Joint Sensing and Communication over Memoryless Broadcast Channels

Mehrasa Ahmadipour*, Michèle Wigger*, Mari Kobayashi†
* LTCI Telecom Paris, IP Paris, 91120 Palaiseau, France, Emails: {mehrasa.ahmadipour,michele.wigger}@telecom-paris.fr
† Technical University of Munich, Munich, Germany, Email: mari.kobayashi@tum.de

***Abstract*—A memoryless state-dependent broadcast channel (BC) is considered, where the transmitter wishes to convey two private messages to two receivers while simultaneously estimating the respective states via generalized feedback. The model at hand is motivated by a joint radar and communication system where radar and data applications share the same frequency band. For physically degraded BCs with i.i.d. state sequences, we characterize the capacity-distortion region tradeoff. For general BCs, we provide inner and outer bounds on the capacity-distortion region, as well as a sufficient condition when it is equal to the product of the capacity region and the set of achievable distortion. Interestingly, the proposed synergetic design significantly outperforms a conventional approach that splits the resource either for sensing or communication.**

I. Introduction

A key-enabler of future high-mobility networks such as Vehicle-to-Everything (V2X) is the ability to continuously track the dynamically changing environment, hereafter called the *state*, and to react accordingly by *exchanging information* between nodes. Although state sensing and communication have been designed separately in the past, power and spectral efficiency as well as hardware costs encourage the integration of these two functions, such that they are operated by sharing the same frequency band and hardware (see e.g. [1]). A typical example of such a scenario is joint radar parameter estimation and communication, where the transmitter equipped with a monostatic radar wishes to convey a message to a receiver and simultaneously estimate the state parameters such as velocity and range from the backscattered signals. [2]. Motivated by such an application, the first information theoretical model for joint sensing and communication has been introduced in [3]. By modeling the backscattered signal as generalized feedback and designing carefully the input signal, the capacity-distortion tradeoff has been characterized for a single-user channel [3], while lower and upper bounds on the rate-distortion region over multiple access channel has been provided in [4].

The current paper extends [3] to the broadcast channel (BC), where the transmitter wishes to convey private messages to two receivers and simultaneously estimate their respective states. For simplicity, the state information is assumed known at each receiver. Although oversimplified, the scenario at hand relates to vehicular networks where a transmitter vehicle, equipped with a monostatic radar, sends (safety-related) messages to multiple vehicles and simultaneously estimates the parameters of these vehicles. It seems challenging to fully characterize the BC capacity-distortion region and to identify the schemes that are jointly optimal for both communication and sensing. In fact, even the optimal schemes for communication only are generally unknow and the simpler BC capacity region with generalized feedback has not been characterised in the general case (see e.g. [5]). Therefore, we first focus on the special case of physically degraded BCs, where generalized feedback is only useful for state sensing but does not increase capacity, as in the single user channel. The capacity-distortion region is completely characterized for this class of BCs. Moreover, closed-form expressions of the region are provided for some binary examples. The numerical evaluations illustrate interesting tradeoffs between the achievable rates and distortions across two receivers. For general BCs, we provide a sufficient condition when the capacity-distortion region is simply the product of the capacity region and the set of all achievable distortions, thus no tradeoff between communication and sensing arises. Furthermore, we provide general inner and outer bounds on the capacity-distortion region, as well as a state-dependent Dueck's example. For all these kinds of BCs, we show though numerical examples that the synergetic design significantly outperforms the resource-sharing scheme that splits the resource either for sensing or communication.

The rest of the paper is organized as follows. Section II introduces our model and Section III presents some cases that yield no tradeoff between sensing and communication. Section IV focuses on the physical degraded broadcast channel and provides some examples. Finally, upper and lower bounds for the general memoryless broadcast channel are provided along with an example in Section V.

II. System Model

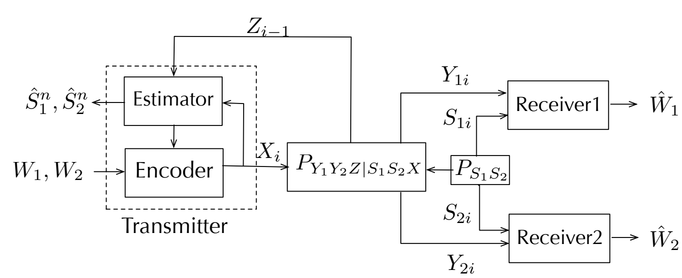

Fig. 1. Broadcast model for joint sensing and communication

Consider a two-user state-dependent memoryless broadcast channel (SDMBC) with two private messages W_1 and W_2 as illustrated in Fig. 1. The model comprises a two-dimensional memoryless state sequence $\{(S_{1,i}, S_{2,i})\}_{i\geq 1}$ whose samples at time i are distributed according to a given joint law $P_{S_1S_2}$ over the state alphabets $\mathcal{S}_1 \times \mathcal{S}_2$. Given input and output alphabets $\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Z}$, input $X_i = x \in \mathcal{X}$ and state-realizations $S_{1,i} = s_1 \in \mathcal{S}_1$ and $S_{2,i} = s_2 \in \mathcal{S}_2$, the SDMBC produces a triple of outputs $(Y_{1,i}, Y_{2,i}, Z_i) \in \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Z}$ according to a given time-invariant transition law $P_{Y_1Y_2Z|S_1S_2X}(\cdot,\cdot,\cdot|s_1,s_2,x)$, for each time i.

A SDMBC is thus entirely specified by the tuple of alphabets and (conditional) pmfs

$$(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Z}, P_{S_1S_2}, P_{Y_1Y_2Z|S_1S_2X}). \tag{1}$$

We will often describe a SDMBC only by the pair of pmfs $(P_{S_1S_2}, P_{Y_1Y_2Z|S_1S_2X})$, in which case, the corresponding alphabets should be clear from the context.

A $(2^{nR_1}, 2^{nR_2}, n)$ code for an SDMBC $P_{Y_1Y_2Z|S_1S_2X}$ consists of

1) two message sets $\mathcal{W}_1 = [1 : 2^{nR_1}]$ and $\mathcal{W}_2 = [1 : 2^{nR_2}]$;
2) a sequence of encoding functions $\phi_i \colon \mathcal{W}_1 \times \mathcal{W}_2 \times \mathcal{Z}^{i-1} \to \mathcal{X}$, for $i = 1, 2, \ldots, n$;
3) for each $k = 1, 2$ a decoding function $g_k \colon \mathcal{S}_k^n \times \mathcal{Y}_k^n \to \mathcal{W}_k$;
4) for each $k = 1, 2$ a state estimator $h_k \colon \mathcal{X}^n \times \mathcal{Z}^n \to \hat{\mathcal{S}}_k^n$, where $\hat{\mathcal{S}}_k$ denotes the given reconstruction alphabet for state sequence $S_k^n = (S_{k,1}, \cdots, S_{k,n})$.

For a given code, we let the random messages W_1 and W_2 be uniform over the message sets $\mathcal{W}_1$ and $\mathcal{W}_2$ and the inputs $X_i = \phi_i(W_1, W_2, Z^{i-1})$, for $i = 1, \ldots, n$. The corresponding outputs $Y_{1,i}Y_{2,i}, Z_i$ at time i are obtained from the states $S_{1,i}$ and $S_{2,i}$ and the input X_i according to the SDMBC transition law $P_{Y_1Y_2Z|S_1S_2X}$. Further, let $\hat{S}_k^n := (\hat{S}_{k,1}, \cdots, \hat{S}_{k,n}) = h_k(X^n, Z^n)$ be the state estimates at the transmitter and let $\hat{W}_k = g_k(S_k^n, Y_k^n)$ be the decoded message by decoder k, for $k = 1, 2$.

The quality of the state estimates $\hat{S}_k^n$ is measured by a given per-symbol distortion function $d_k \colon \mathcal{S}_k \times \hat{\mathcal{S}}_k \mapsto [0, \infty)$, and we will be interested in the *expected average per-block distortion*

$$\Delta_k^{(n)} \triangleq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[d_k(S_{ki}, \hat{S}_{ki})], \quad k = 1, 2. \tag{2}$$

For the decoded messages $\hat{W}_1$ and $\hat{W}_k$ we focus on their joint probability of error:

$$p^n(\text{error}) := \Pr\left(\hat{W}_1 \neq W_1 \quad \text{or} \quad \hat{W}_2 \neq W_2\right). \tag{3}$$

Definition 1. *A rate-distortion tuple (R_1, R_2, D_1, D_2) is said achievable if there exists a sequence (in n) of $(2^{nR_1}, 2^{nR_2}, n)$ codes that simultaneously satisfy*

$$\lim_{n\to\infty} p^{(n)}(\text{error}) = 0 \tag{4a}$$

$$\varlimsup_{n\to\infty} \Delta_k^{(n)} \leq D_k, \quad \text{for } k = 1, 2. \tag{4b}$$

The closure of the union of all achievable rate-distortion tuples (R_1, R_2, D_1, D_2) is called the capacity-distortion region and is denoted $\mathcal{CD}$. The current work aims at specifying the tradeoff between the achievable rates and distortions. As we will see in Sections III and V, there is no such tradeoff in some cases, and the resulting region $\mathcal{CD}$ is the product of SDMBC's capacity region:

$$\mathcal{C} \triangleq \{(R_1, R_2) \colon (R_1, R_2, D_1, D_2) \in \mathcal{CD} \text{ for } D_1, D_2 \geq 0\}, \tag{5}$$

and its distortion region:

$$\mathcal{D} \triangleq \{(D_1, D_2) \colon (R_1, R_2, D_1, D_2) \in \mathcal{CD} \text{ for } R_1, R_2 \geq 0\}. \tag{6}$$

Before presenting our results on the tradeoff region $\mathcal{CD}$ in the following sections, we describe the optimal choice of the estimators h_1 and h_2.

Lemma 1. *For $k = 1, 2$ and any $i = 1, \ldots, n$, whenever $X_i = x$ and $Z_i = z$, the optimal estimator h_k that minimizes the average expected distortion $\Delta_k^{(n)}$ is given by*

$$\hat{s}_{k,i}^*(x, z) \triangleq \arg\min_{s' \in \hat{\mathcal{S}}_k} \sum_{s_k \in \mathcal{S}_k} P_{S_{k,i}|X_iZ_i}(s_k|x, z) d(s_k, s'). \tag{7}$$

In above definition (7), *ties can be broken arbitrarily.*

Notice that the lemma implies in particular that a symbolwise estimator that estimates $S_{k,i}$ only based on (X_i, Z_i) is optimal; there is no need to resort to previous or past observations (X^{i-1}, Z^{i-1}) or (X_{i+1}^n, Z_{i+1}^n).

Proof of Lemma 1: Recall that $\hat{S}_k^n$ is a function of X^n, Z^n and write for each $i = 1, \cdots, n$:

$$\begin{aligned}
&\mathbb{E}\left[d_k(S_{k,i}, \hat{S}_{k,i})\right] \\
&= \mathbb{E}_{X^n,Z^n}\left[\mathbb{E}[d_k(S_{k,i}, \hat{S}_{k,i})|X^n, Z^n]\right] &(8)\\
&\overset{(a)}{=} \sum_{x^n,z^n} P_{X^nZ^n}(x^n, z^n) \sum_{\hat{s}_k \in \mathcal{S}_k} P_{\hat{S}_{k,i}|X^nZ^n}(\hat{s}_k|x^n, z^n) \\
&\qquad \cdot \sum_{s_k} P_{S_{k,i}|X_iZ_i}(s_k|x_i, z_i) d(s_k, \hat{s}_k) &(9)\\
&\geq \sum_{x^n,z^n} P_{X^nZ^n}(x^n, z^n) \\
&\qquad \cdot \min_{\hat{s}_k \in \mathcal{S}_k} \sum_{s_k} P_{S_{k,i}|X_iZ_i}(s_k|x_i, z_i) d(s_k, \hat{s}_k) \\
&= \mathbb{E}[d(S_{k,i}, \hat{s}_k^*(X_i, Z_i))], &(10)
\end{aligned}$$

where (a) holds by the Markov chain

$$\left(X^{i-1}, X_{i+1}^n, Z^{i-1}, Z_{i+1}^n, \hat{S}_{k,i}\right) \multimap\!- (X_i, Z_i) \multimap\!- S_{k,i}.$$

■

III. Absence of Rate-Distortion Tradeoff

We first consider degenerate cases where the rate-distortion tradeoff region is given by the Cartesian product between the capacity region $\mathcal{C}$ and the distortions region $\mathcal{D}$.

Proposition 2 (No Distortions-Rate Tradeoff)**.** *Consider a SDMBC $(P_{S_1S_2}, P_{Y_1Y_2Z|S_1S_2X})$ and let $(X, S_1, S_2, Y_1, Y_2, Z) \sim P_X P_{S_1S_2} P_{Y_1Y_2Z|S_1S_2X}$ for a given input law P_X. If there exist functions ψ_1 and ψ_2 with domain $\mathcal{Z}$ such that for all P_X the Markov chains*

$$(S_k, \psi_k(Z)) \perp X, \tag{11}$$

$$S_k \multimap\!\!- \psi_k(Z) \multimap\!\!- (Z, X), \quad k \in \{1, 2\}, \tag{12}$$

hold, then for the SDMBC under consideration:

$$\mathcal{CD} = \mathcal{C} \times \mathcal{D}. \tag{13}$$

In this case, there is no tradeoff between the achievable rate pairs (R_1, R_2) and the achievable distortion pairs (D_1, D_2).

Proof: Notice that under the given Markov chains:

$$P_{S_{k,i}|X_i Z_i}(s_k|x, z) = P_{S_{k,i}|\psi_k(Z_i)}(s_k|\psi_k(z)). \tag{14}$$

By Lemma 1, the optimal estimators depend only on the sequences $\{\psi_k(Z_i)\}_{i=1}^n$, for $k = 1, 2$. Then, by (11), the optimal estimators and their performances are independent of the chosen encoding scheme and we conclude (13). ■

The following example satisfies conditions (11) and (12) in Proposition 2 for an appropriate choice of ψ_1 and ψ_2 .

A. Example: Erasure BC with Noisy Feedback

Let the joint law $\mathrm{P}_{S_1 S_2 E_1 E_2}(s_1, s_2, e_1, e_2)$ over $\{0, 1\}^4$ be arbitrary but given, and $(E_1, E_2, S_1, S_2) \sim \mathrm{P}_{S_1 S_2 E_1 E_2}$. Consider the state-dependent erasure BC

$$Y_k = \begin{cases} X & \text{if } S_k = 0, \\ ? & \text{if } S_k = 1, \end{cases}, \qquad k = \{1, 2\}, \tag{15}$$

where the feedback signal $Z = (Z_1, Z_2)$ is given by

$$Z_k = \begin{cases} Y_k & \text{if } E_k = 0, \\ ? & \text{if } E_k = 1, \end{cases}, \qquad k = \{1, 2\}. \tag{16}$$

Further consider the Hamming distortion measure $d_k(s, \hat{s}) = s \oplus \hat{s}$, for $k = 1, 2$. For the choice

$$\psi_k(Z) = \begin{cases} 1, & \text{if} \quad Z_k = ? \\ 0, & \text{else}, \end{cases} \tag{17}$$

the described SDMBC satisfies the conditions in Proposition 2 and its capacity-distortion region is thus given by

$$\mathcal{CD} = \mathcal{C} \times \mathcal{D}. \tag{18}$$

Remark 1. *For the case of output feedback $Z = (Y_1, Y_2)$ or $E_1 = E_2 = 0$, the transmitter can perfectly estimate the state (S_1, S_2), yielding $D_1 = D_2 = 0$ regardless of the rate pair $(R_1, R_2) \in \mathcal{C}$. The capacity region $\mathcal{C}$ of the erasure broadcast channel with output feedback is still unknown in general.*

IV. Physically Degraded BCs

In this section, by focusing on the physically degraded SDMBC, we fully characterize the capacity-distortion region. Then, we discuss two binary physically degraded SDMBCs to illustrate the rate-distortion tradeoff between the two receivers.

Definition 2. *An SDMBC $(P_{S_1 S_2}, P_{Y_1 Y_2 Z|S_1 S_2 X})$ is called* physically degraded *if there are conditional laws $P_{Y_1|X S_1}$ and $P_{Y_2 S_2|S_1 Y_1}$ such that*

$$P_{Y_1 Y_2|S_1 S_2 X} P_{S_1 S_2} = P_{S_1} P_{Y_1|S_1 X} P_{Y_2 S_2|S_1 Y_1}. \tag{19}$$

That means for any arbitrary input P_X, if a tuple $(X, S_1, S_2, Y_1, Y_2) \sim P_X P_{S_1 S_2} P_{Y_1 Y_2|S_1 S_2 X}$, then it satisfies the Markov chain

$$X \multimap\!\!- (S_1, Y_1) \multimap\!\!- (S_2, Y_2). \tag{20}$$

Proposition 3. *The capacity-distortion region $\mathcal{CD}$ of a physically degraded SDMBC is the closure of the set of all quadruples (R_1, R_2, D_1, D_2) for which there exists a joint law P_{UX} so that the tuple $(U, X, S_1, S_2, Y_1, Y_2, Z) \sim P_{UX} P_{S_1 S_2} P_{Y_1 Y_2 Z|S_1 S_2 X}$ satisfies the two rate constraints*

$$R_1 \le I(X; Y_1 \mid S_1, U) \tag{21}$$
$$R_2 \le I(U; Y_2|S_2), \tag{22}$$

and the distortion constraints

$$\mathbb{E}[d_k(S_k, \hat{s}_k^*(X, Z)))] \le D_k, \quad k \in \{1, 2\}, \tag{23}$$

where

$$\hat{s}_k^*(x, z) \triangleq \arg \min_{s' \in \hat{\mathcal{S}}_k} \sum_{s_k \in \mathcal{S}_k} P_{S_k|XZ}(s_k|x, z) d(s_k, s'). \tag{24}$$

Moreover, one can restrict to random variables U over alphabets $\mathcal{U}$ satisfying $|\mathcal{U}| \le \min\{|\mathcal{X}|, |\mathcal{Y}_1| \cdot |\mathcal{S}_1|, |\mathcal{Y}_2| \cdot |\mathcal{S}_2|\} + 1$.

Proof: The converse follows as a special case of Theorem 6 ahead where one can ignore constraints (33c) and (33d). Notice that constraint (33b) is equivalent to (22) because (U, X) is independent of (S_1, S_2) and because for a physically degraded DMBC the Markov chain (20) holds. The cardinality bound can be proved using Carathéodory's theorem.

Achievability is obtained by simple superposition coding and using the optimal estimator described in Lemma 1. ■

We consider two binary state-dependent channels. For the binary states, we consider the Hamming distortion measure.

A. Example: Binary BC with Multiplicative States

Consider the physically degraded SDMBC with binary input/output alphabets $\mathcal{X} = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ and binary state alphabets $\mathcal{S}_1 = \mathcal{S}_2 = \{0, 1\}$. The channel input-output relation is described by

$$Y_k = X \cdot S_k, \qquad k = 1, 2, \tag{25}$$

with the joint state pmf

$$P_{S_1 S_2}(s_1, s_2) = \begin{cases} 1 - q, & \text{if } (s_1, s_2) = (0, 0) \\ 0, & \text{if } (s_1, s_2) = (0, 1) \\ q \cdot \gamma, & \text{if } (s_1, s_2) = (1, 1) \\ q \cdot (1 - \gamma) & \text{if } (s_1, s_2) = (1, 0), \end{cases} \tag{26}$$

for $\gamma, q \in [0, 1]$. Notice that S_2 is a degraded version of S_1. We consider output feedback $Z = (Y_1, Y_2)$.

Corollary 4. *The capacity-distortions region $\mathcal{CD}$ of the binary physically degraded SDMBC in* (25)–(26) *parameterized by (q, γ) is the set of all quadruples (R_1, R_2, D_1, D_2) satisfying*

$$R_1 \le q \cdot H_{\mathrm{b}}(p) \cdot r \tag{27a}$$
$$R_2 \le \gamma \cdot q \cdot H_{\mathrm{b}}(p) \cdot (1 - r) \tag{27b}$$

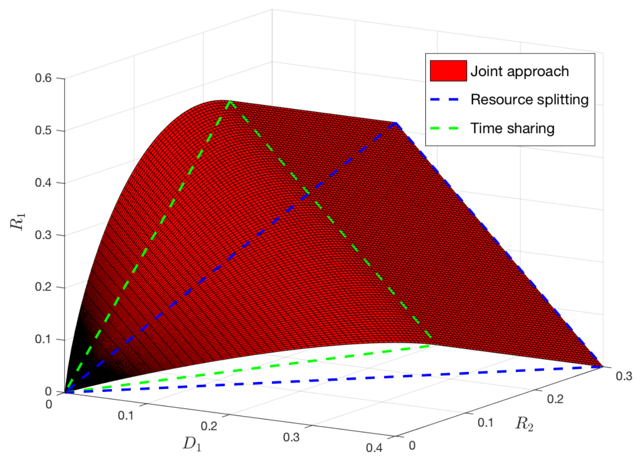

Fig. 2. Boundary of the capacity-distortion region $\mathcal{CD}$ for the example in Subsection IV-A.

$$D_1 \geq (1-p) \cdot \min\{q, 1-q\} \tag{27c}$$
$$D_2 \geq (1-p) \cdot \min\{\gamma \cdot q, 1-\gamma \cdot q\}, \tag{27d}$$

for some choice of the parameters $r, p \in [0,1]$.

Proof. It suffices to evaluate the rate-constraints (21) and (22) for $X = V \oplus U$ when U and V are independent Bernoulli distributed random variables. In (27), we choose the parameter $p = \Pr[X = 1]$ and $r = \frac{H(V)}{H_b(p)}$. To calculate the distortion, we determine the optimal estimator $\hat{s}_k^*(x, y_1, y_2)$ from (24) as

$$\hat{s}_k^*(1, y_1, y_2) = y_k, \tag{28a}$$
$$\hat{s}_k^*(0, y_1, y_2) = \mathbf{1}\{P_{S_k}(1) > 1/2\}. \quad \square \tag{28b}$$

Remark 2. *Fixing $r = 1$, the capacity-distortion region in* (27) *reduces to the capacity-distortion tradeoff of a single user channel [3, Proposition 1]. Similarly to the single-user case, we observe the tension between the minimum distortion by choosing $p = 1$ (always sending $X = 1$) and the maximum rate by choosing $p = 1/2$. In the BC, the resource is shared between the two users via the time-sharing parameter r.*

It is worth comparing the capacity-distortion region $\mathcal{CD}$, achieved by the proposed *co-design scheme* that uses a common waveform for both sensing and communication tasks, with the rate-distortion region achieved by two baseline schemes: i) a resource splitting scheme that performs either state estimation via feedback or broadcasting that ignores the feedback; ii) a time-sharing scheme that performs either state estimation via feedback or broadcasting with feedback.

Fig. 2 shows in red colour the dominant boundary points of the projection of the tradeoff region $\mathcal{CD}$ onto the 3-dimensional plane (R_1, R_2, D_1) when $\gamma = 1/2$ and $q = 0.6$. The tradeoff with D_2 is omitted because D_2 is a scaled version of D_1. We compare our proposed scheme with the two aforementioned baseline schemes. From Fig. 2, we observe that both resource-splitting and time-sharing approaches fail to achieve the entire tradeoff region $\mathcal{CD}$.

So far, there was no tradeoff between the two distortion constraints D_1 and D_2. This is different in the next example, which otherwise is very similar.

B. Example: Binary BC with Flipping Inputs

Reconsider the same state pmf $P_{S_1S_2}$ as in the previous example, but now a SDMBC with transition law

$$Y_1 = X \cdot S_1, \qquad Y_2 = (1-X) \cdot S_2. \tag{29}$$

Consider output feedback $Z = (Y_1, Y_2)$.

Corollary 5. *The capacity-distortion region $\mathcal{CD}$ of the binary SDMBC with flipping inputs in* (29) *and output feedback is the set of all quadruples (R_1, R_2, D_1, D_2) satisfying*

$$R_1 \leq q \cdot H_b(p) \cdot r \tag{30a}$$
$$R_2 \leq \gamma \cdot q \cdot H_b(p) \cdot (1-r) \tag{30b}$$
$$D_1 \geq (1-p) \cdot \min\{q(1-\gamma), (1-q)\} \tag{30c}$$
$$D_2 \geq p \cdot q \min\{\gamma, 1-\gamma\} \tag{30d}$$

for some choice of the parameters $r, p \in [0,1]$.

Proof. To achieve this region, we can consider the same choices of (U, X) as in the previous example. The optimators are given by (28a) and

$$\hat{s}_2^*(x = 0, y_1, y_2) = y_2, \tag{31}$$
$$\hat{s}_2^*(x = 1, y_1, y_2) = \mathbf{1}\{P_{S_2}(1) > 1/2\}. \quad \square \tag{32}$$

In contrast to the previous example, here we observe a tradeoff between the achievable distortions D_1 and D_2.

V. General Bounds

Reconsider the general SDMBC (not necessarily physically degraded). We provide an inner and an outer bound on the capacity-distortion region.

Theorem 6. *If (R_1, R_2, D_1, D_2) is achievable on a SDMBC $(P_{S_1S_2}, P_{Y_1Y_2Z|S_1S_2X})$, then there exists for each $k = 1, 2$ a conditional pmf $P_{U_k|X}$ such that the random tuple $(U_k, X, S_1, S_2, Y_1, Y_2, Z) \sim P_{U_k|X}P_XP_{S_1S_2}P_{Y_1Y_2Z|S_1S_2X}$ satisfies the rate constraints*

$$R_1 \leq I(U_1; Y_1 \mid S_1), \tag{33a}$$
$$R_1 + R_2 \leq I(X; Y_1, Y_2 \mid S_1, S_2, U_1), \tag{33b}$$
$$R_1 + R_2 \leq I(X; Y_1, Y_2 \mid S_1, S_2, U_2), \tag{33c}$$
$$R_2 \leq I(U_2; Y_2 \mid S_2) \tag{33d}$$

and the average distortion constraints

$$\mathbb{E}[d_k(S_k, \hat{s}_k^*(X, Z)))] \leq D_k, \quad k \in \{1, 2\}, \tag{34}$$

where the function $\hat{s}_k^*(\cdot, \cdot)$ is defined in (24).

Proof: Omitted. See [5]. ∎

The next inner bound is obtained by combining the achievable region in [6] with the optimal estimator in Lemma 1.

Proposition 7. *Consider a SDMBC $(P_{S_1S_2}, P_{Y_1Y_2Z|S_1S_2X})$. For any (conditional) pmfs $P_{U_0U_1U_2X}$ and $P_{V_0V_1V_2|U_0U_1U_2Z}$ and tuple $(U_0, U_1, U_2, X, S_1, S_2, Y_1, Y_2, Z, V_0, V_1, V_2) \sim P_{U_0U_1U_2X}P_{S_1S_2}P_{Y_1Y_2Z|S_1S_2X}P_{V_0V_1V_2|U_0U_1U_2Z}$, the closure of*

$$
\begin{aligned}
R_1 &\leq I(U_0, U_1; Y_1, V_1 \mid S_1) - I(U_0, U_1, U_2, Z; V_0, V_1 | S_1, Y_1) && \text{(36a)}\\
R_2 &\leq I(U_0, U_2; Y_2, V_2 \mid S_2) - I(U_0, U_1, U_2, Z; V_0, V_2 | S_2, Y_2) && \text{(36b)}\\
R_1 + R_2 &\leq I(U_1; Y_1, V_1 | U_0, S_1) + I(U_2; Y_2, V_2 | U_0, S_2) + \min_{i \in \{1,2\}} I(U_0; Y_i, V_i \mid S_i) - I(U_1; U_2 | U_0)\\
&\quad - I(U_0, U_1, U_2, Z; V_1 | V_0, S_1, Y_1) - I(U_0, U_1, U_2, Z; V_2 | V_0, S_2, Y_2) - \max_{i \in \{1,2\}} I(U_0, U_1, U_2, Z; V_0 | S_i, Y_i) && \text{(36c)}
\end{aligned}
$$

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the set of all quadruples (R_1, R_2, D_1, D_2) satisfying inequalities (36) on top of this page and the distortion constraints

$$\mathbb{E}[d_k(S_k, \hat{s}_k^*(X, Z)))] \leq D_k, \quad k \in \{1, 2\}, \tag{35}$$

for $\hat{s}_k^*(\cdot, \cdot)$ defined in (24), is achievable.

A. Example: Dueck's BC with Binary States

Consider a state-dependent version of Dueck's BC [7] with input $X = (X_0, X_1, X_2) \in \{0, 1\}^3$, outputs

$$Y_k = (X_0, Y_k', S_1, S_2), \qquad k = 1, 2, \tag{37}$$

and states $S_1, S_2 \in \{0, 1\}$, and

$$Y_k' = S_k(X_k \oplus N), \qquad k = 1, 2, \tag{38a}$$

where the noise N is Bernoulli-$\frac{1}{2}$ independent of the inputs. Assume i.i.d. states such that $P_{S_1 S_2}(s_1, s_2) = P_S(s_1) P_S(s_2)$ for a given pmf P_S. The feedback signal is $Z = (Y_1', Y_2')$.

Corollary 8. *The capacity-distortion region $\mathcal{CD}$ of Dueck's state-dependent BC is included in the set of quadruples (R_1, R_2, D_1, D_2) satisfying the four rate-constraints*

$$
\begin{aligned}
R_1 &\leq 1 - p && \text{(39a)}\\
R_2 &\leq p + (P_S(1))^2 \cdot H_b(\beta) && \text{(39b)}\\
R_1 &\leq q + (P_S(1))^2 \cdot H_b(\beta) && \text{(39c)}\\
R_2 &\leq 1 - q && \text{(39d)}
\end{aligned}
$$

and the two distortion-constraints for k=1,2

$$
\begin{aligned}
D_k \geq\ & \frac{1}{2}(1-\beta) \cdot \min\{P_S(1), P_S(0) \cdot (1 + P_S(0))\}\\
&+ \frac{1}{2}\beta \big[P_S(0) P_S(1) + P_S(1) \cdot \min\{P_S(0), P_S(1)\}\big] && \text{(39e)}
\end{aligned}
$$

for some choice of the parameters $p, q, \beta \in [0, 1]$.

Moreover, it includes the set of all quadruples (R_1, R_2, D_1, D_2) that for some $\beta \in [0, 1]$ satisfy

$$
\begin{aligned}
R_k &\leq 1, \qquad k \in \{1, 2\}, && \text{(40a)}\\
R_1 + R_2 &\leq 1 + P_S(1) \cdot (H_b(\beta) - P_S(0)) && \text{(40b)}
\end{aligned}
$$

and the two distortion constraints in (39e).

Proof: To obtain the inner bound (40), evaluate Proposition 7 for: X_0, X_1, X_2 are Bernoulli-$\frac{1}{2}$ with X_0 independent of (X_1, X_2) and $X_1 = X_2 = x$ with probability $\frac{\beta}{2}$ for all $x \in \{0, 1\}$; $U_i = X_i$, for $i = 0, 1, 2$; and $V_1 = (X_0, X_1)$, $V_2 = (X_0, X_2)$, and either $V_0 = X_1 \oplus Y_1'$ or $V_0 = X_2 \oplus Y_2'$. The outer bound is based on Theorem 6. ∎

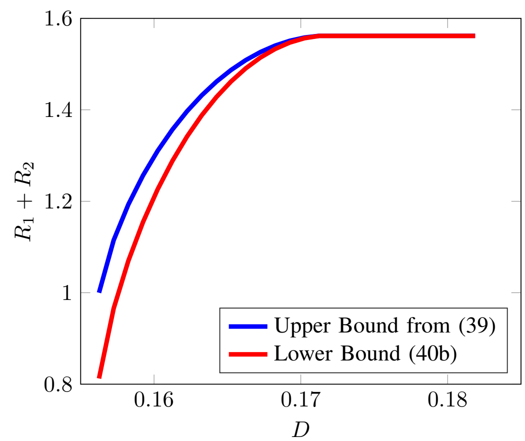

Fig. 3. Upper and lower bounds of Corollary 8 on the maximum achievable sum-rate $(R_1 + R_2)$ in function of the admissible distortion $D_1 = D_2$ for the state-dependent Dueck BC when $P_S(1) = 3/4$ and $P_S(0) = 1/4$.

The inner and outer bounds of Corollary 8 do not coincide in general. In particular, Fig. 3 shows the largest sum-rate $R_1 + R_2$ that our inner and outer bounds admit in function of the admissible distortion $D_1 = D_2$ when $P_S(1) = \frac{3}{4}$ and $P_S(0) = \frac{1}{4}$. (Notice that the minimum distortion is $D_{\min} = \frac{5}{32}$.) In contrast, when $P_S(1) \leq P_S(0)$, the distortion constraint (39e) simplifies to $D_k \geq \frac{1}{2} P_S(1)$. In this case, the choice $\beta = 1/2$ is optimal for both the inner and outer bounds, in which case the bounds coincide and are equal to $\mathcal{C} \times \mathcal{D}$. There is thus no rate-distortion tradeoff.

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