upper bounds by duality-based capacity expression and lower bounds by the EPI

- High-and low-SNR asymptotic capacity

- With limited bits of channel state feedback, the perfect CSI capacity can be well approximated