

Wavelet and image compression

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Plan

1 Introduction

2 Discrete Wavelet Transform

- Filter banks
- Multiresolution analysis

3 Images compression

- EZW
- JPEG 2000

Plan

1 Introduction

2 Discrete Wavelet Transform

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3 Images compression

- EZW
- JPEG 2000

Digital images representation

- Image as an $N \times M$ matrix of “pixels”
- Color images: three channels i.e. components
- Each channel (red, green, blue) is in turns an $N \times M$ matrix



Digital image representation

Color images: YCbCr format

- Coordinates transformation:
 $[R, V, B] \rightarrow [Y, C_b, C_r]$
- Luminance Y and chrominance C_b, C_r



Why compression

- Photos : 5 Megapixel
- Three channels
- One byte per channel
- 15 MB per photo
- Video SD, 720×576 pixel, 25 images per second
 - Rate: 125 Mbps
 - 2 hours: more than 100 GB
- Video HD, 1920×1080 pixel, 50 images per second
 - Rate: 1250 Mbps
 - 2 hours: more than 1 TB

Compression and redundancy

- Statistical redundancy
 - Images are homogeneous
 - Statistical dependency among pixels
- Psycho-visual redundancy
 - Spatial frequency sensitivity
 - Masking effects
 - Contours importance
 - Other limits of the HVS
- Transform coding: we pass from NM luminance values (all equally important) to only $n \ll NM$ significant transform coefficients

Compression algorithms

- *Lossless* algorithms
 - Perfect reconstruction
 - Based on statistical redundancy
 - Low compression ratio
- *Lossy* algorithms
 - Decoded image $\neq$ original image
 - Based on transform and quantization
 - “Visually lossless”
 - High compression ration

Wavelets and images : Motivations

- Image model : *trends + anomalies*



Wavelets and images : Motivations

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Wavelets and images : Motivations

- Image model : *trends + anomalies*



Wavelets and images : Motivations

- *Anomalies* :
 - Abrupt variations of the signal
 - High frequency contributions
 - Objects' contours
 - Good spatial resolution
 - Rough frequency resolution
- *Trends* :
 - Slow variations of the signal
 - Low frequency contributions
 - Objects' texture
 - Rough spatial resolution
 - Good frequency resolution

Wavelets and images : Motivations

Time-frequency boxes of basis signals

Time analysis:

$$\theta_\gamma(t) = \delta(t - t_0)$$

Frequency analysis:

$$\theta_\gamma(t) = e^{2i\pi f_0 t}$$

Short Time Fourier Transform
(STFT):

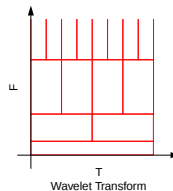
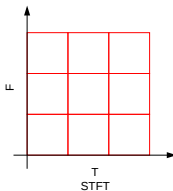
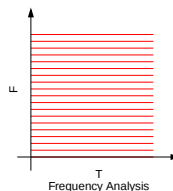
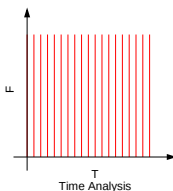
$$\theta_\gamma(t) = g(t - t_0)e^{2i\pi f_0 t}$$

σ_t, σ_ξ independent from γ

Wavelet Transform:

$$\theta_\gamma(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right)$$

$$\sigma_{t(a,b)} = a\sigma_{t,(1,0)} \quad \sigma_{\xi(a,b)} = a^{-1}\sigma_{\xi,(1,0)}$$



Wavelets and images : Motivations

Signal model: an image row



Wavelets and images : Motivations

Signal model: an image row



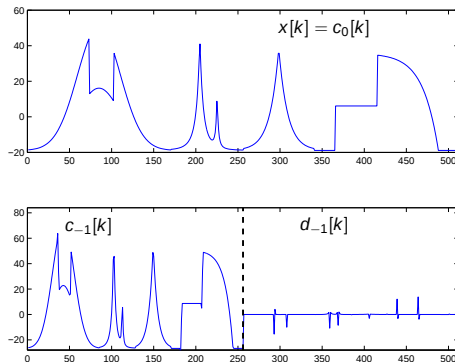
Wavelets and images : Motivations

Signal model: an image row



Wavelets and Multiple resolution analysis

- Approximation: low resolution version
- “Details”: zeros when the signal is polynomial



Plan

1 Introduction

2 Discrete Wavelet Transform

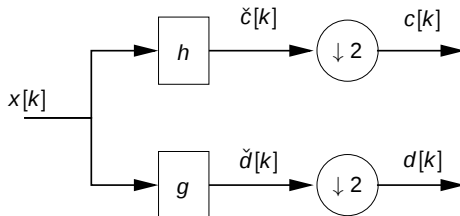
- Filter banks
- Multiresolution analysis

3 Images compression

- EZW
- JPEG 2000

1D filter banks

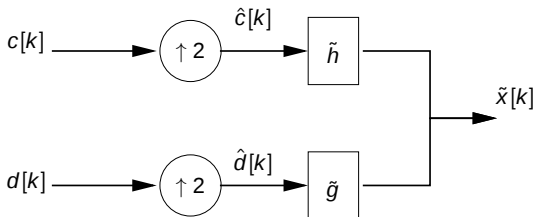
Decomposition



Analysis filter bank

$2 \downarrow$: decimation : $c[k] = \check{c}[2k]$

Reconstruction



Synthesis filter bank

$2 \uparrow$: interpolation operator, doubles the sample number

$$\hat{c}[k] = \begin{cases} c[k/2] & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Filter properties

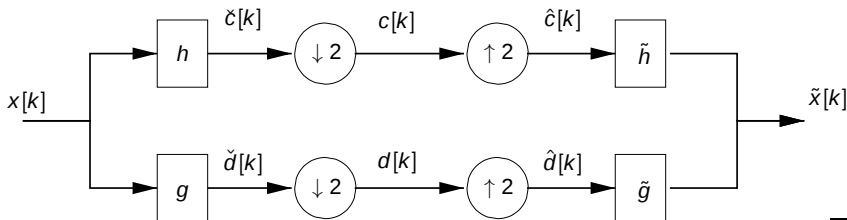
- Perfect reconstruction (PR)
- FIR
- Orthogonality
- Vanishing moments
- Symmetry

Perfect reconstruction conditions

We want PR after synthesis and analysis filter banks :

$\forall k \in \mathbb{Z}$,

$$\tilde{x}_k = x_{k+\ell} \iff \tilde{X}(z) = z^{-\ell} X(z)$$



Z-domain relationships

filter $\check{C}(z) = \sum_{n=-\infty}^{\infty} \check{c}_n z^{-n} = H(z) X(z)$

decimation $C(z) = \frac{1}{2} \left[\check{C}(z^{1/2}) + \check{C}(-z^{1/2}) \right]$

interpolation $\hat{C}(z) = C(z^2)$

output $\tilde{X}(z) = \tilde{H}(z) C(z^2) + \tilde{G}(z) D(z^2)$

$$\begin{aligned} \tilde{X}(z) &= \frac{1}{2} \left[\tilde{H}(z) H(z) + \tilde{G}(z) G(z) \right] X(z) \\ &\quad + \frac{1}{2} \left[\tilde{H}(z) H(-z) + \tilde{G}(z) G(-z) \right] X(-z) \end{aligned}$$

PR conditions in Z

$$\forall k \in \mathbb{Z},$$

$$\tilde{X}_k = x_{k+\ell} \iff \tilde{X}(z) = z^{-\ell} X(z)$$

$$\Updownarrow$$

$$\tilde{H}(z)H(z) + \tilde{G}(z)G(z) = 2z^{-\ell} \quad \text{Non distortion}$$

$$\tilde{H}(z)H(-z) + \tilde{G}(z)G(-z) = 0 \quad \text{Non aliasing}$$

Perfect reconstruction conditions

Matrix form

For simplicity, we ignore the delay, $\ell = 0$

If the analysis filter bank is given, the synthesis one is determined by:

$$\begin{bmatrix} H(z) & G(z) \\ H(-z) & G(-z) \end{bmatrix} \cdot \begin{bmatrix} \tilde{H}(z) \\ \tilde{G}(z) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

We assume that the *modulation matrix* is invertible.

Perfect reconstruction conditions

Synthesis filter bank

Modulation matrix determinant :

$$\Delta(z) = H(z)G(-z) - G(z)H(-z)$$

$$\tilde{H}(z) = \frac{2}{\Delta(z)}G(-z)$$

$$\tilde{G}(z) = -\frac{2}{\Delta(z)}H(-z)$$

Perfect reconstruction with FIR filters

Finite impulse response filters:

It can be shown that in this case the PR condition is equivalent to the altering signs condition. Example:

$$h(k) = \begin{bmatrix} a & b & c \end{bmatrix}$$

$$\tilde{h}(k) = \begin{bmatrix} p & -q & r & -s & t \end{bmatrix}$$

$$g(k) = \begin{bmatrix} p & q & r & s & t \end{bmatrix}$$

$$\tilde{g}(k) = \begin{bmatrix} -a & b & -c \end{bmatrix}$$

Orthogonality

Orthogonality assures energy conservation:

$$\sum_{k=-\infty}^{\infty} (x_k)^2 = \sum_{k=-\infty}^{\infty} (c_k)^2 + \sum_{k=-\infty}^{\infty} (d_k)^2$$

⇒ reconstruction error = quantization error on DWT coefficients
For non orthogonal filters, the reconstruction errors is a weighted sum of the quantization errors on the DWT subbands, with suitable weights ω_i

Vanishing moments

- Vanishing moments (VM) represent filter ability to reproduce polynomials: a filter with p VM can represent polynomials with degree $< p$
- The High-pass filter will not respond to a polynomial input with degree $< p$
- In this case all the signal information is preserved in the approximation signal (half the samples)
- A filter with p VM has at least $2p$ taps

Haar filter

$$h(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$g(k) = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

$$\tilde{h}(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\tilde{g}(k) = \begin{bmatrix} -1 & 1 \end{bmatrix}$$

- Symmetric
- Orthogonal
- $VM = 1$
 - Only capable to represent piecewise constant signals

Orthogonality and biorthogonality

Perfect reconstruction for finite duration signals

- Convolution involves coefficient expansion
- Solution: circular convolution
 - Circular convolution allows to reconstruct an N -samples signal with N wavelet coefficients
 - The periodization generates borders discontinuities, i.e. spurious high frequencies coefficients that demand a lot of coding resources
- Solution: Symmetric periodization
 - No discontinuities
 - Does it double the coefficient number?
 - No, if the filter is **symmetric**!

Bad news: the only orthogonal symmetric FIR filter is Haar!

Biorthogonal filters

Cohen-Daubechies-Fauveau filters

With biorthogonal (i.e. PR) filters, if h has p VM and $\tilde{h}$ has $\tilde{p}$ VM, the filter has at least $p + \tilde{p} - 1$ taps.

The CDF filters have the following properties:

- They are symmetric (linear phase)
- They maximize the VM for a given filter length
- They are close to orthogonality (weights ω_i are close to one)

They are by far the most popular in image compression

9/7 biorthogonal filters

Filter coefficients:

n	0	± 1	± 2	± 3	± 4
$h[l]$	0.852699	0.377403	-0.110624	-0.023849	0.037828
$\tilde{h}[l]$	0.788486	0.418092	-0.040689	-0.064539	

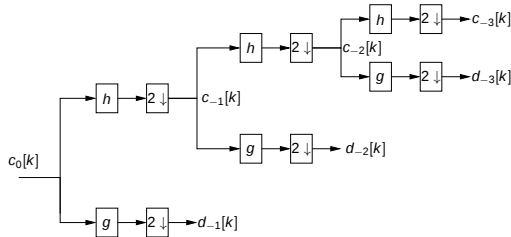
Impulse response of low-pass filters

For high pass filters, we have (altering sign condition):

$$g[l] = (-1)^{l+1} \tilde{h}[l-1] \quad \text{and} \quad \tilde{g}[l] = (-1)^{l-1} h[l+1].$$

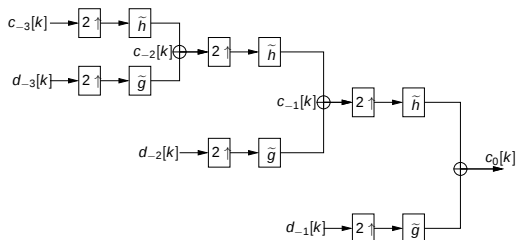
1D Multiresolution analysis

Decomposition



Three level wavelet decomposition structure

Reconstruction

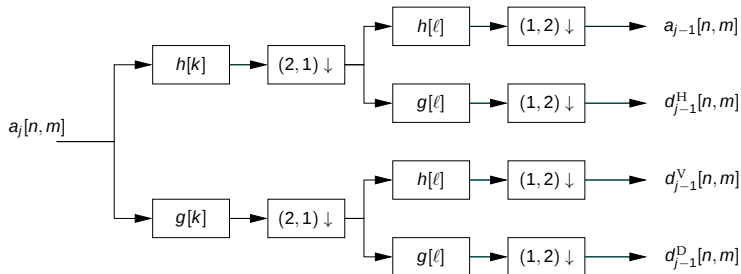


Reconstruction from wavelet coefficients

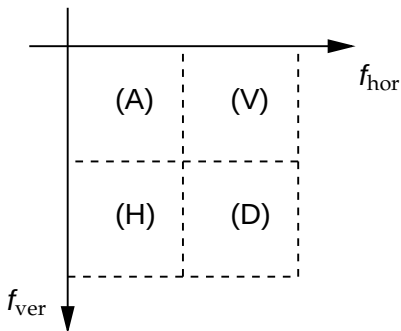
2D AMR

2D Filter banks for separable transform

One decomposition level

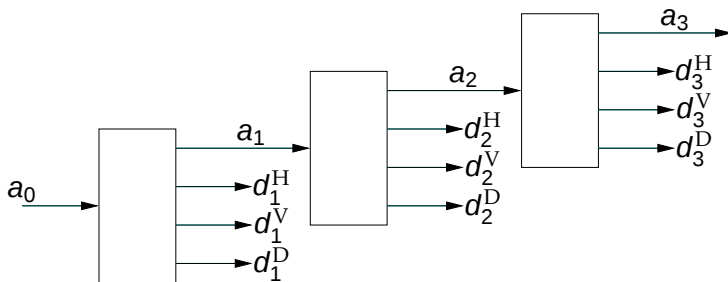


2D-DWT subbands: orientations



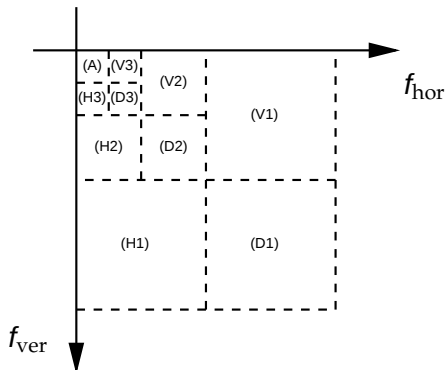
(A), (H), (V) and (D) respectively correspond to approximation coefficients, horizontal, vertical and diagonal detail coefficients.

2D AMR: multiple levels

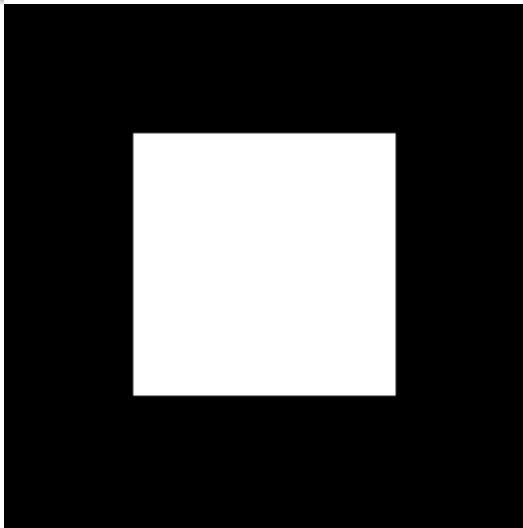


Three levels of separable 2D-AMR.

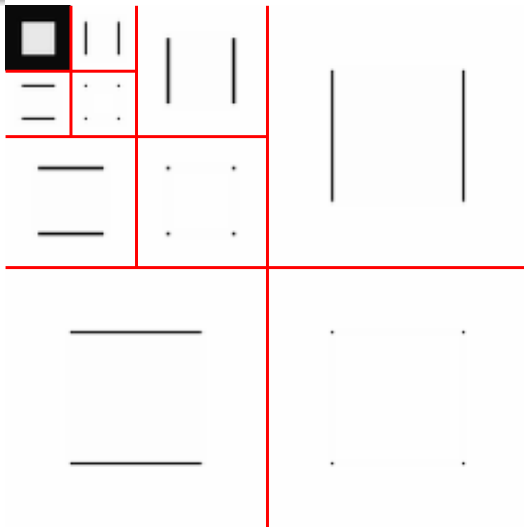
2D-DWT subbands: orientations



Example



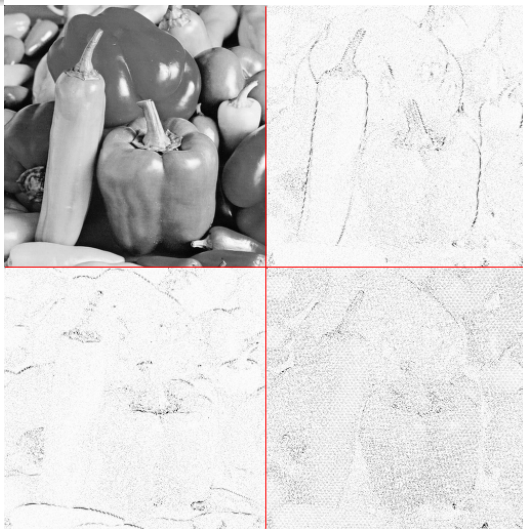
Example



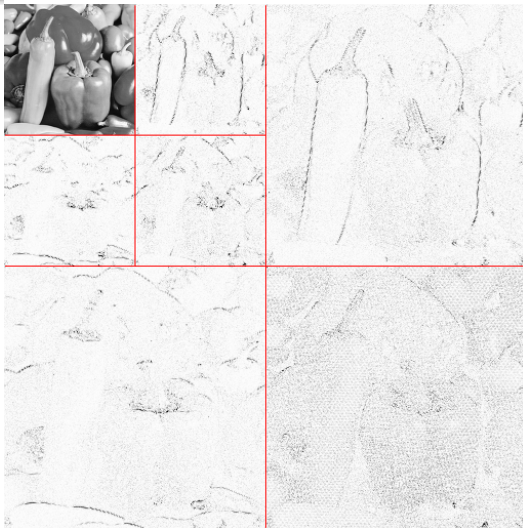
Example



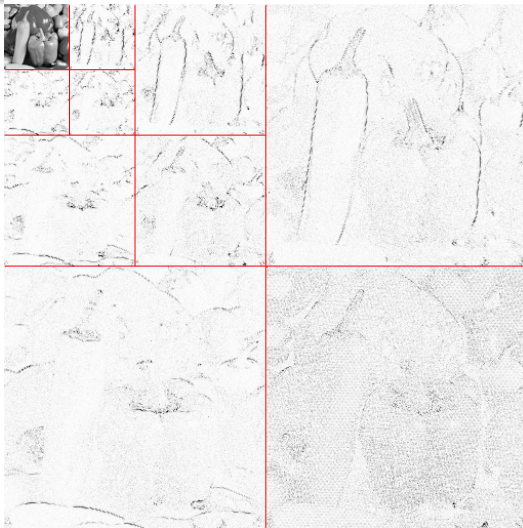
Example



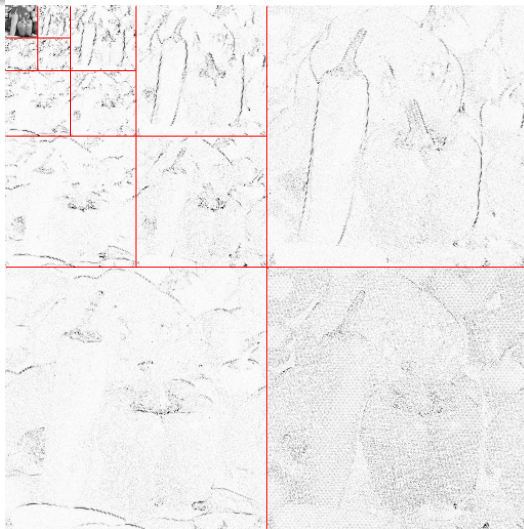
Example



Example



Example

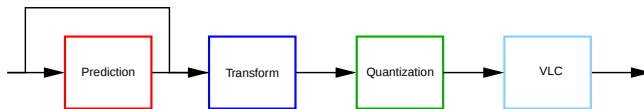


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Basic tools for compression

- Transform
 - It concentrates information in a few coefficients
- Prediction
 - Alternative (and sometimes additional) method for information concentration
- Quantization
 - Rate reduction: rough representation of less important coefficients
- Lossless coding, or variable length coding (VLC)
 - Residual redundancy reduction



Performance criteria

Rate

Compression ratio:

- $T = \frac{B_{in}}{B_{out}} = \frac{R_{in}}{R_{out}}$

Coding Rate

- Image: $R = \frac{B_{out}}{NM}$ [bpp]
- Video, sound: $R = \frac{B_{out}}{T}$ [bps]

Performance criteria

Quality and distortion

Rate alone is not sufficient for evaluate a lossy coding algorithm.
We need to assess quality or distortion for the reconstructed image

- The **Objective criteria** are mathematical functions of
 - $f_{n,m}$: the original image; and
 - $\tilde{f}_{n,m}$: the reconstructed image
- Non-perceptual objective criteria: they do not take into account the characteristics of the Human Visual System (HVS)
- Perceptual objective criteria: try to model the HVS

Performance criteria

Non-perceptual objective criteria

- Error image: $\mathcal{E}(f, \tilde{f}) = f - \tilde{f}$
- **Mean Square Error (MSE) $\mathcal{D}$:**

$$\mathcal{D}(f, \tilde{f}) = \frac{1}{NM} \|\mathcal{E}\|^2 = \frac{1}{NM} \sum_{n=1}^N \sum_{m=1}^M \mathcal{E}_{n,m}^2$$

- **Peak Signal-to-Noise Ratio:**
 $\text{PSNR}(f, \tilde{f}) = 10 \log_{10} \left(\frac{255^2}{\mathcal{D}(f, \tilde{f})} \right)$
- Simple, derivable, related to the $\mathcal{L}^2$ norm

Performance criteria

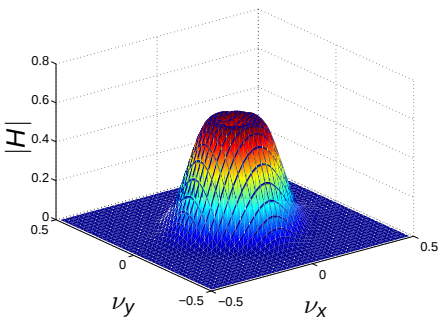
Perceptual objective criteria

Weighted PSNR: Given a frequency weighting function (a linear filter) h :

$$\text{WPSNR}(f, \tilde{f}) = 10 \log_{10} \left(\frac{255^2}{\mathcal{D}_W(f, \tilde{f})} \right)$$

where

$$\mathcal{D}_W(f, \tilde{f}) = \frac{1}{NM} \|h * \mathcal{E}\|^2$$



Performance criteria

Perceptual objective criteria

Structural Similarity Index (SSIM Index) between two blocks x et y :

$$\text{SSIM}(x, y) = [l(x, y)]^\alpha \cdot [c(x, y)]^\beta \cdot [s(x, y)]^\gamma$$

$$l(x, y) = \frac{2\mu_x\mu_y + C_1}{\mu_x^2 + \mu_y^2 + C_1} \quad \text{Luminance}$$

$$c(x, y) = \frac{2\sigma_x\sigma_y + C_2}{\sigma_x^2 + \sigma_y^2 + C_2} \quad \text{Contrast}$$

$$s(x, y) = \frac{\sigma_{xy} + C_3}{\sigma_x\sigma_y + C_3} \quad \text{Structure}$$

for simplicity, $\alpha = \beta = \gamma = 1$, $C_3 = C_2/2$

$$\text{SSIM} = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}$$

Le SSIM between two images is the average block SSIM

Quality evaluation: Example

Image Originale, 24 bpp



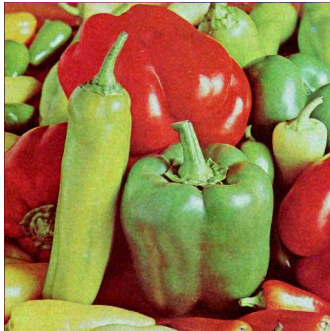
Quality evaluation: Example

White noise $\sigma = 4$

Image Originale, 24 bpp



Erreur dispersée, MSE = 16.00



Quality evaluation: Example

Moise concentrated on 100×100 pixels

Image Originale, 24 bpp



Erreur concentrée, MSE = 16.00



Quality evaluation: Example

Bruit concentré sur contours (estimated by the Sobel filter)

Contours de l'image



Erreur concentrée sur les contours, MSE = 16.00



Quality evaluation: Example

Noise on high spatial frequencies

Image Originale, 24 bpp



Erreur en frequence, MSE = 16.00



Quality evaluation: Example

Chrominance subsampling

Image Originale, 24 bpp



Sous-échantillonnage en couleur, MSE = 21.27



Performance criteria

Quality and distortion

- The **subjective criteria** are based on human observation of images
 - Need for result statistical analysis
 - Time-consuming, difficult, expensive process
- In conclusion, the non perceptual objective criteria are the most used in practice:
 - They are simple to implement and interpret
 - They are related to a geometric norm
 - They allow analytical optimization
 - Relationship with subjective quality is sometimes questioned

Performance criteria

Complexity, delay and robustness

- An algorithm complexity could be limited by:
 - Application related constraints (i.e. real-time operation)
 - Hardware limitations
 - Economic cost
- The delay, usually measured at the encoder side
 - is related to the complexity
 - Influenced by the coding order
- Robustness: algorithm sensitivity to alterations to the encoded data

Performance criteria

Summary

Contrasting needs:

↑↑ Quality	↓↓ Rate
↑↑ Robustness	↓↓ Complexity
	↓↓ Delay

Compression with DWT

- Methods based on inter-scale dependencies:
 - EZW (Embedded Zerotrees of Wavelet coefficients),
 - SPIHT (Set Partitioning in Hierarchical Trees)
 - Tree-based representation of dependencies
 - Advantages: good exploitation of inter-scale dependencies, low complexity
 - Disadvantage: no resolution scalability
- Methods not based on inter-scale dependencies
 - Explicit bit-rate allocation among subbands
 - Entropy coding of coefficients
 - Advantages: Good exploitation of intra-scale dependencies, random access, resolution scalability
 - Disadvantage: no exploitation of inter-scale dependencies

Embedded Zerotrees of Wavelet coefficients

Main characteristics

- Quality scalability (i.e. progressive representation)
- Lossy-to-lossless coding
- Small complexity
- Rate-distortion performance much better than JPEG **above all at small rates**

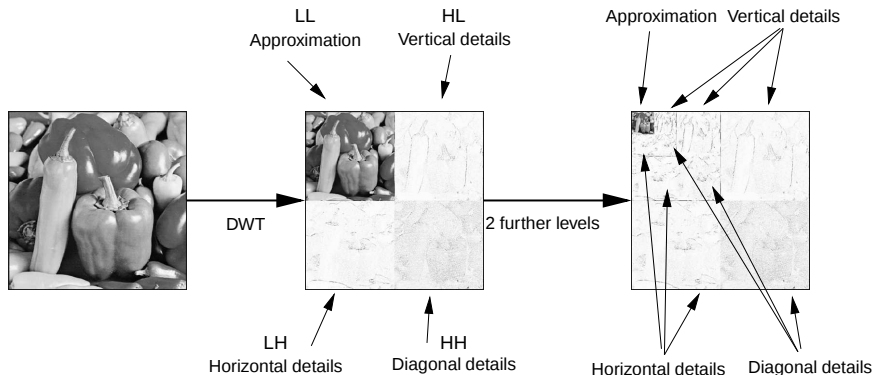
Progressive representation of DWT coefficients

- Each new coding bit must convey the maximum of information



- Each new coding bit must reduce as much as possible distortion
- We first send the largest coefficients
- Problem: localization overhead

Example: an image and its wavelet coefficients



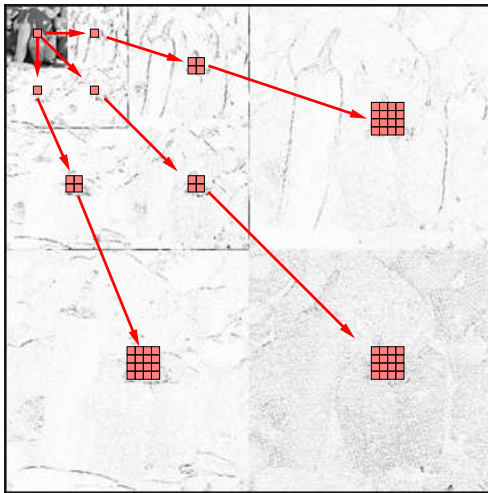
Progressive representation: subband order



EZW Algorithm

- The subband scan order alone is not enough to assure that largest coefficients are sent first
- We need to localize the largest coefficients
- Without having to send explicit localization information
- Idea: to exploit the inter-band correlation to predict the position of non-significant coefficients
- If the prediction is correct we save many coding bits (for all the predicted coefficients)

Zero-tree of wavelet coefficients



EZW idea

- **Auto-similarity** : When a coefficient is small (below a threshold) it is probable that its descendants are small as well
- In this case we use a single coding symbol to represent the coefficient and all its descendants. If c and all its descendants are smaller than the threshold, c is called a zero-tree root
- With just one symbol, (ZT) we code $(1 + 4 + 4^2 + \dots + 4^{N-n})$ coefficients
- The localization information is implicit in the significance information

EZW algorithm

- 1 $k = 0$
- 2 $n = \lfloor \log_2 (|c|_{\max}) \rfloor$
- 3 $T_k = 2^n$
- 4 while (rate < available rate)
 - Dominant pass
 - Refining pass
- 5 $T_{k+1} \leftarrow T_k / 2$
- 6 $k \leftarrow k + 1$ end

Dominant pass

- For each coefficient c (in the scan order)
- If $|c| \geq T_n$, the coefficient is significant
 - If $c > 0$ we encode SP (Significant Positive)
 - If $c < 0$ we encode SN (Significant Negative)
- If $|c| < T_n$, we compare all its descendants with the threshold
 - If no descendant is significant, c is coded as a zero-tree root (ZT)
 - Otherwise the coefficient is coded as Isolated Zero (IZ)

Refining pass

- We encode a further bit for all significant coefficients
- This is equivalent to halve the quantization step

Iteration and termination

- The k -th dominant pass allows to encode the k -th bit-plane
- A significant coefficient c is such that $2^k < |c| < 2^{k+1}$
- For the next step we halve the threshold: it is equivalent to pass to the next bitplane
- Algorithm stops when
 - the bit budget is exhausted; or when
 - all the bitplanes have been coded

EZW Algorithm: summary

- Bitplane coding: at the k -th pass, we encode the bitplane $\log_2 T_k$
- Progressive coding: each new bitplane allows refining the coefficients quantization
- Lossless coding of significance symbols
- Lossless-to-lossy coding: When an integer transform is used, and all the bitplanes are coded, the original image can be restored with zero distortion

JPEG2000

- JPEG2000 aims at challenges unresolved by previous standards:
- Low bit-rate coding: JPEG has low quality for $R < 0.25$ bpp
- Synthetic images compression
- Random access to image parts
- Quality and resolution scalability

New functionalities

- Region-of-interest (ROI) coding
- Quality and resolution scalability
- Tiling
- Exact coding rate
- Lossy-to-lossless coding

Algorithm

JPEG2000 is made up of two *tiers*

- First tier
 - DWT and quantization
 - Lossless coding of codeblocks
- Second tier
 - **EBCOT**: embedded block coding with optimized truncation
 - Scalability (quality, resolution) and ROI management

Quantization in JPEG2000

- DWT coefficients are encoded with a very fine quantization step
- For the lossless coding case, DWT coefficients are integers, and they are not quantified
- In summary, it is not in the quantization step that the really lossy operations are performed
- The lossy coding is performed by the bitstream truncation of Tier 2

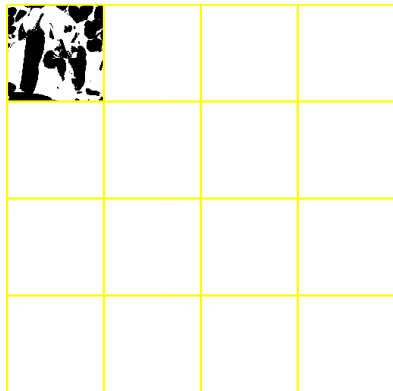
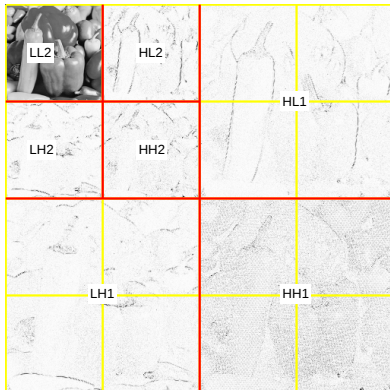
EBCOT

Embedded Block Coding with Optimized Truncation

- Each subband is split in equally sized blocks of coefficients, called codeblocks
- The codeblocks are losslessly and independently coded with an arithmetic coder
- We generate as much bitstreams as codeblocks in the image

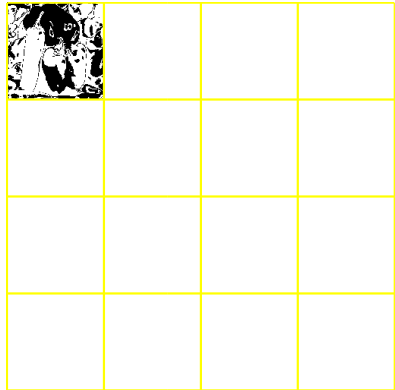
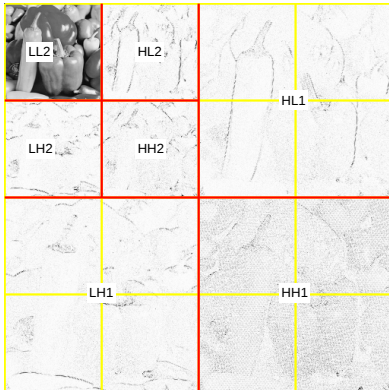
Bitplane coding

Most significant bitplane



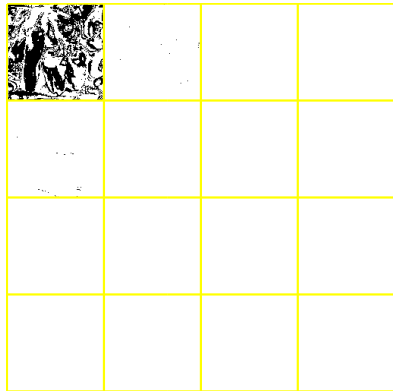
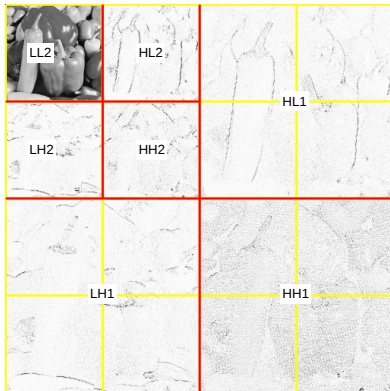
Bitplane coding

Second bitplane



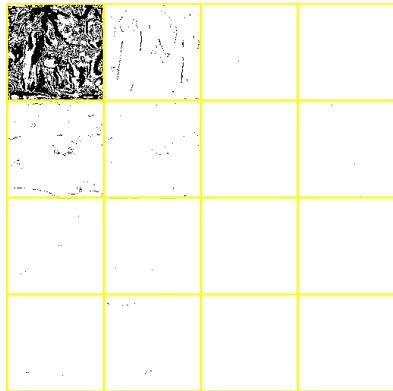
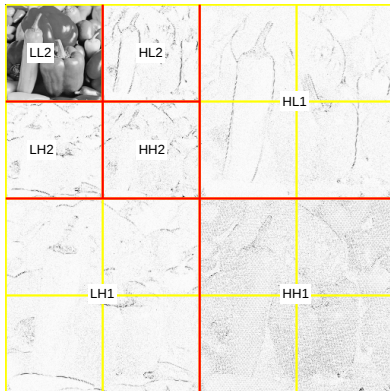
Bitplane coding

Third bitplane



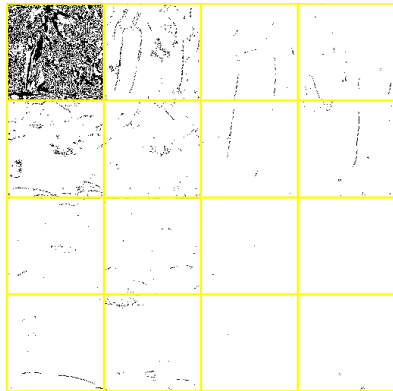
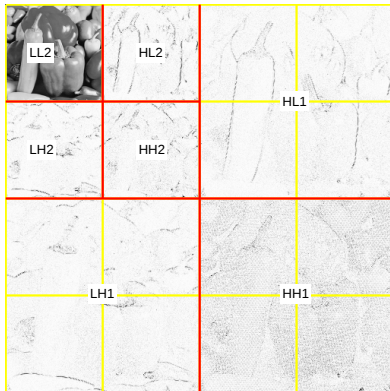
Bitplane coding

Fourth bitplane

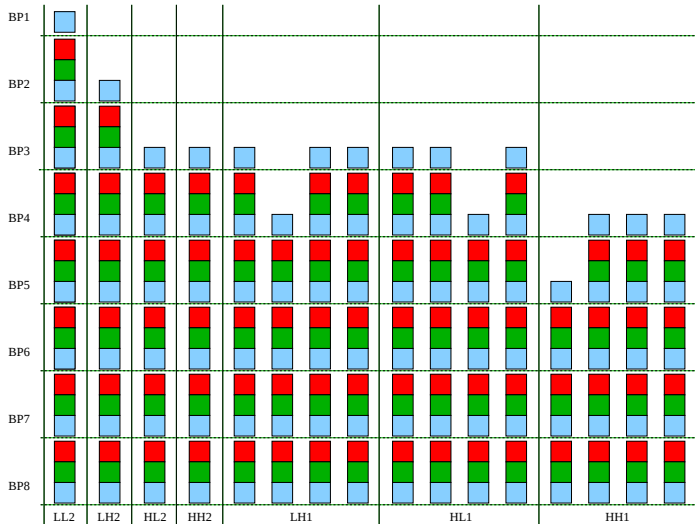


Bitplane coding

Fifth bitplane



Example of bitstreams associated to codeblocks



EBCOT

Optimization

- If we keep all the bitstreams of all the codeblocks, we end up with a huge bitrate
- We have to truncate the bitstream to attain the target bit-rate
- Problem: how to truncate the bitstreams with a minimum resulting distortion?

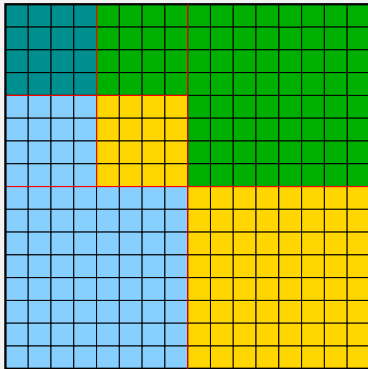
$$\min \sum_i D_i$$
$$\sum_i R_i \leq R_{\text{tot}}$$

- Solution : Lagrange multiplier

$$J = \sum_i D_i + \lambda \left(\sum_i R_i - R \right)$$

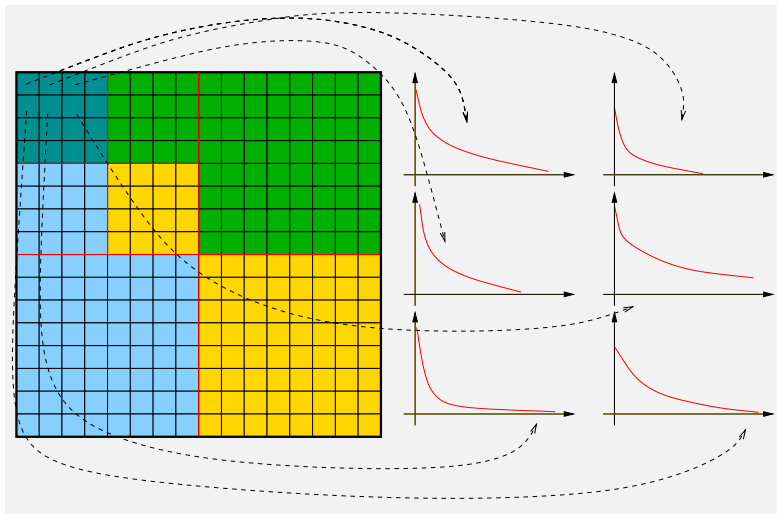
EBCOT

Rate-distortion curve per each codeblock



EBCOT

Rate-distortion curve per each codeblock



EBCOT

Embedded block coding with optimized truncation

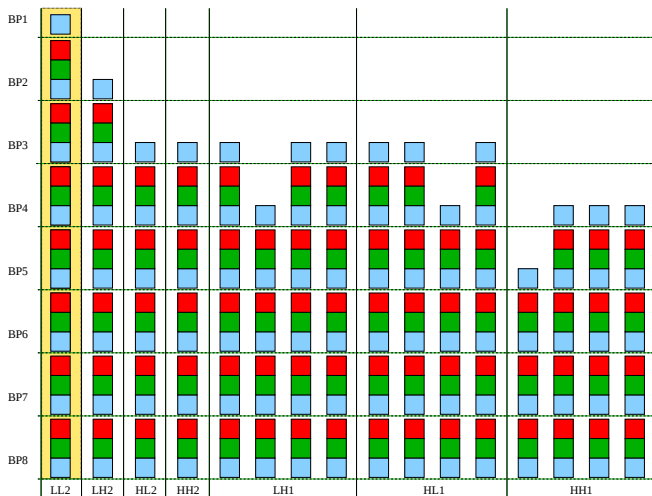
- Optimal truncation point:

$$\frac{\partial D_i}{\partial R_i} = -\lambda$$

- The value of the Lagrange multiplier can be find by an iterative algorithm.
- We can have several truncations for several target rates (quality scalability)

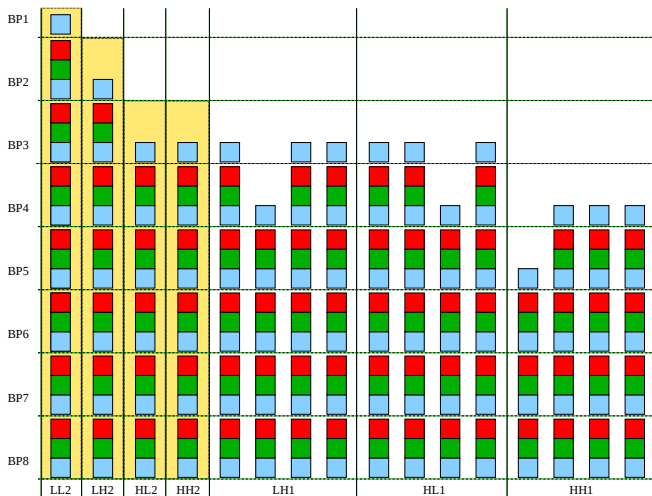
Example of bit-rate allocation with EBCOT

Allocation for maximal quality and minimal resolution



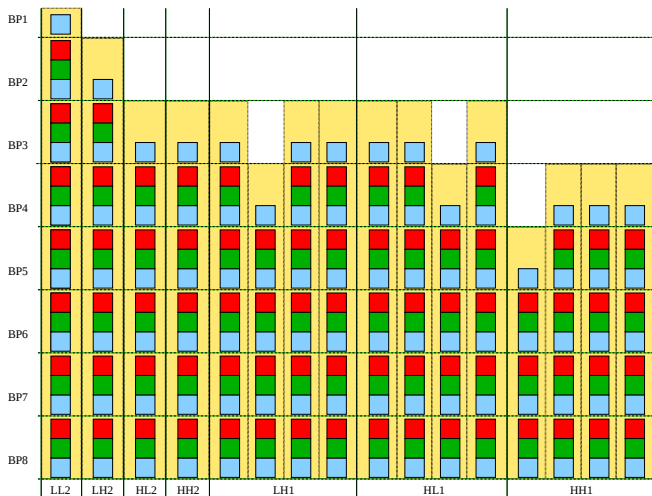
Example of bit-rate allocation with EBCOT

Allocation for maximal quality and medium resolution



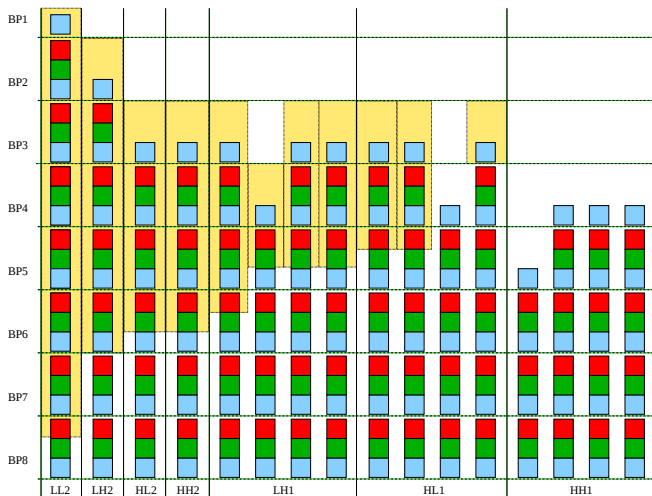
Example of bit-rate allocation with EBCOT

Allocation for maximal quality and maximal resolution



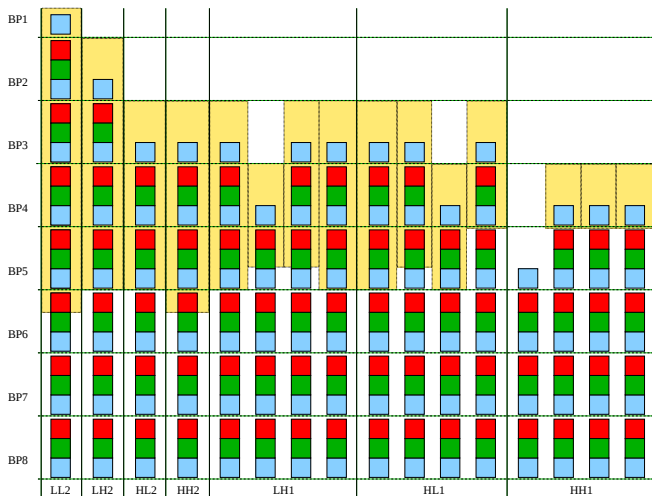
Example of bit-rate allocation with EBCOT

Allocation for perceptual quality and maximal resolution



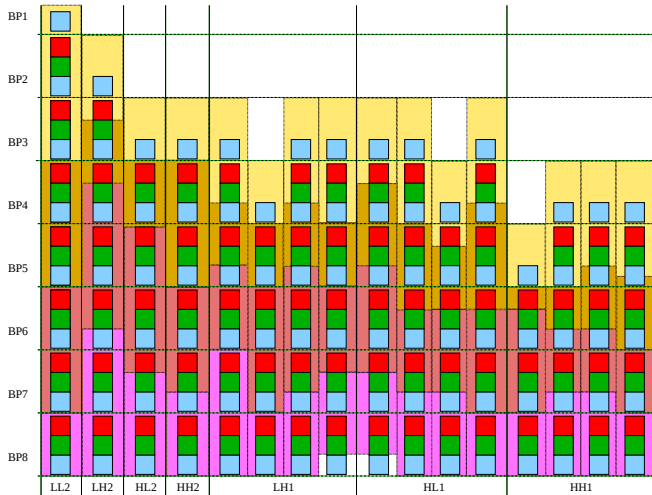
Example of bit-rate allocation with EBCOT

Allocation for a given bit-rate, maximal quality and resolution



Example of bit-rate allocation with EBCOT

Allocation pour several layers and maximal resolution



JPEG

Comparison JPEG / JPEG2000

Image Originale, 24 bpp



Comparison JPEG / JPEG2000

Rate: 1bpp



Comparison JPEG / JPEG2000

Rate: 0.75bpp



Comparison JPEG / JPEG2000

Rate: 0.5bpp



Comparison JPEG / JPEG2000

Rate: 0.3bpp



Comparison JPEG / JPEG2000

Rate: 0.2bpp



Comparison JPEG / JPEG2000

Rate: 0.2bpp pour JPEG, 0.1 pour JPEG2000



Error effect: JPEG



JPEG, $p_E = 10^{-4}$



JPEG, $p_E = 10^{-4}$

Error effect: JPEG and JPEG 2000



JPEG, $p_E = 10^{-4}$



JPEG 2000, $p_E = 10^{-4}$

Error effect: JPEG and JPEG 2000



JPEG, $p_E = 10^{-3}$



JPEG 2000, $p_E = 10^{-3}$

Image coding and robustness

- Markers insertion
- Markers period
- Marker emulation prevention
- Trade-off between robustness and rate

Error robustness in JPEG2000

- Data prioritization is possible
- No dependency among codeblocks
 - No error propagation
- No block-based transform
 - No *blocking* artifacts