An Achievable Region for the Discrete Memoryless Broadcast Channel with Feedback

Michèle Wigger (michele.wigger@telecom-paristech.fr)

joint work with Ofer Shayevitz (ofersha@ucsd.edu)

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When and Why Does Feedback Increase Capacity?

- No gain for memoryless point-to-point channels

  $\rightarrow$ transmitter learns only about *past* channel realizations!

- Gain for point-to-point channels with memory

  $\rightarrow$ transmitter learns about *future* channel realizations!

- Gain for multiple-access channels

  $\rightarrow$ transmitters learn about other transmitter's message

- Gain for interference channels

  $\rightarrow$ transmitters learn about other transmitter's message

Memoryless broadcast channels (BC)?

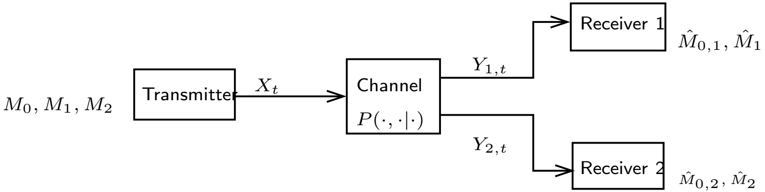
Previous Results on Capacity of BCs with Feedback

- El Gamal'78: No feedback-gain for physically degraded BCs
- Dueck'80, Kramer'00: Feedback-gain for specific discrete memoryless BCs
- Ozarow'85: Feedback-gain for some white Gaussian noise BCs
- Kramer'00: Multi-letter achievable region for general discrete memoryless BCs with noisy or noise-free feedback

In this talk

Single-letter achievable region for discrete memoryless BCs with noise-free or noisy feedback

Discrete Memoryless Broadcast Channel

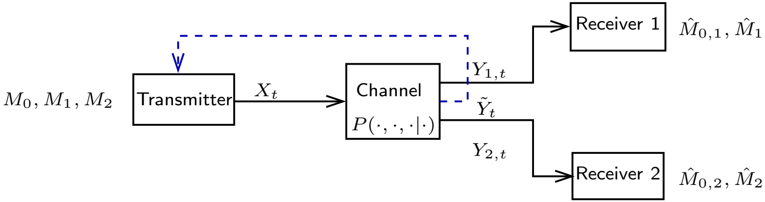


- Rx i wants to learn $M_0 \in \{1, \ldots, \lfloor 2^{nR_0} \rfloor\}$ and $M_i \in \{1, \ldots, \lfloor 2^{nR_i} \rfloor\}$
- Finite input and output alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2$
- Channel memoryless

$$(Y_{1,t}, Y_{2,t}) \multimap\!\!- X_t \multimap\!\!- (X^{t-1}, Y_1^{t-1}, Y_2^{t-2})$$

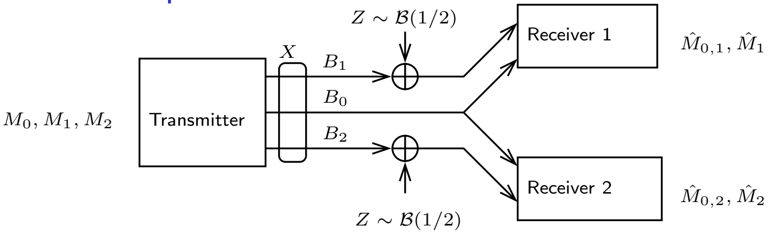
- Channel law $P(y_1, y_2|x)$ of observing y_1 and y_2 for input x

Generalized Feedback



- Third output $\tilde{Y}_t \in \tilde{\mathcal{Y}}$ observed at transmitter
- Inputs: $X_t = f_t(M_0, M_1, M_2, \tilde{Y}^{t-1})$
- Special cases:
  - Noise-free output feedback: $\tilde{Y}_t = (Y_{1,t}, Y_{2,t})$
  - Noisy output feedback: $\tilde{Y}_t = (Y_{1,t} + W_{1,t}, W_{2,t} + Z_{2,t})$

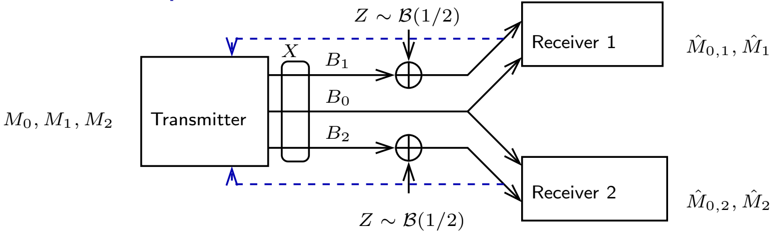
Dueck's Example



Without feedback:

- Top and bottom links useless
- No-feedback capacity: $0 \leq R_0 + R_1 + R_2 \leq 1$

Dueck's Example



Noise-free feedback:

- Transmitter learns noise and sends $B_{0,t} = Z_{t-1}$
- Feedback capacity: $0 \leq R_0 + R_1, R_0 + R_2 \leq 1$

Intuition why feedback helps

- "Actions" of channels $X \to Y_1$ and $X \to Y_2$ correlated
- Can send information useful to both receivers

Our Coding Scheme

- Block-Markov strategy
- Update info. about previous channel "actions" learned via feedback
- Fresh data/update info. sent with Marton's no-fb scheme
- Backward decoding:
  1. Block-b outputs improved with block-$(b+1)$ update info.
  2. Marton-decoding based on improved outputs

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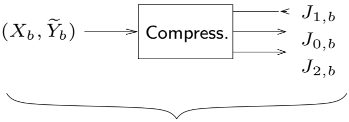
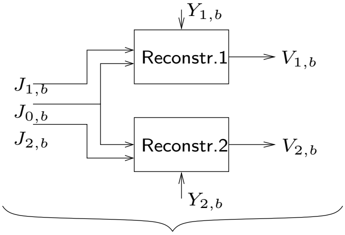
Update info.: Lossy GW-Compression of Channel Actions

Update info sent in block $(b+1)$:



Generation at transmitter



Processing at receivers

- Indices $(J_{0,b}, J_{1,b}, J_{2,b})$ describe lossy compression of $(X_b, \widetilde{Y}_b)$
- Goal: $(V_{i,b}, X_b, \tilde{Y}_b)$ jointly typical $\sim P_{V_i,X,\tilde{Y}}$
- $V_{1,b}, V_{2,b}$: lossy reconstructions of channel "actions"
- Improved block-b outputs: $(V_{i,b}, Y_{i,b})$

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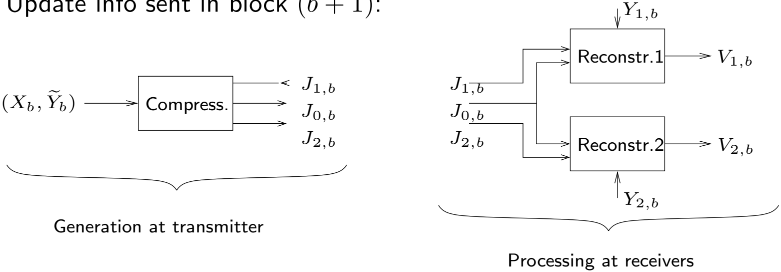

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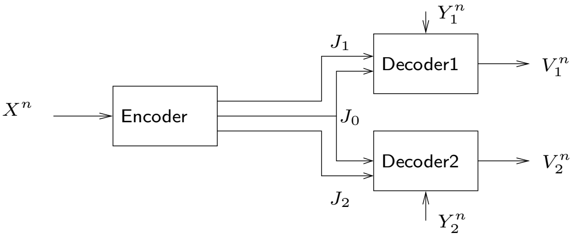
Due to side-info, even independent $V_{1,b}$ and $V_{2,b}$ have interesting J_0

Lossy Gray-Wyner Source Coding with Side-Info.

Given: $P_{XY_1Y_2}P_{V_1V_2|X}$; $(X^n, Y_1^n, Y_2^n) \sim$ IID $P_{XY_1Y_2}$

Goal: V_i^n jointly typical with X^n according to P_{X,V_i}



A triplet (R_0, R_1, R_2) is achievable, if

$$R_0 > \max_i I(X; V_0|Y_i),$$
$$R_1 > I(X; V_1|V_0, Y_1)$$
$$R_2 > I(X; V_2|V_0, Y_2)$$

for some V_0 s.t. $(V_0, V_1, V_2) - \circ - X - \circ - (Y_1, Y_2)$ forms a Markov chain.

Our Achievable Region for the DMBC with Feedback

Theorem

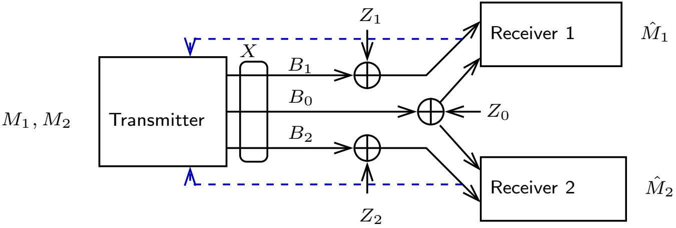
Nonnegative triplet (R_0, R_1, R_2) *achievable if,*

$$
\begin{aligned}
R_0 &\leq \min_i I(U_0; Y_i, V_i) - \max_i I(V_0; X, \widetilde{Y} | Y_i) \\
R_0 + R_1 &\leq I(U_0, U_1; Y_1, V_1) - I(X, \widetilde{Y}; V_1 | V_0, Y_1) - \max_i I(V_0; X, \widetilde{Y} | Y_i) \\
R_0 + R_2 &\leq I(U_0, U_2; Y_2, V_2) - I(X, \widetilde{Y}; V_2 | V_0, Y_2) - \max_i I(V_0; X, \widetilde{Y} | Y_i) \\
R_0 + R_1 + R_2 &\leq I(U_1; Y_1, V_1 | U_0) + I(U_2; Y_2, V_2 | U_0) + \min_i I(U_0; Y_i, V_i) \\
&\quad - I(U_1; U_2 | U_0) - I(X, \widetilde{Y}; V_1 | V_0, Y_1) - I(X, \widetilde{Y}; V_2 | V_0, Y_2) \\
&\quad - \max_i I(V_0; X, \widetilde{Y} | Y_i)
\end{aligned}
$$

for some $(U_0, U_1, U_2, V_0, V_1, V_2)$ *such that*

$$
\begin{aligned}
(U_0, U_1, U_2) &-\!\circ\!- \quad X \quad -\!\circ\!- (Y_1, Y_2, \tilde{Y}) \\
(V_0, V_1, V_2) &-\!\circ\!- (X, \tilde{Y}) -\!\circ\!- (Y_1, Y_2, U_0, U_1, U_2)
\end{aligned}
$$

Capacity of Generalized Dueck-Example with Noise-Free Fb

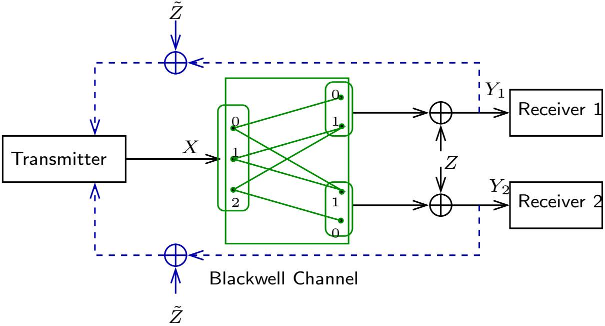


- $B_0, B_1, B_2, Z_0, Z_1, Z_2$ binary
- Assumption: $H(Z_0, Z_1) \leq 1$ and $H(Z_0, Z_2) \leq 1$
- Noise-free feedback capacity: all pairs (R_1, R_2) s.t.

$$\begin{aligned} R_1 &\leq 2 - H(Z_0, Z_1) \\ R_2 &\leq 2 - H(Z_0, Z_2) \\ R_1 + R_2 &\leq 3 - H(Z_0, Z_1, Z_2). \end{aligned}$$

$\rightarrow$ feedback helps unless $Z_1 \multimap\!\!\!\!\circ\, Z_0 \multimap\!\!\!\!\circ\, Z_2$

Another Example: Noisy Blackwell Channel with Noisy Fb



- Noises Z and $\tilde{Z}$ independent and $\sim \mathcal{B}(p)$ and $\mathcal{B}(q)$
- Both channel outputs corrupted by same noise
- Both feedback outputs corrupted by same noise

Our Achievable Region for Noisy Blackwell Channel

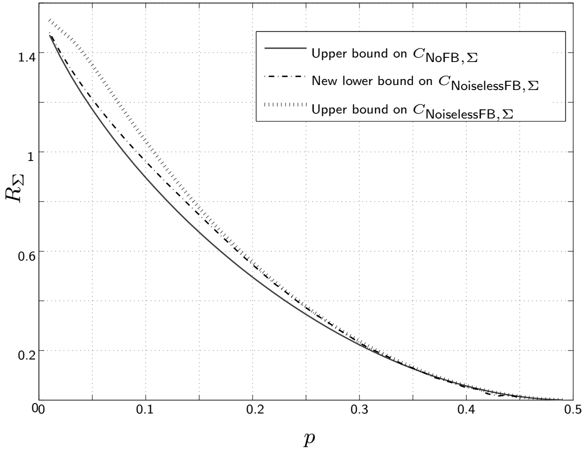
Our Achievable Region

$$
\begin{aligned}
R_0 &\le h_b\left(\frac{\alpha+\beta}{2}\right) - \frac{1}{2}(h_b(\alpha)+h_b(\beta)) - \lambda(p,q,\alpha,\beta) \\
R_0 + R_1 &\le h_b\left(\frac{\alpha+\beta}{2}\right) - \lambda(p,q,\alpha,\beta) - h_b(q) \\
R_0 + R_2 &\le h_b\left(\frac{\alpha+\beta}{2}\right) - \lambda(p,q,\alpha,\beta) - h_b(q) \\
R_0 + R_1 + R_2 &\le h_b\left(\frac{\alpha+\beta}{2}\right) + \frac{1-\beta}{2} h_b\left(\frac{\alpha}{1-\beta}\right) \\
&\quad + \frac{1-\alpha}{2} h_b\left(\frac{\beta}{1-\alpha}\right) - \lambda(p,q,\alpha,\beta) - 2h_b(q)
\end{aligned}
$$

where

$$
\lambda(p,q,\alpha,\beta) \triangleq h_b(p \star q) + h_b\left(\frac{\alpha+\beta}{2}\right) - h_b\left(\left(\frac{\alpha+\beta}{2}\right) \star p \star q\right)
$$

Sum-Capacity of Noisy Blackwell Chan. with Noise-free Fb



Usefulness of Feedback

- For most p noise-free feedback beneficial
- For small q even noisy feedback beneficial

Improved Coding Scheme

- *Channels of interest* in Marton's scheme: $(U_0, U_1) \to Y_1$ and $(U_0, U_2) \to Y_2$
- Update info: lossy compression of these channel "actions", i.e., of $(U_0, U_1, U_2, \tilde{Y})$
- At least as good as before. Better?

Summary/Future Work

Summary:

- Proposed coding schemes for general DMBC with generalized feedback
- Derived new single-letter achievable regions
- Simple example where our scheme yields noise-free feedback-capacity
- Noisy Blackwell channel: scheme improves on no-feedback capacity; even for noisy feedback

Future Work:

- Examine more channels & compare our two regions