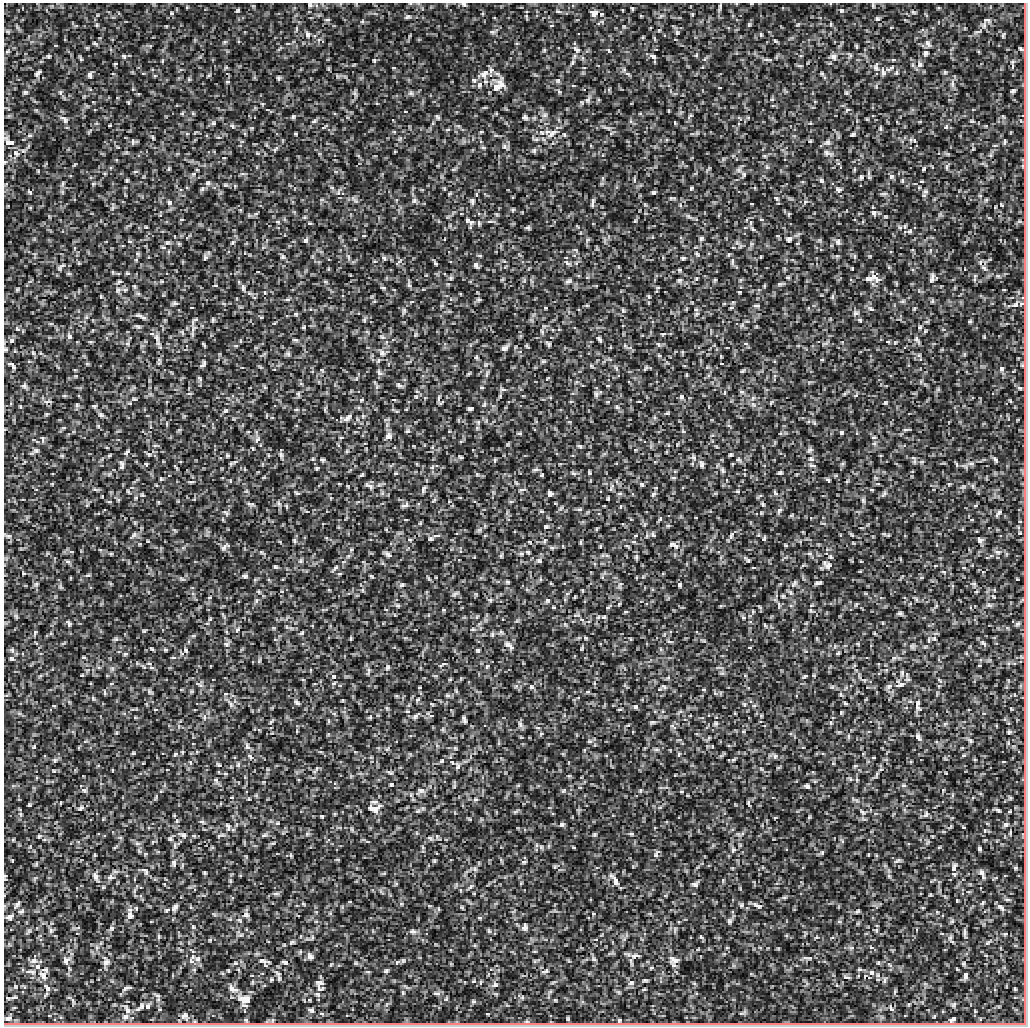
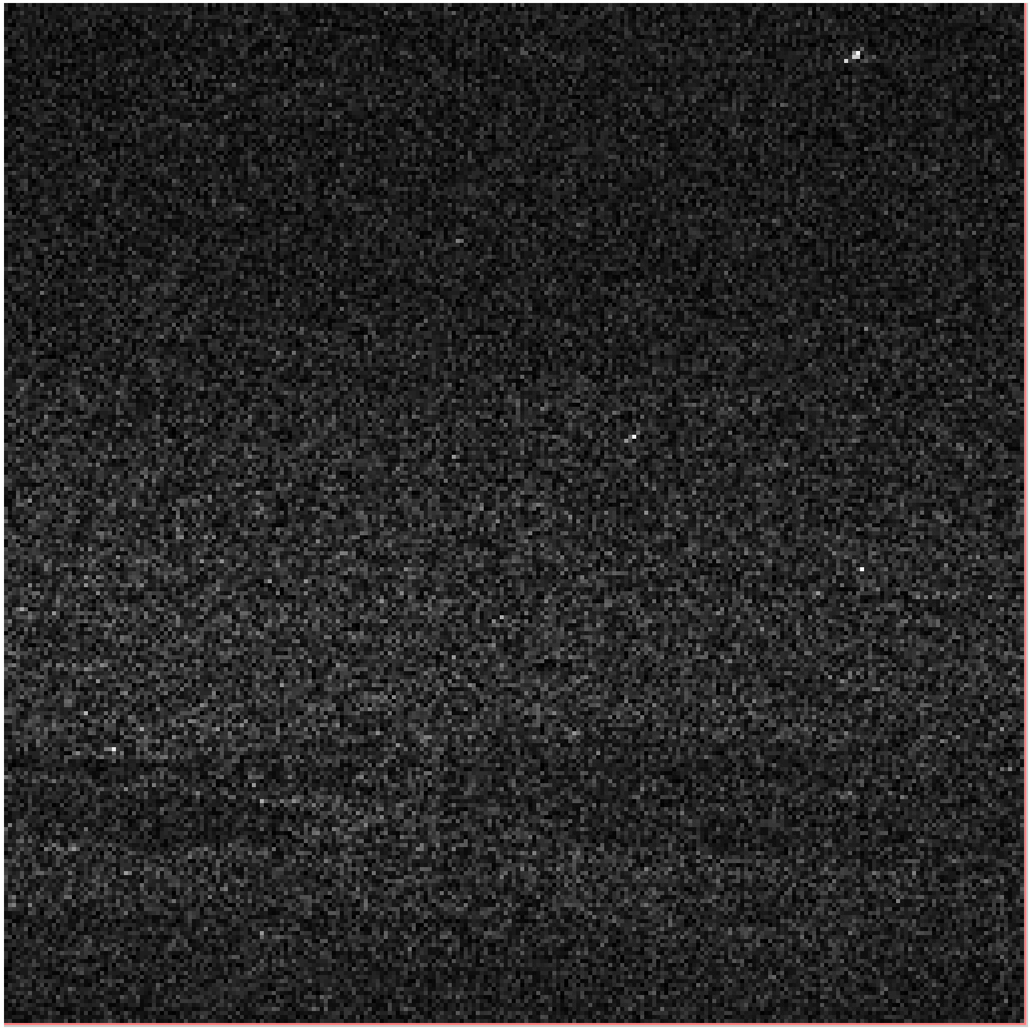
Statistical models for SAR data

Jean-Marie Nicolas
Florence Tupin

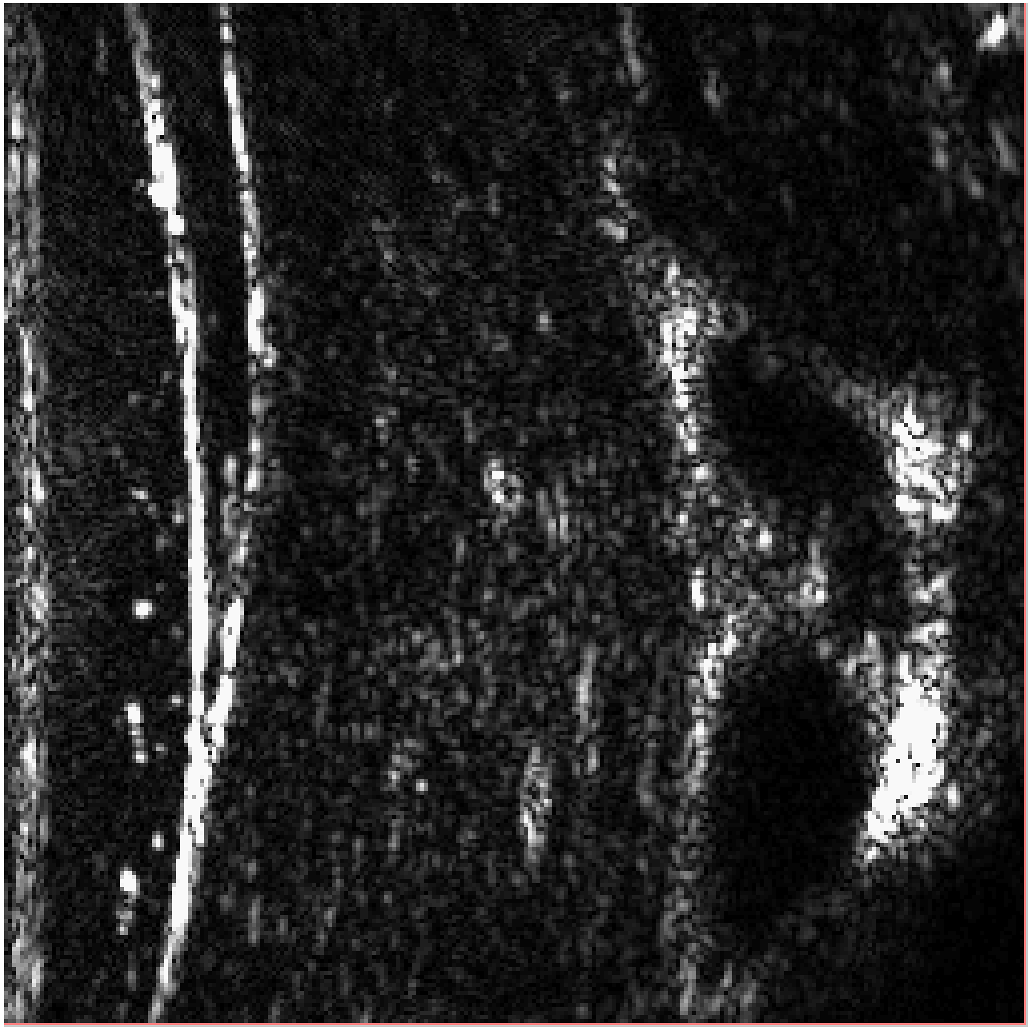
Desert in Australia (Terrasar-X)



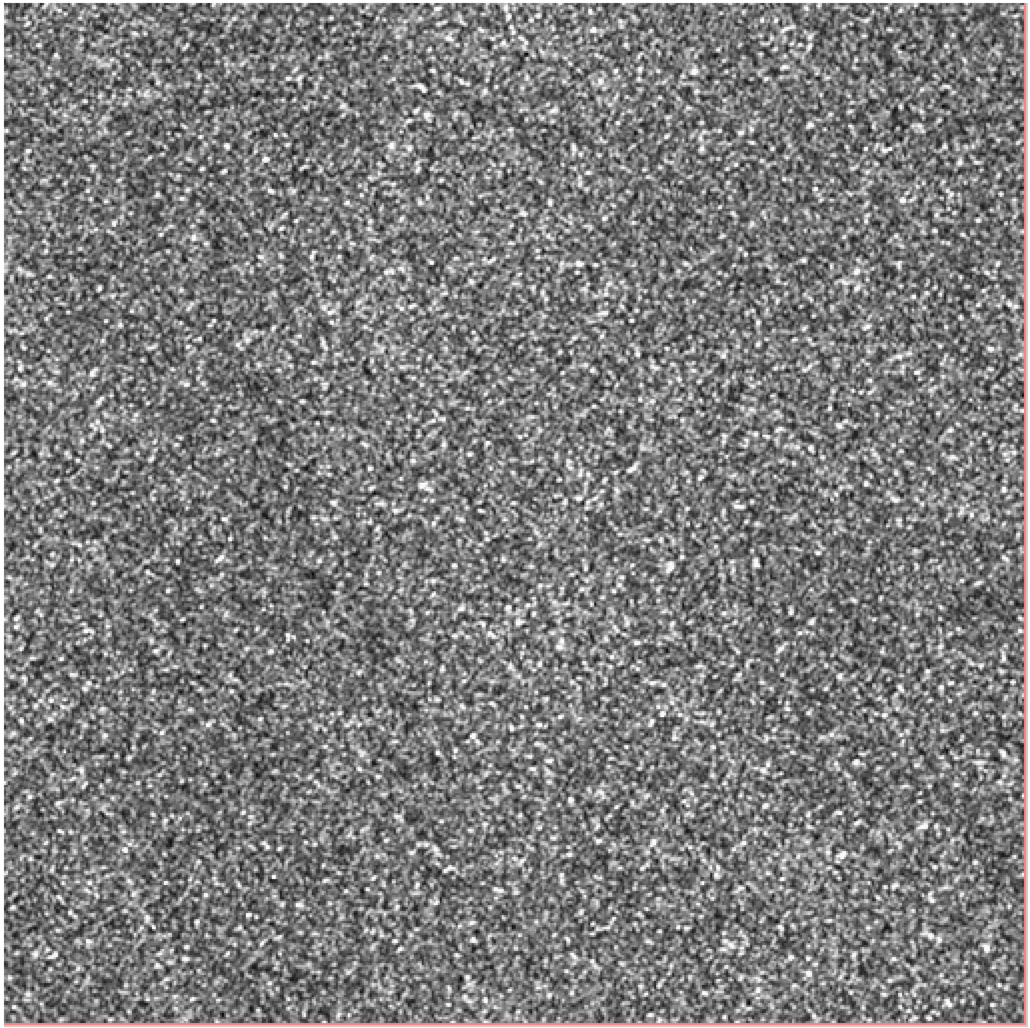
Sonar: underwater ground



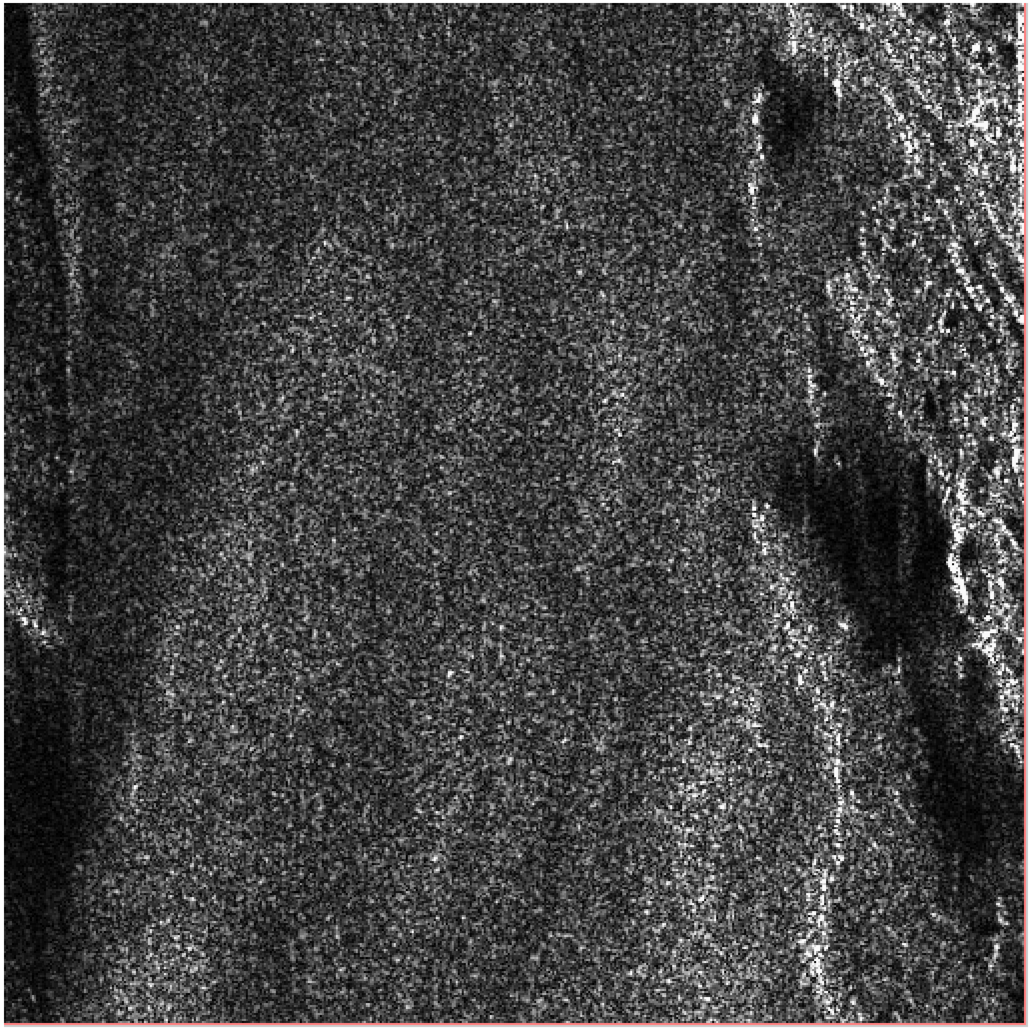
Echography



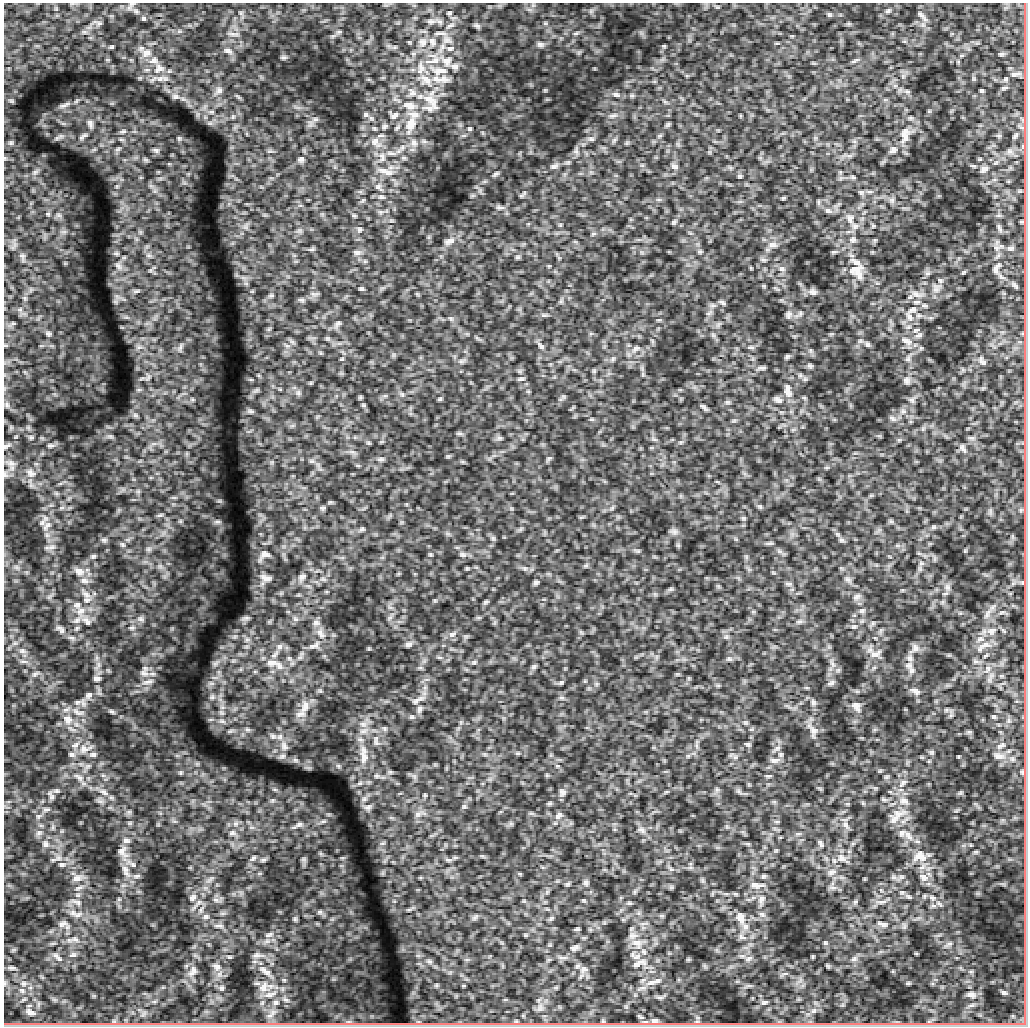
Desert of Mauritania (ERS)



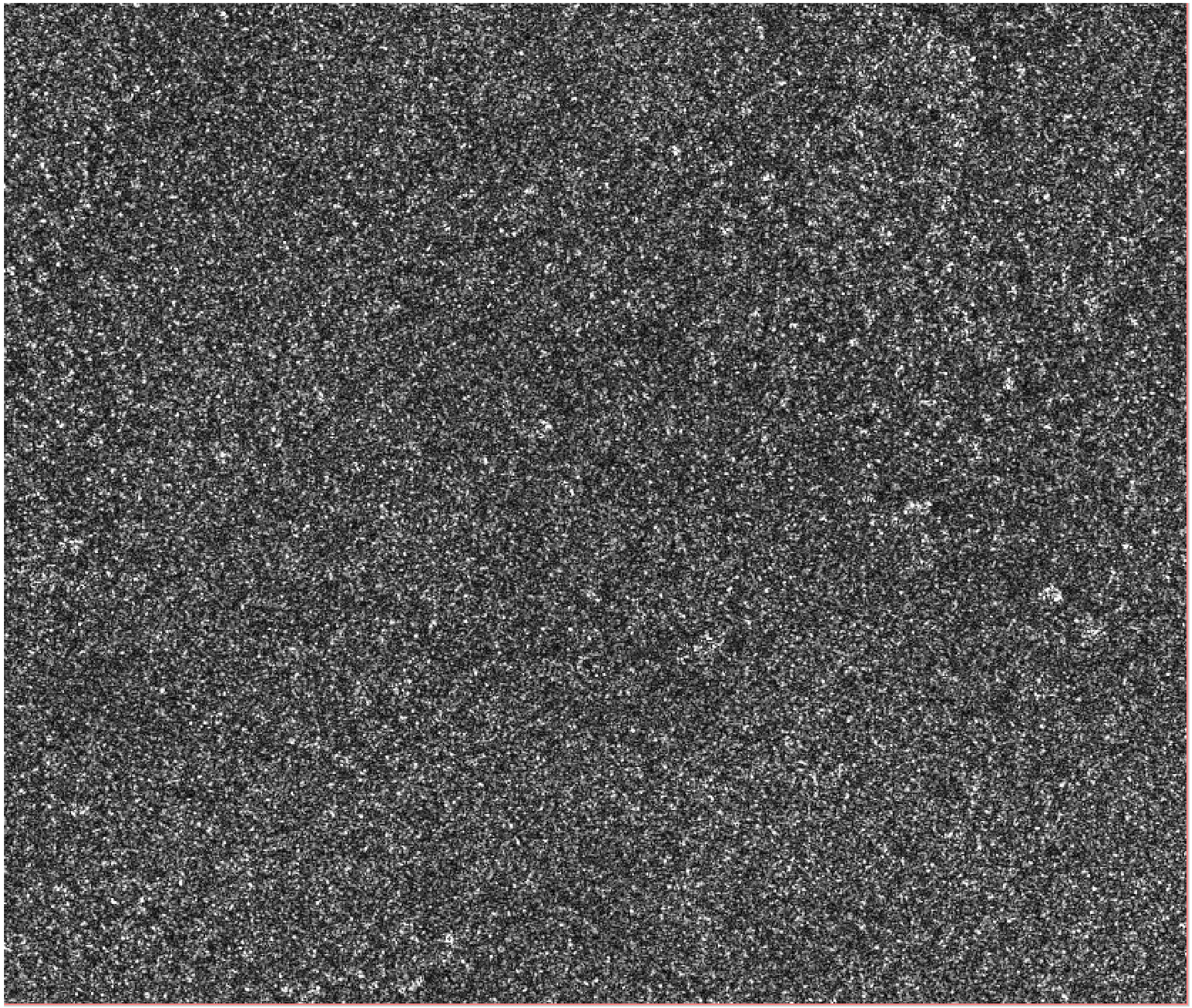
Sea – near Bergen (Terrasar-X Spotlight)



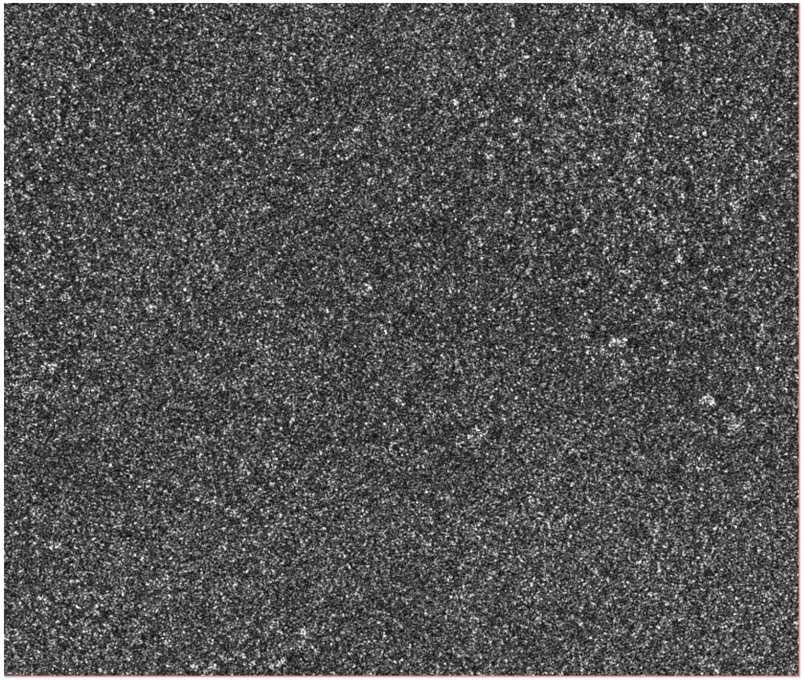
Forest in Guyana (ERS)

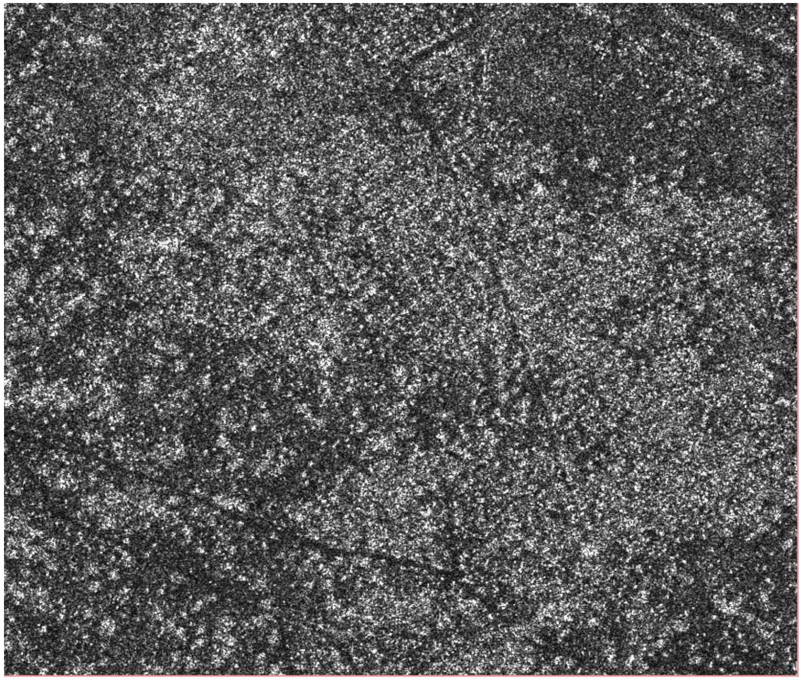


Phenomenological analysis (Terrasar-X)



Two areas in Australia



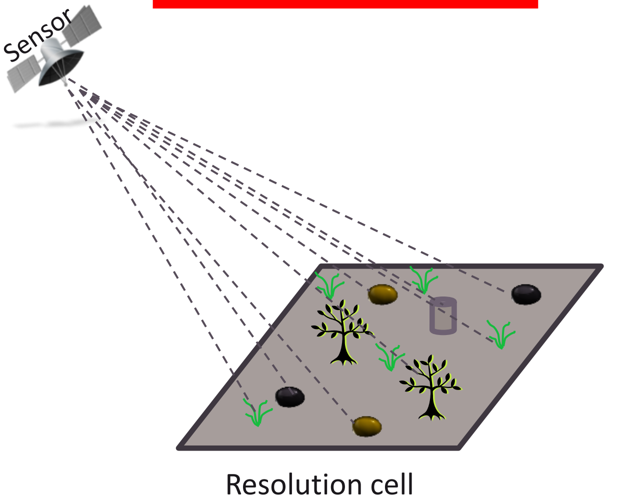


Statistical models for SAR data

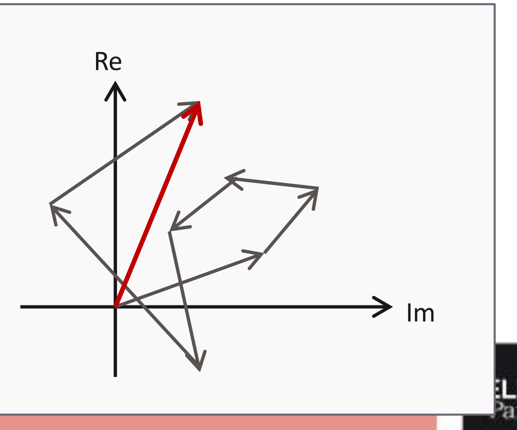
1. Speckle principle
2. Fully developped single look speckle
3. Multi-look processing
4. Multiplicative model
5. Mellin transform and log-cumulants
6. Extension to vectorial data

Speckle principle

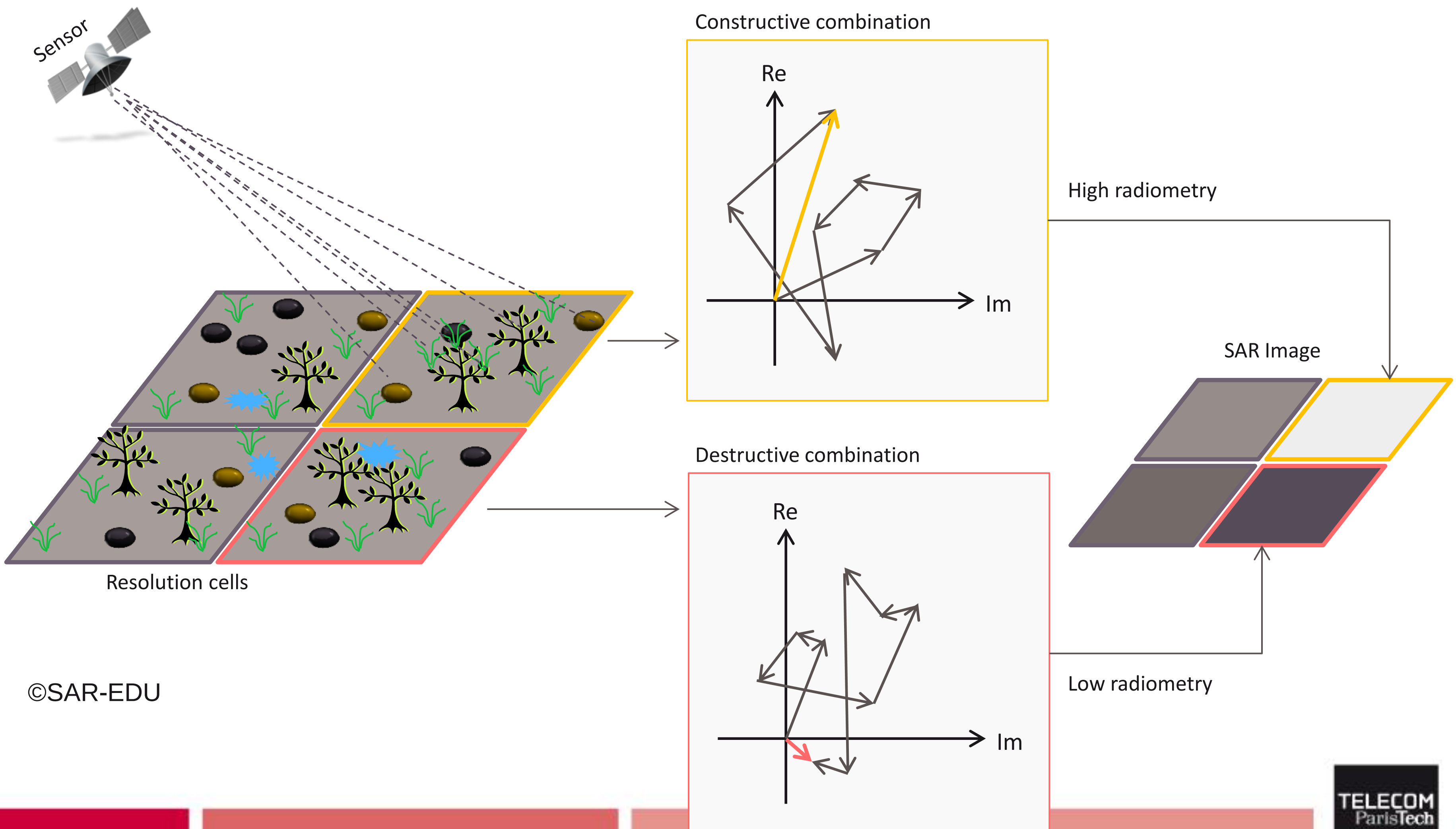
- **Size of resolution cell >> λ**
  - Elementary scatterers inside the resolution cell



- **Coherent sum of the waves:**
  - Each scatterer backscatters the e.m wave
  - Phenomenon of interferences
  - Vectorial addition in the complex plane



What is Speckle?



Speckle principle

- **No acces to the scatterers inside the resolution cells (even if it is deterministic!)**
- **Random variable modeling:**

  ⇒ **backscattering modeled by a r.v !**

- **Why developing models for the backscattered field ?**
  - Prediction of the performances of image processings
  - Choice of the thresholds
  - Developement of model based methods

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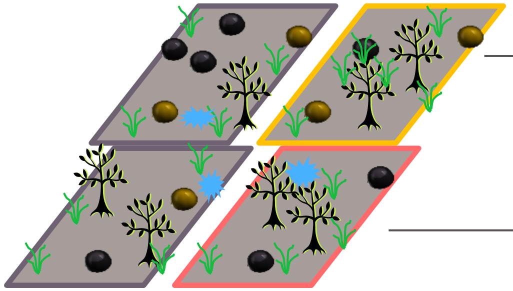
Goodman model (75)



- **Fully developed speckle :**
  - N scatterers and Goodman hypothese:
    - Independency of phase and amplitude of each scatterer
    - Amplitudes and phases are i.i.d
    - Phases are uniformy distributed on [-π; π] (**rough surface** for λ)

$$E = \sum_{i=1}^{N} a_i e^{j\varphi_i}$$

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Goodman model

$$E = \sum_{i=1}^{N} a_i (\cos \varphi_i + j \sin \varphi_i) = \sum_{i=1}^{N} a_i \cos \varphi_i + j \sum_{i=1}^{N} a_i \sin \varphi_i$$

$$p(Re(E), Im(E)) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{Re(E)^2 + Im(E)^2}{2\sigma^2} \right)$$

$$I = A^2 = Re(E)^2 + Im(E)^2$$

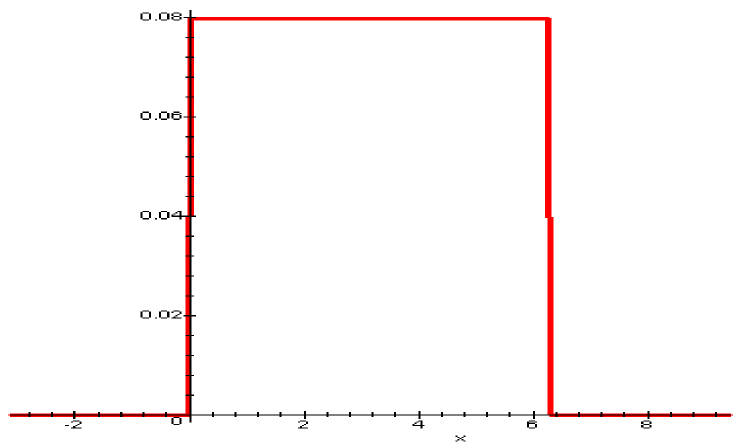
$$\phi = tan^{-1}\left(\frac{Im(E)}{Re(E)} \right)$$

$$p(I, \phi) = \frac{1}{4\pi\sigma^2} e^{\left(\frac{-I}{2\sigma^2}\right)}$$

Goodman model

$$p(\phi) = \int_0^{+\infty} p(I,\phi)\,dI$$

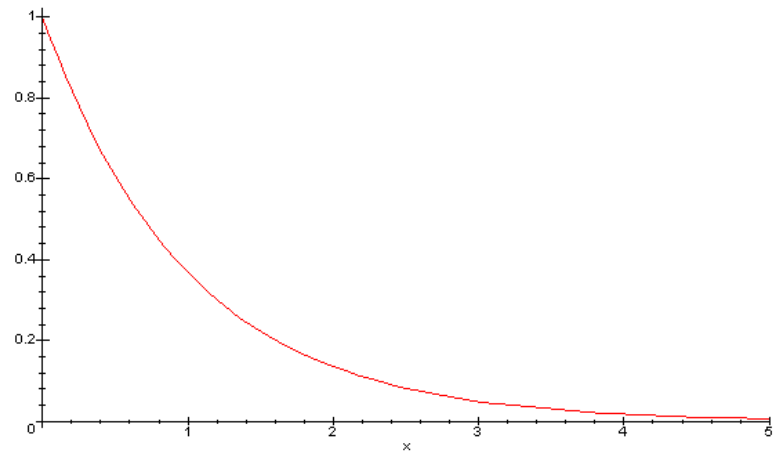
$$p(\phi) = \frac{1}{2\pi}$$



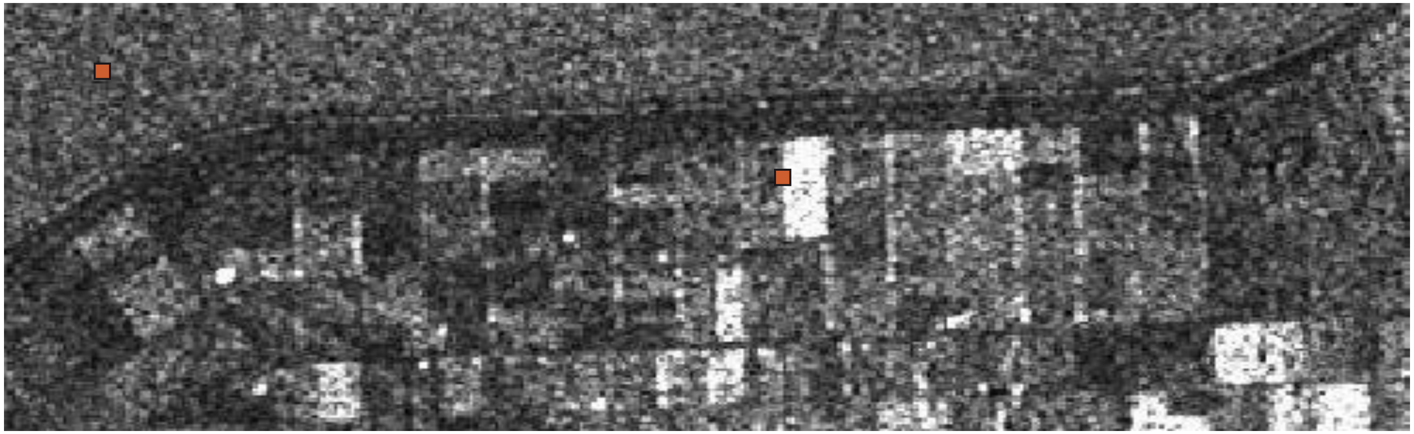
$$p(I) = \int_{-\pi}^{+\pi} p(I,\phi)\,d\phi$$

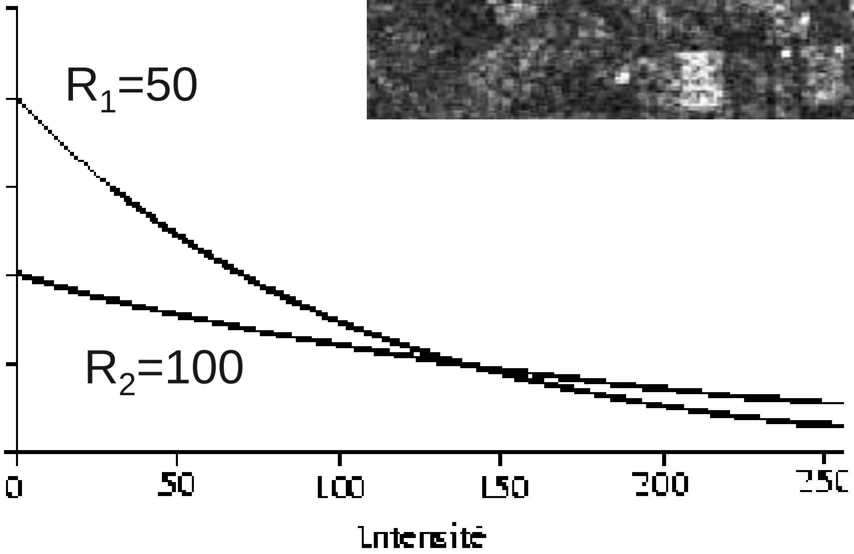
$$p(I) = \frac{1}{R} e^{\left(-\frac{I}{R}\right)}$$

$$R = 2\sigma^2 \propto \sigma^0$$



Single look intensity data…



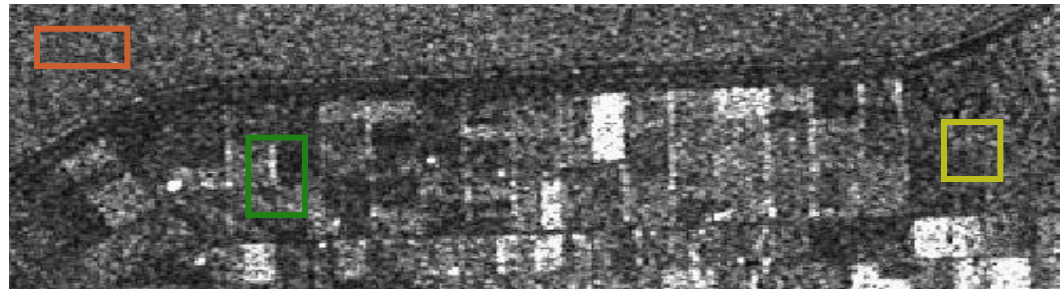


Coefficient of variation

$$p(I) = \frac{1}{R} e^{\left(-\frac{I}{R}\right)}$$

$$\mu_I = \sigma_I = R$$

$$\gamma_I = \frac{\sigma_I}{\mu_I} = 1$$

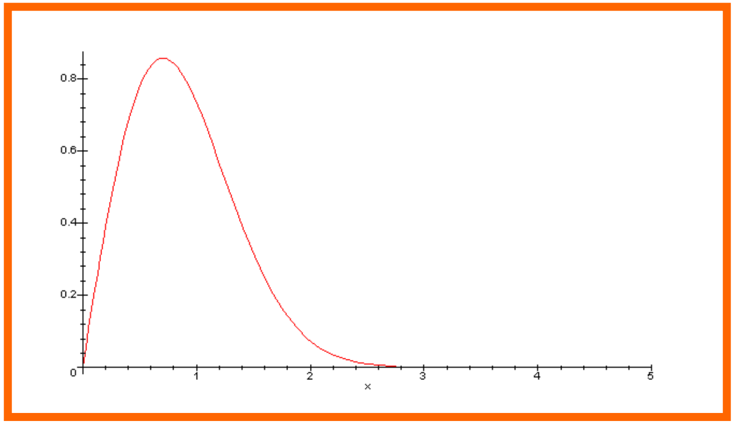


hererogeneity measure = coefficient of variation

Goodman model - amplitude

Amplitude distribution:

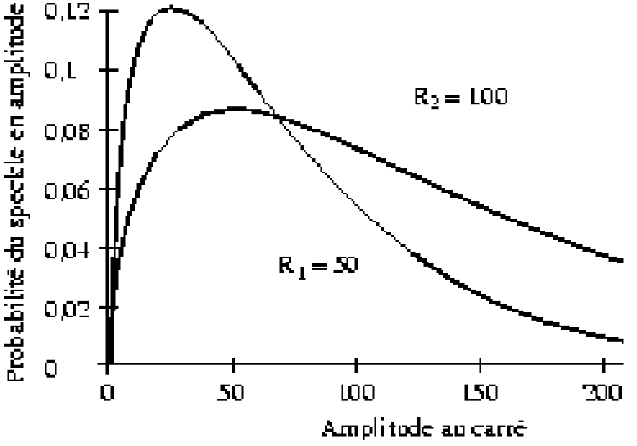
$$p(A) = \frac{2A}{R} e^{\left(-\frac{A^2}{R}\right)}$$



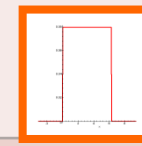
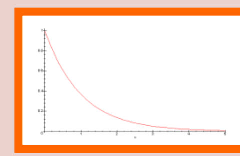
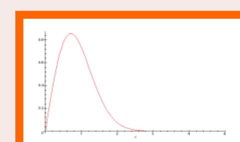
Rayleigh pdf

$$R = 2\sigma^2 \propto \sigma^0$$

- $\mu_A = \sqrt{\frac{\pi R}{4}}$
- $\gamma_A = \frac{\sigma_A}{\mu_A} = \sqrt{\frac{4}{\pi} - 1} \approx 0.523$



Goodman model (homogenous area)

Data	Pdf
Real part Imaginary part	Gaussian pdf 0 mean $R = 2\sigma^2 \propto \sigma^0$ 
Phase	Uniform pdf 
Intensity	Negative exponential pdf $\mu_I = \sigma_I = R$ 
Amplitude	Rayleigh pdf $\mu_A = \sqrt{\frac{\pi R}{4}}$ 

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Multi-look processing

- **Averaging N i.i.d samples reduces the variance by N**

$X_1, X_2, ..., X_N$ samples

$$X^{ML} = \frac{1}{N}\sum_i X_i$$

$$\begin{cases} E(X^{ML}) = E(X_i) \\ \mathrm{Var}X^{ML} = \dfrac{\mathrm{Var}X_i}{N} \end{cases}$$

- **Which data ?**
  - Complex data ? $z_1, z_2, ..., z_N$
  - Intensity data ? $I_1, I_2, ..., I_N$
  - Amplitude data ? $A_1, A_2, ..., A_N$

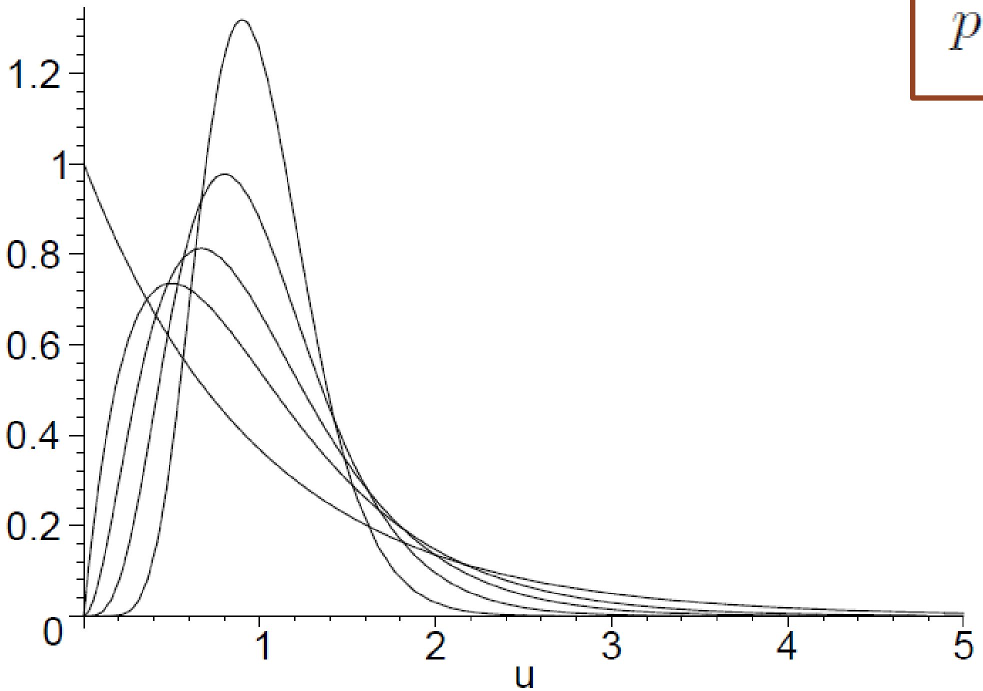
Intensity multi-looking

- **Averaging of L intensity samples:**
  - Convolution of neg. exp. pdf : Gamma pdf
  - L: number of looks

$$p(I|R) = \frac{L^{L-1}}{\Gamma(L)} \frac{I^{L-1}}{R^L} \exp(-L\frac{I}{R})$$

$$\mu_I = R \qquad \sigma_I = \frac{R}{\sqrt{L}}$$

$$\gamma_I = \frac{1}{\sqrt{L}}$$

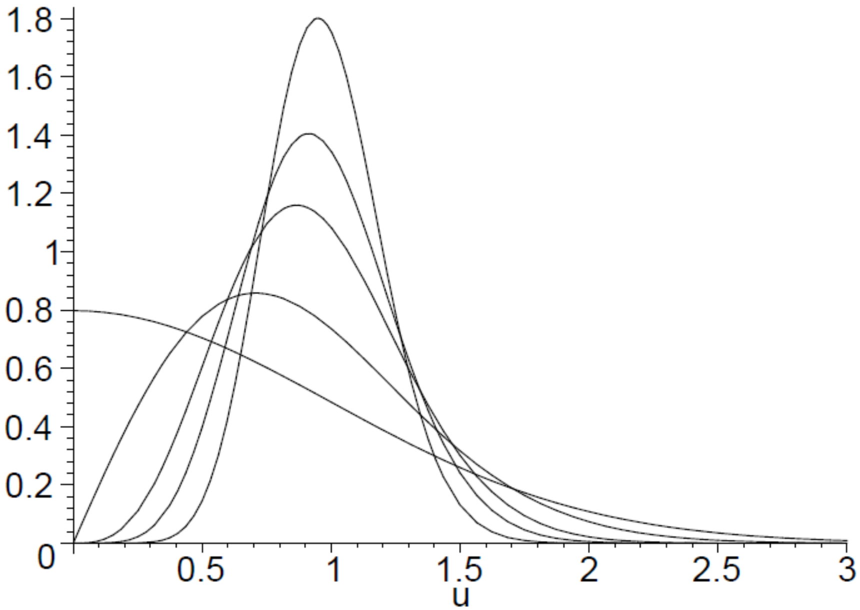


Amplitude multi-looking

- **Square root of the average of L intensity samples**
  - Nakagami pdf
  - L: number of looks

$$p(A|R) = \frac{2L^L}{\Gamma(L)} \frac{A^{2L-1}}{R^L} \exp(-L\frac{A^2}{R})$$

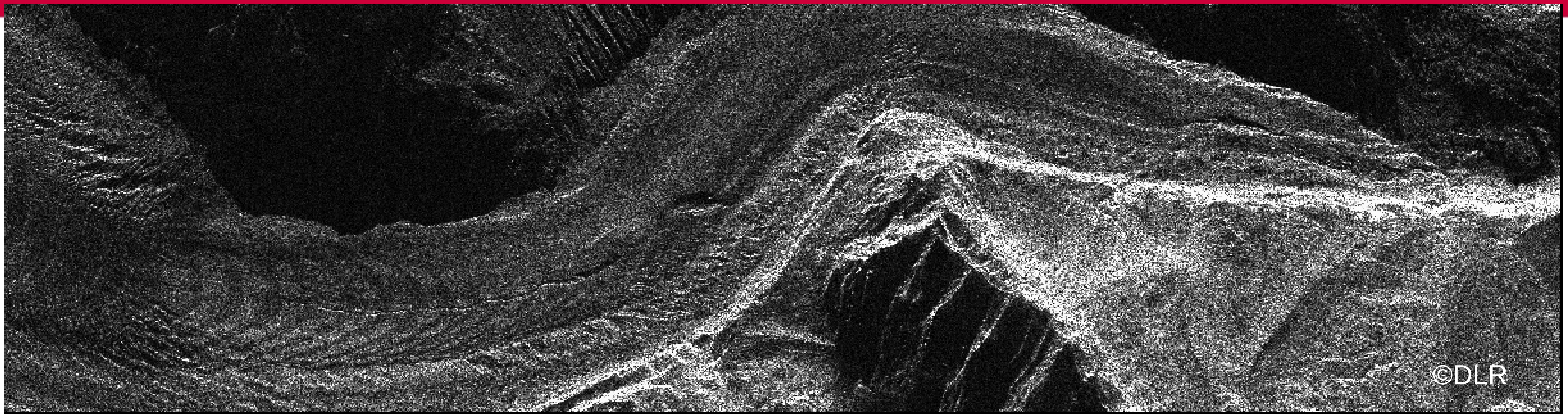
$$\gamma_A = \frac{0.523}{\sqrt{L}}$$

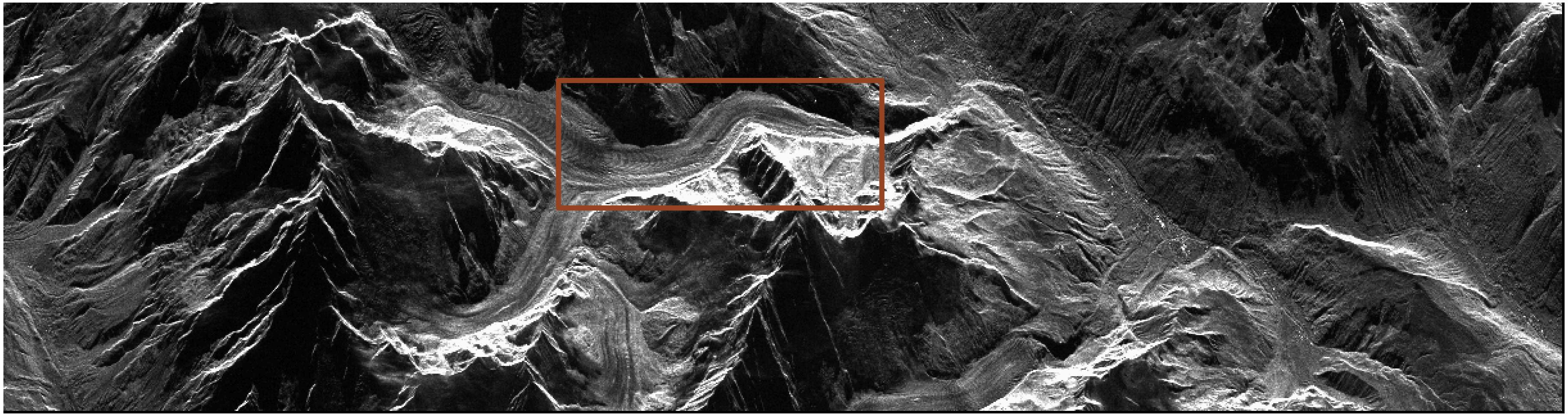


Multi-looking

- **Which samples ?**
  - Historically :
    - Azimuth sub-band decomposition of the complex spectrum
    - Decrease of spatial resolution to improve radiometric resolution
  - Spatial samples
    - Mean filter
    - Loss of spatial resolution
  - Temporal samples
    - Not iid ?

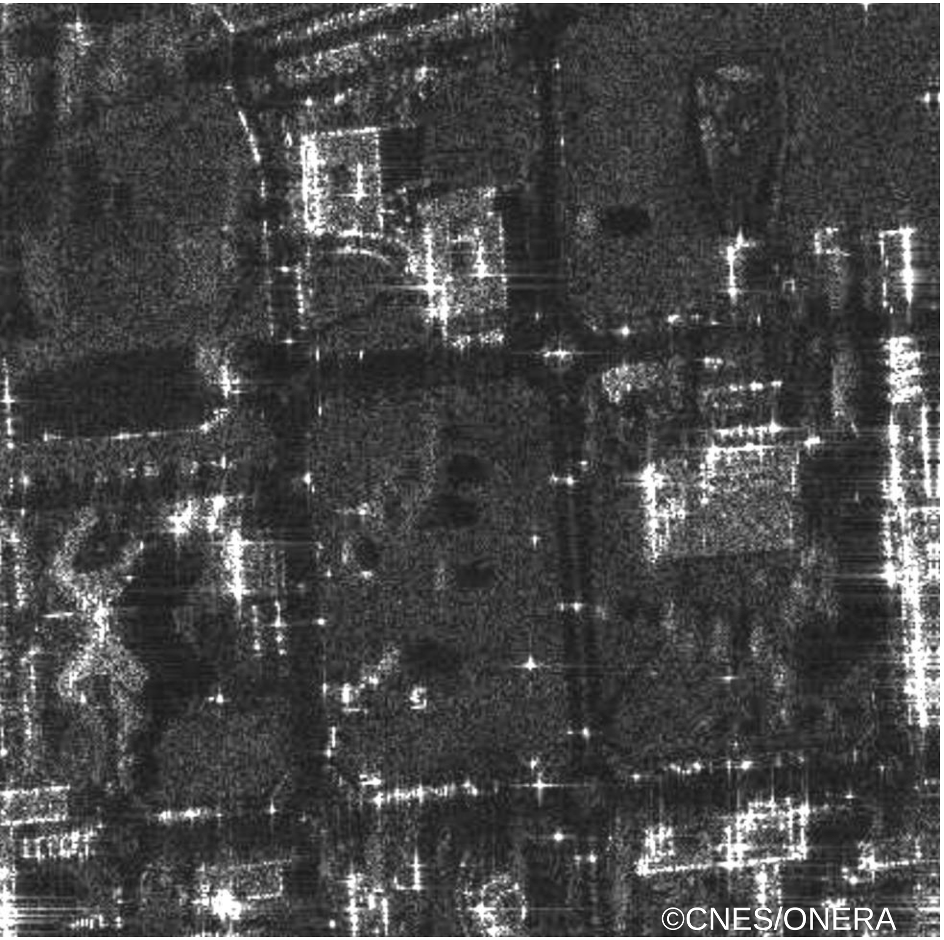
Multi-looking : less speckle, less resolution

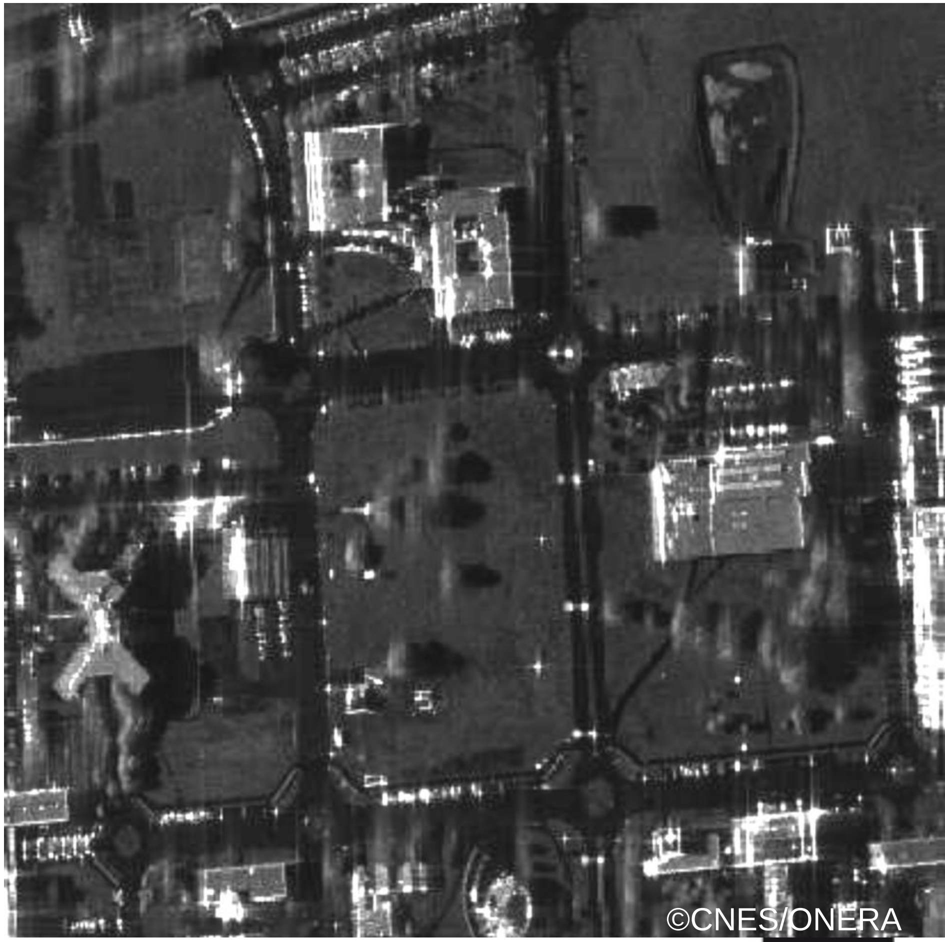




- 10x10 multi-looking : easier image interpretation

Spatial multi-looking





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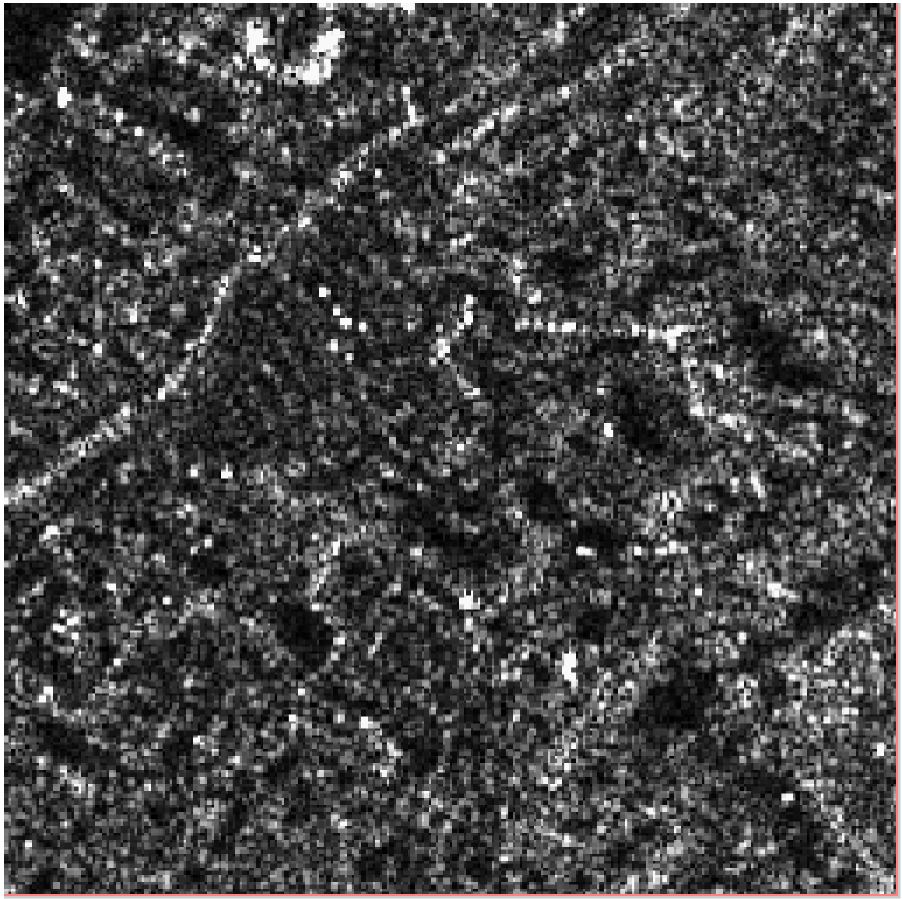
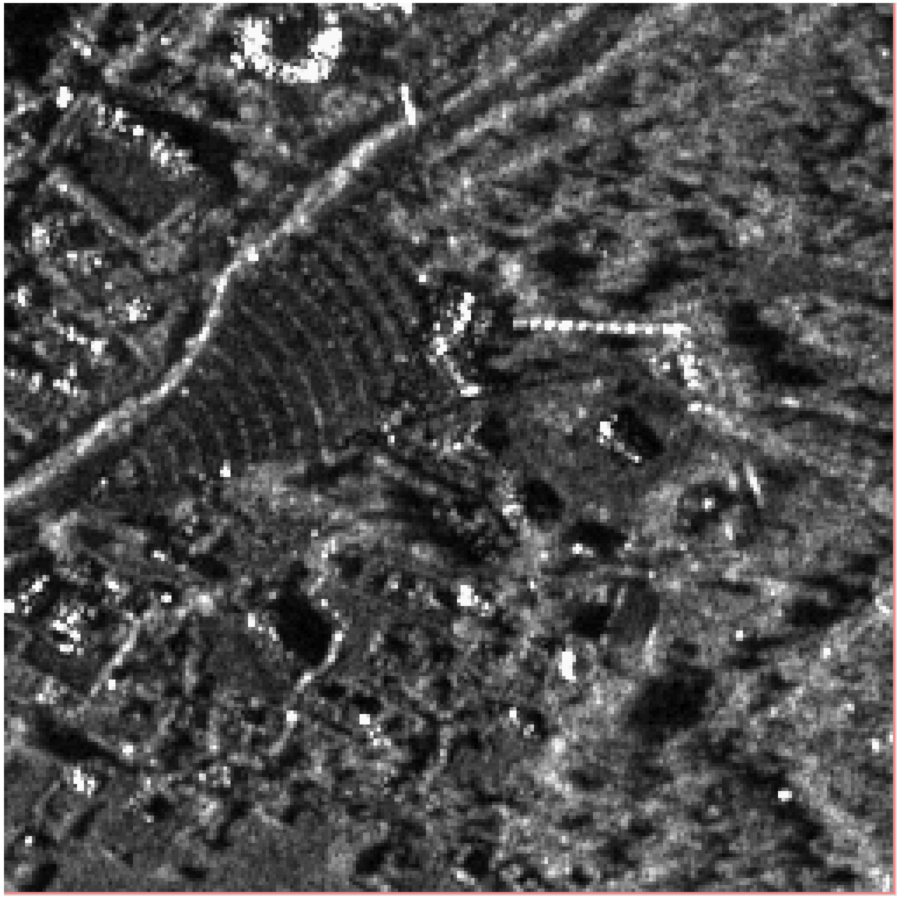
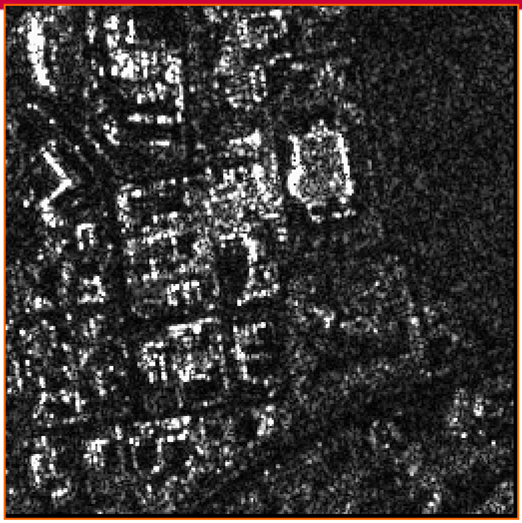
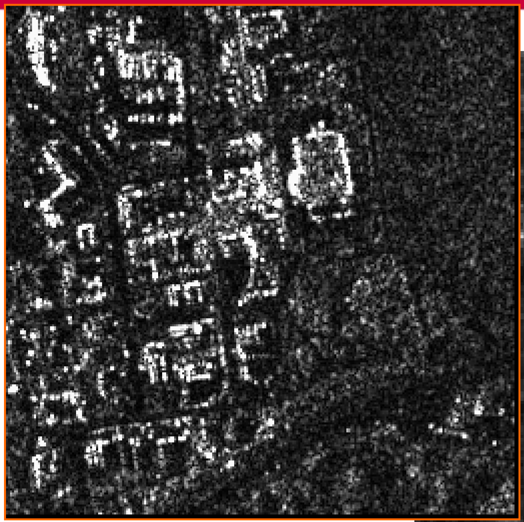
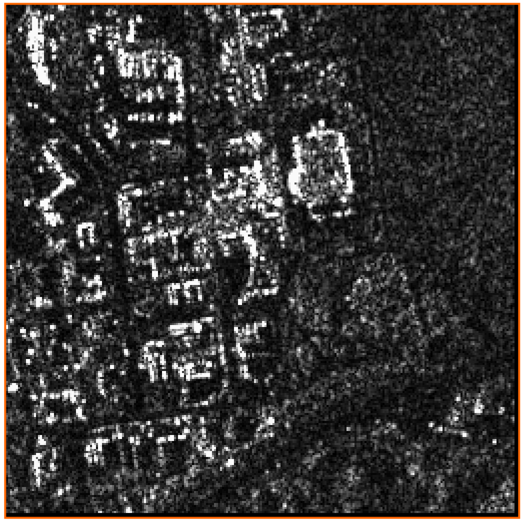
Temporal multi-looking (13 images)



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Multiplicative model

- **Modeling of the speckle noise as multiplicative:**

$$I = R.S$$

$$E(S) = 1$$

$$\mathrm{Var}(S) = \frac{1}{L}$$

$$p(S) = \frac{L^{L-1}}{\Gamma(L)} S^{L-1} \exp(-LS)$$

- **Texture modeling with scene pdf:**

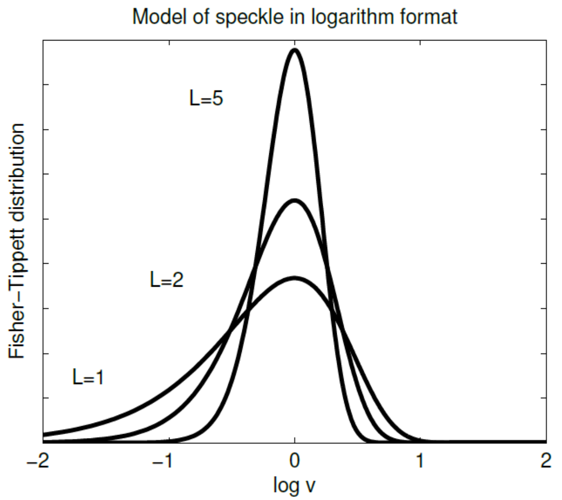
$$p(I = RS) = \int p(R)p(S = \frac{I}{R})\frac{1}{R}dR$$

$$p(I) = \int p(R)p(I|R))dR$$

(« Homomorphic » approaches)

Principle

- Logarithmic transform of the image
- Fisher-Tipett distribution
- Processing of log-image with additive noise models
- Exponential transform of the filtered image and debiasing



$$\mathbb{E}[s(V)|u] = \ln u + \psi(L) - \log L$$

$$\hat{u}^{(debiased)} = \frac{L}{\exp\psi(L)}\hat{u}^{(biased}$$

Mellin convolution and associated tools

- Mellin convolution for positive r.v:

$$r(I) = \int_0^\infty p(R)\,\gamma\left(\frac{I}{R}\right)\frac{dR}{R} = p \,\hat{*}\, \gamma$$

- Modeling of many textures on SAR:

S distribution	R distribution	I distribution
Gamma	dirac	Gamma
Gamma	Gamma	K
Gamma	Gamma inverse	Fisher

Mellin convolution

- **Convolution and Fourier transform:**

$$r = p * q = \int_{-\infty}^{\infty} p(u)q(x-u)du \Leftrightarrow TF(r) = TF(p).TF(q)$$

  - Adapted to additive noise

- **Mellin convolution and Mellin transform:**

$$r = p \hat{*} q = \int_{0}^{\infty} p(u)q\left(\frac{x}{u}\right)\frac{du}{u} \Leftrightarrow TM(r) = TM(p).TM(q)$$

  - Adapted to multiplicative noise

Statistics et log-statistics

- **Statistics : pdf defined on $\Re$**
  - Use of the Fourier transform
  - Convolution: additive noise
  - Characteristic functions
  - Gaussian pdf: defined on $\Re$
- **Log-statistics : pdf defined on $\Re^+$**
  - Use of Mellin transform
  - Mellin convolution: multiplicative noise
  - Characteristic function of «second kind»
  - Gamma pdf : defined on $\Re^+$

Characteristic functions, moments and cumulants

$$\Phi(f) = TF(p) = \int_{-\infty}^{+\infty} e^{2j\pi xf}\, p(x)\,dx$$

$$\tilde{\Phi}(s) = \mathrm{TM}(p) = \int_{0}^{\infty} x^{s-1}\, p(x)\,dx$$

$$m_k = (-j)^k \left.\frac{d^k\Phi(f)}{df^k}\right|_{f=0}$$

$$\tilde{m}_k = \left.\frac{d^k\tilde{\Phi}(s)}{ds^k}\right|_{s=1}$$

$$\kappa_k = (-j)^k \left.\frac{d^k \log\left(\Phi(f)\right)}{df^k}\right|_{f=0}$$

$$\tilde{\kappa}_k = \left.\frac{d^k \log\left(\tilde{\Phi}(s)\right)}{ds^k}\right|_{s=1}$$

Convolution and Mellin convolution

$$r = p * q = \int_0^{\infty} p(u) q(x-u) du$$

$$TF(r) = TF(p).TF(q)$$

$$\Phi[r] = \Phi[p]\Phi[q]$$

$$r = p \hat{*} q = \int_0^{\infty} p(u) q\left(\frac{x}{u}\right) \frac{du}{u}$$

$$TM(r) = TM(p).TM(q)$$

$$\tilde{\Phi}[r] = \tilde{\Phi}[p]\tilde{\Phi}[q]$$

Estimation of moments and log-moments

$$\Phi(f) = TF(p) = \int_{-\infty}^{+\infty} e^{2j\pi x f}\, p(x)\,dx$$

$$m_k = (-j)^k \left.\frac{d^k \Phi(f)}{df^k}\right|_{f=0}$$

$$= \int_{-\infty}^{+\infty} x^k\, p(x)\,dx$$

$$\hat{m}_k = \frac{1}{N}\sum_{i=1}^{N} (x_i)^k$$

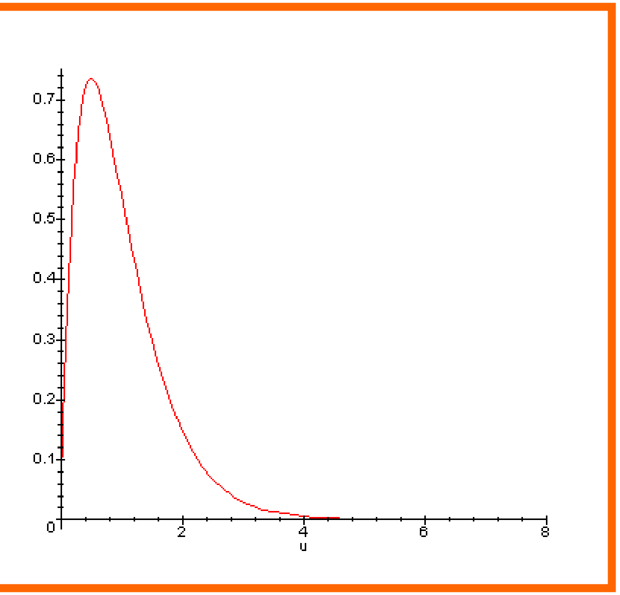
$$\tilde{\Phi}(s) = TM(p) = \int_{0}^{\infty} x^{s-1}\, p(x)\,dx$$

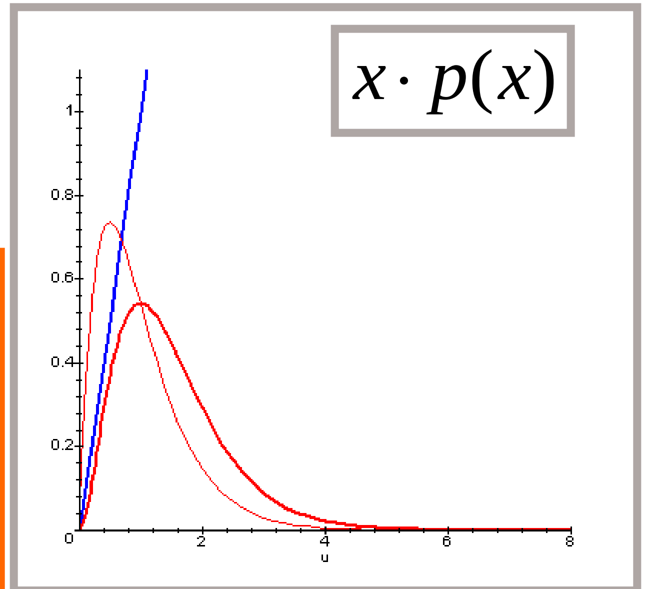
$$\tilde{m}_k = \left.\frac{d^k \tilde{\Phi}(s)}{ds^k}\right|_{s=1}$$

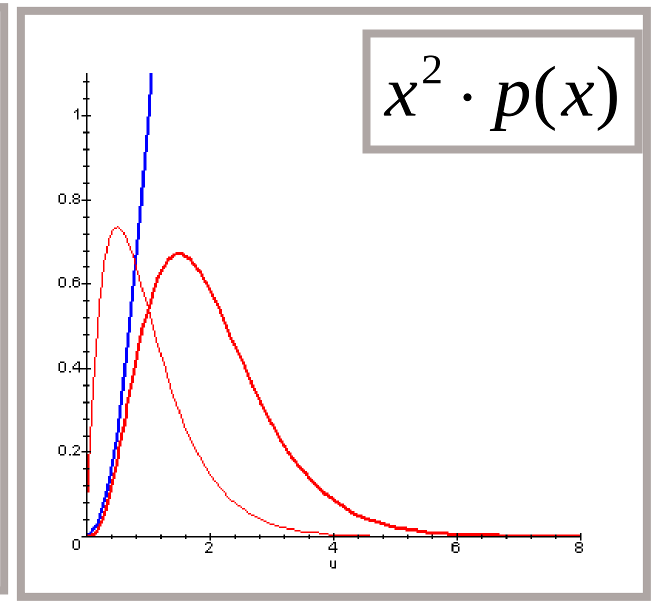
$$= \int_{0}^{+\infty} (\log(x))^k\, p(x)\,dx$$

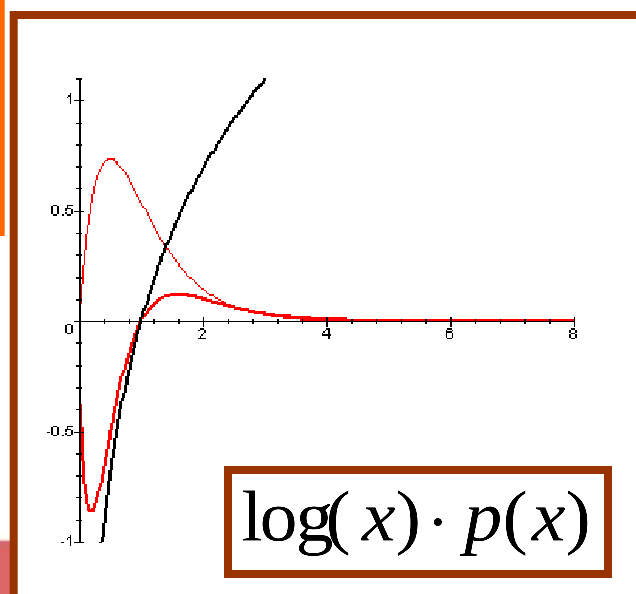
$$\hat{\tilde{m}}_k = \frac{1}{N}\sum_{i=1}^{N} \log(x_i)^k$$

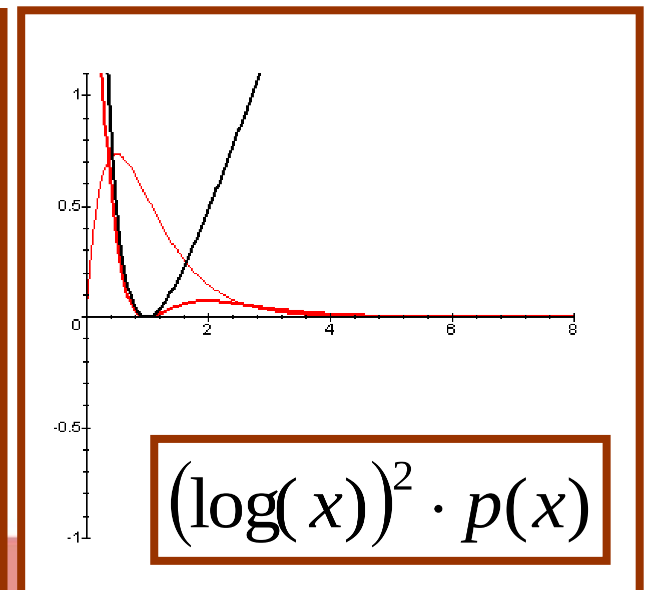
Moments et log-moments











L=2

Example of Rayleigh-Nakagami pdf

$$G[L,\mu](x) = \frac{1}{\Gamma(L)}\frac{1}{\mu}\left(\frac{Lx}{\mu}\right)^{L-1} e^{-\frac{Lx}{\mu}}$$

$$RN[L,\mu](x) = \frac{2}{\Gamma(L)}\frac{\sqrt{L}}{\mu}\left(\frac{\sqrt{L}x}{\mu}\right)^{2L-1} e^{-\left(\frac{\sqrt{L}x}{\mu}\right)^2}$$

$$\tilde{\Phi}_{\mathrm{G}}(s) = \mu^{s-1}\frac{\Gamma(L+s-1)}{L^{s-1}\Gamma(L)}$$

$$\tilde{\Phi}_{\mathrm{RN}}(s) = \mu^{s-1}\frac{\Gamma\left(L+\frac{s-1}{2}\right)}{L^{\frac{s-1}{2}}\Gamma(L)}$$

$$\tilde{\kappa}_{1,\mathrm{G}} = \log(\mu) + \Psi(L) - \log(L)$$
$$\tilde{\kappa}_{2,\mathrm{G}} = \Psi(1,L)$$

$$\tilde{\kappa}_{1,\mathrm{RN}} = \log(\mu) + \frac{1}{2}\left(\Psi(L) - \log(L)\right)$$
$$\tilde{\kappa}_{2,\mathrm{RN}} = \frac{1}{4}\Psi(1,L)$$

$$\tilde{\kappa}_k = \Psi(k-1,L)$$

$$\tilde{\kappa}_{k,\mathrm{RN}} = \left(\frac{1}{2}\right)^k \Psi(k-1,L)$$

Practical use of log-cumulants to analyze textures

- **Computation of log-cumulants:**

$$\widehat{\kappa_1} = \frac{1}{N}\sum_{i=1}^{N} \ln(x_i)$$

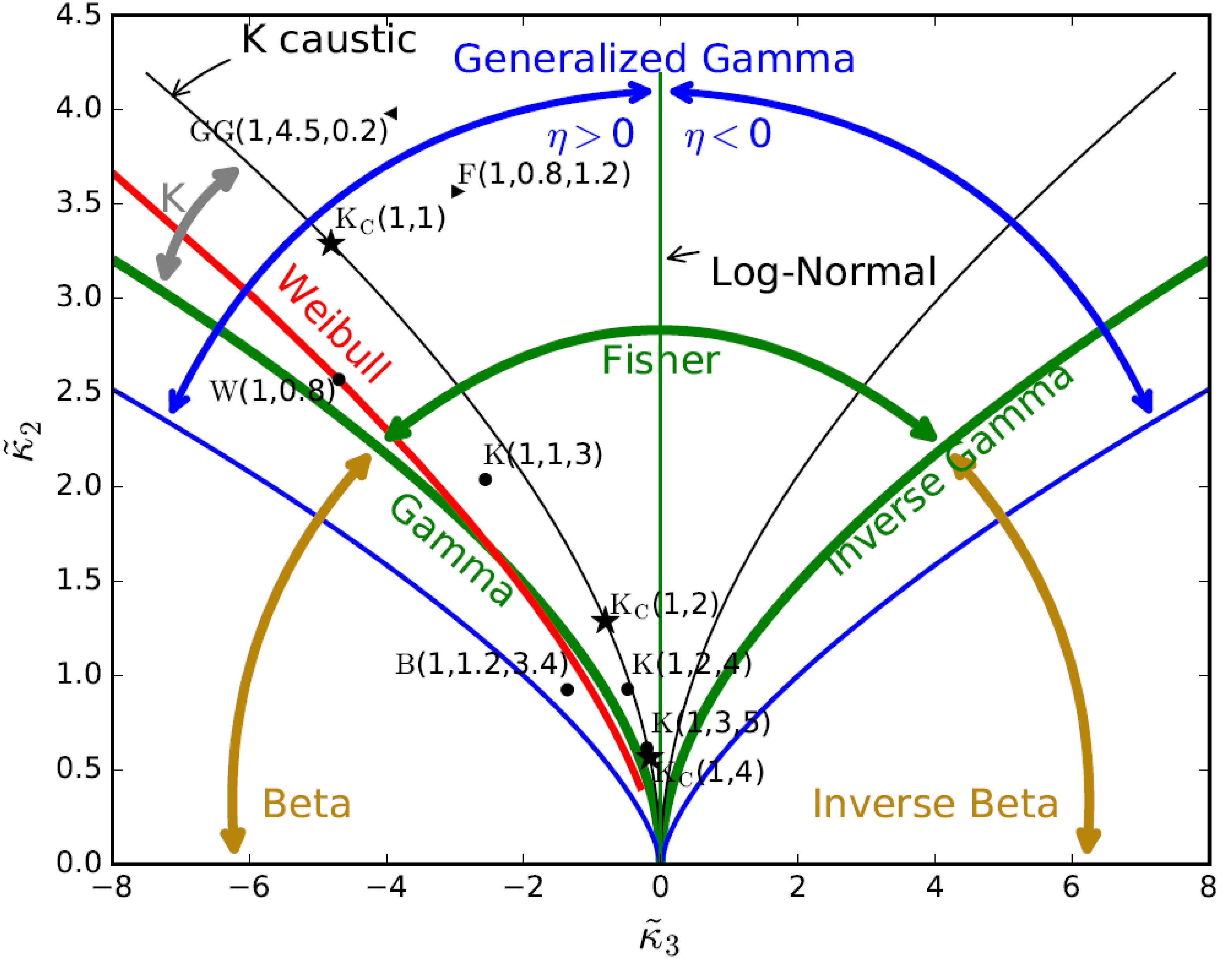
$$\widehat{\kappa_2} = \frac{1}{N}\sum_{i=1}^{N} (\ln(x_i))^2 - \frac{1}{N^2}\left(\sum_{i=1}^{N} \ln(x_i)\right)^2$$

$$\begin{aligned}\widehat{\kappa_3} = & \frac{1}{N}\sum_{i=1}^{N} (\ln(x_i))^3 \\ & -\frac{3}{N^2}\left(\sum_{i=1}^{N} \ln(x_i)\right)\left(\sum_{i=1}^{N} (\ln(x_i))^2\right) \\ & +\frac{2}{N^3}\left(\sum_{i=1}^{N} \ln(x_i)\right)^3\end{aligned}$$

- **For known pdf : inversion to recover pdf parameters**

- **For unknown pdf: positioning in the log-cum3 / log-cum2 diagram**

Log-cum3 / log-cum2 diagram



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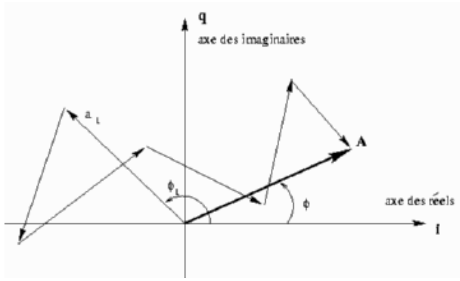
4. Multiplicative model
5. Mellin transform and log-cumulants
6. Extension to vectorial data

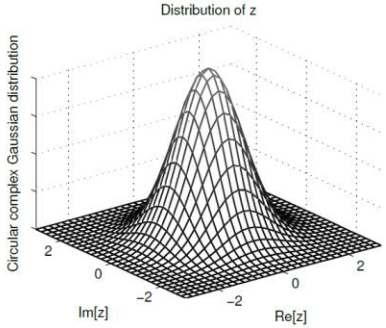
SAR data and statistics

- **Data: complex electro-magnetic field** $z = Ae^{j\varphi}$ **(amplitude** $A = |z|$**, intensity** $I = A^2$ **)**

- **Speckle: coherent imagery, interferences**
  - Goodman model (rough surfaces)

$$p(z|\sigma^2) \triangleq p(\mathrm{Re}[z], \mathrm{Im}[z]|\sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|z|^2}{2\sigma^2}\right)$$

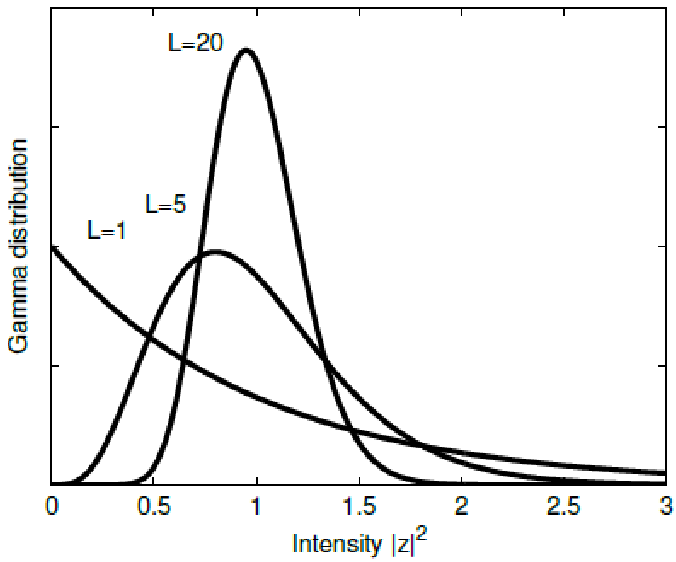


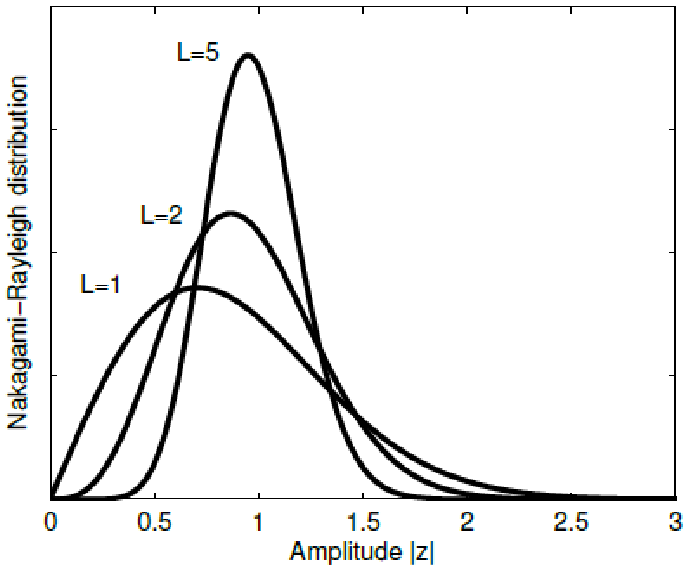


$$2\sigma^2 = R$$

SAR data and statistics

- **One channel, Goodman model:**
  - Multi-look images: $I = \frac{1}{L}\sum_{i=1}^{L} |z_i|^2$
  - Intensity distribution: Gamma
  - Amplitude distributions: Rayleigh-Nakagami





SAR data and statistics

D channels, Goodman model:

- Vectorial data: $\mathbf{k} = (z_1, \dots, z_D)^t$
- Circular complex Gaussian distribution:

$$\mathrm{p}(\boldsymbol{k}|\boldsymbol{\Sigma}) = \frac{1}{\pi^D \det(\boldsymbol{\Sigma})} \exp\left(-\boldsymbol{k}^\dagger \, \boldsymbol{\Sigma}^{-1} \, \boldsymbol{k}\right)$$

$$\boldsymbol{\Sigma} = \mathbb{E}\{\boldsymbol{k}\boldsymbol{k}^\dagger\}$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} R_1 & \sqrt{R_1}\sqrt{R_2}\gamma_{1,2}\exp(j\psi_{1,2}) & \cdots & \sqrt{R_1}\sqrt{R_D}\gamma_{1,D}\exp(j\psi_{1,D}) \\ \sqrt{R_1}\sqrt{R_2}\gamma_{1,2}\exp(-j\psi_{1,2}) & R_2 & & \sqrt{R_2}\sqrt{R_D}\gamma_{2,D}\exp(j\psi_{2,D}) \\ \vdots & & \ddots & \vdots \\ \sqrt{R_1}\sqrt{R_D}\gamma_{1,D}\exp(-j\psi_{1,D}) & \sqrt{R_2}\sqrt{R_D}\gamma_{2,D}\exp(-j\psi_{2,D}) & & R_D \end{pmatrix}$$

SAR data and statistics

- **Multi-look data, Goodman model: Wishart distribution**

$$\boldsymbol{C} = \frac{1}{L}\sum_{i=1}^{L} \boldsymbol{k}_i \boldsymbol{k}_i^{\dagger}$$

$$p(\boldsymbol{C}|\boldsymbol{\Sigma}) = \frac{L^{LD}|\boldsymbol{C}|^{L-D}}{\Gamma_D(L)|\boldsymbol{\Sigma}|^L} \exp\left(-L\,\mathrm{tr}(\boldsymbol{\Sigma}^{-1}\boldsymbol{C})\right)$$

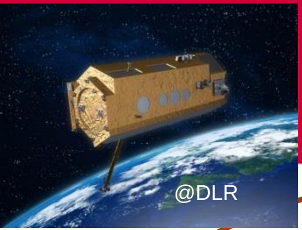
$$I_k = A_k^2 = \frac{1}{L}\sum_{i=1}^{L} |z_{i,k}|^2$$

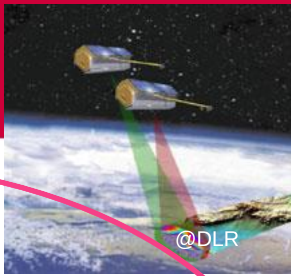
$$d_{k,l}e^{j\phi_{k,l}} = \frac{\sum_{i=1}^{L} z_{i,k} z_{i,l}^*}{\sqrt{\sum_{i=1}^{L} |z_{i,k}|^2 \sum_{i=1}^{L} |z_{i,l}|^2}}$$

coherence ($d_{k,l}$)

phase ($\phi_{k,l}$)

SAR data and statistics





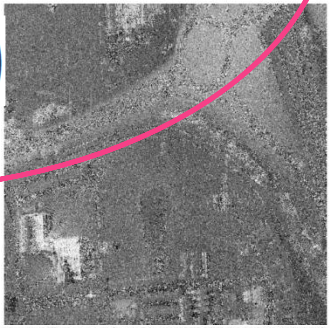
D=1 $z = Ae^{j\varphi}$

Amplitude data
(classification, object recognition,…)

$$\mathbf{k} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

D=2
different incidence angles
Interferometric data:
geometric information
(elevation, movement)

$$\mathbf{\Sigma} = R \begin{pmatrix} 1 & \gamma_{1,2} e^{j\psi_{1,2}} \\ \gamma_{1,2} e^{-j\psi_{1,2}} & 1 \end{pmatrix}$$



D=3 $\boldsymbol{k} = (z_{hh}, z_{vv}, \sqrt{2} z_{hv})^t$

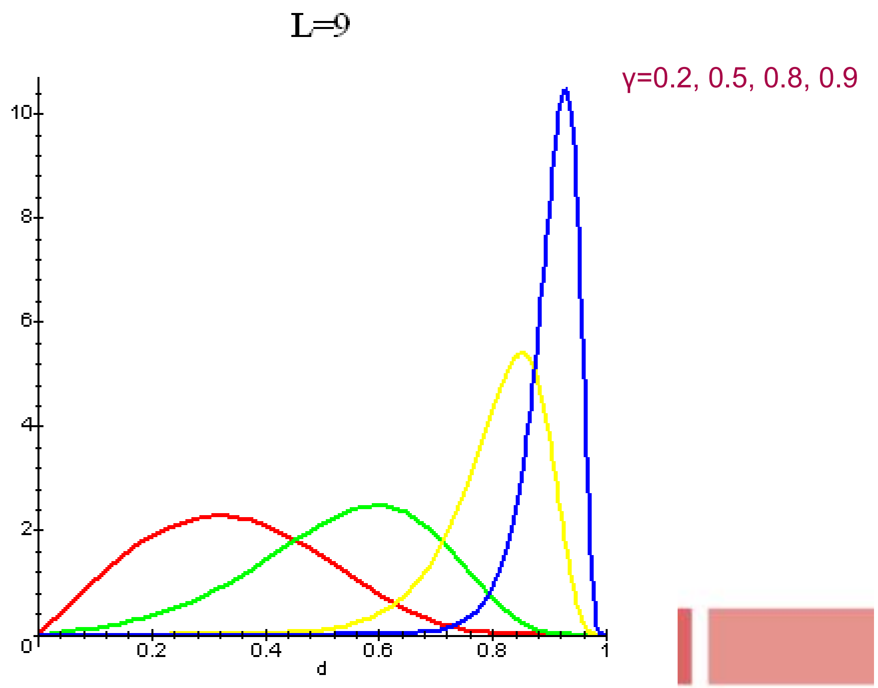
different polarizations
Polarimetric data
Backscattering mechanisms
(classification, object recognition,…)

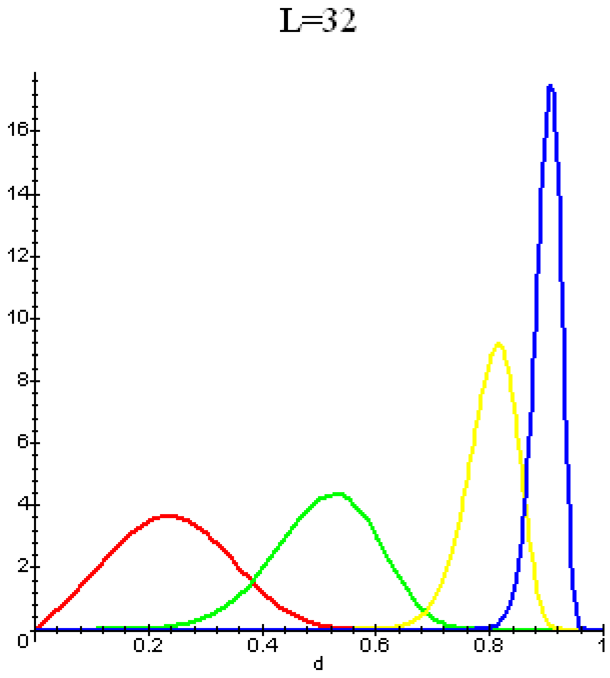
Interferometric data

- **2-D covariance matrix :** $\Sigma = \begin{pmatrix} R_1 & \sqrt{R_1}\sqrt{R_2}\gamma e^{j\psi} \\ \sqrt{R_1}\sqrt{R_2}\gamma e^{-j\psi} & R_2 \end{pmatrix}$

- **Pdf of empirical coherence with L looks**

$$p(d/\gamma, L) = 2(L-1)(1-\gamma^2)^L d(1-d^2)^{L-2} {}_1F_2(L; L, 1; d^2\gamma^2)$$





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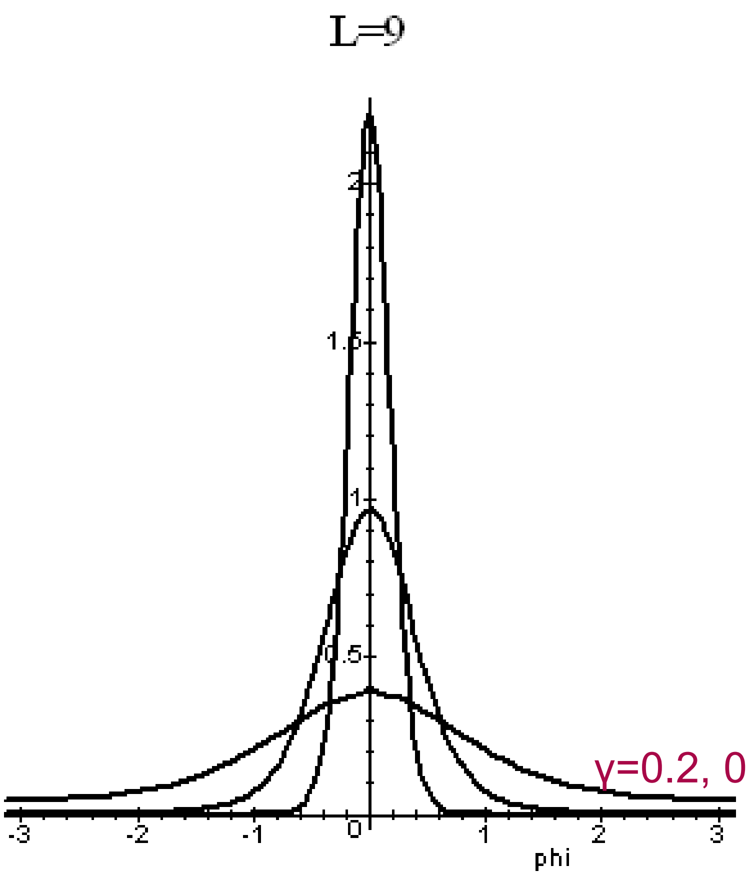
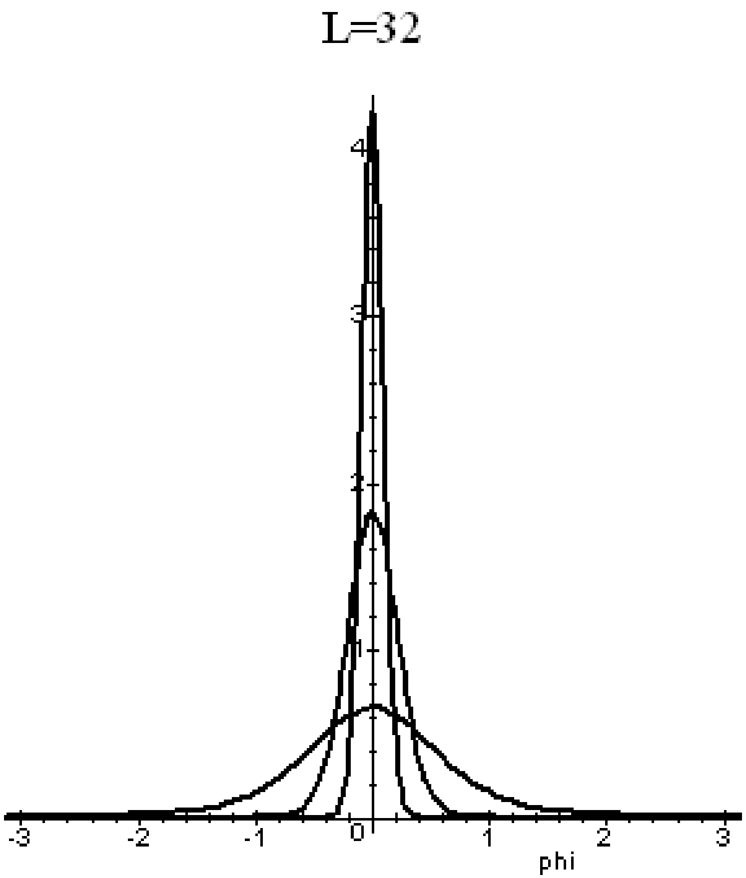
Interferometric phase pdf

$$p(\varphi/\gamma,L\ )=\frac{\left(1-\gamma^2\right)^L}{2\pi(2L+1)}{}_2F_1\left(2,2L;L+\frac{3}{2};\frac{1+\gamma\cos(\varphi)}{2}\right)$$



γ=0.2, 0.5, 0.8, 0.9



Summary

- **Goodman model (rough surfaces, coherent imaging system)**
  - Amplitude: Rayleigh - Nakagami
  - Intensity: Gamma
  - Covariance matrix: Wishart distributed
- **Multi-looking: L**
  - Incoherent (intensity)
  - Hermitian product (interferometry, polarimetry, polInSAR)
- **Statistical models: to be taken into account to process these data**