Mathematical Morphology

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Introduction

- Origin: study of porous media
- Principle: study of objects (images) based on:
  - shape, geometry, topology
  - grey levels, colors
  - neighborhood information
- Mathematical bases:
  - set theory
  - topology
  - geometry
  - algebra (lattice theory)
  - probabilities, random closed sets
  - functions
- Main characteristics:
  - non linear
  - non invertible
  - strong properties
  - associated algorithms

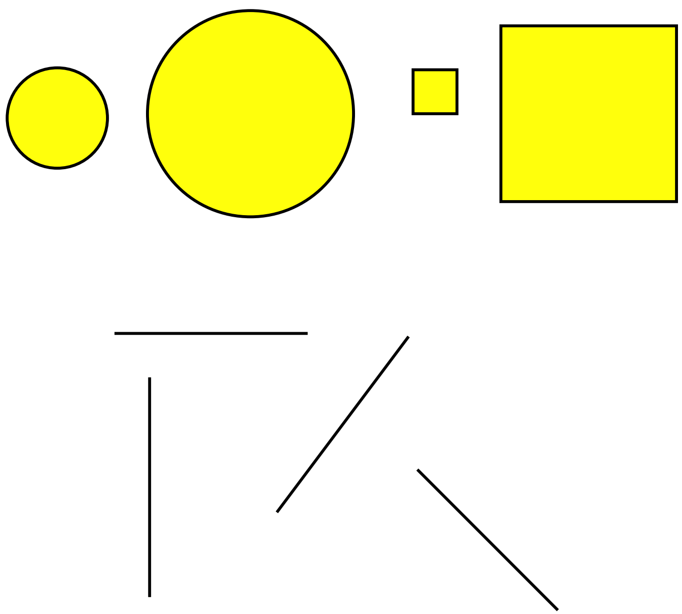
Shape or spatial relationships ?

Simplifying and selecting relevant information...

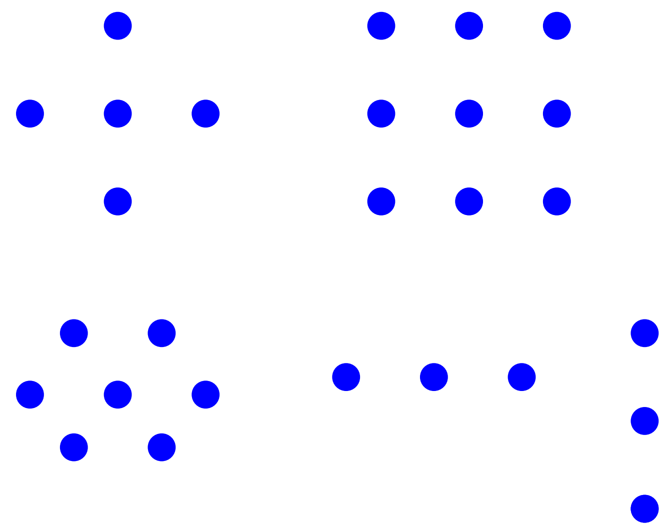
Structuring element

- shape
- size
- origin (not necessarily in B)
- examples:

Continuous:



Digital:



Binary dilation

- Minkowski addition:

$$X \oplus Y = \{x + y \; / \; x \in X, y \in Y\}$$

- Binary dilation:

$$\begin{aligned} D(X,B) &= X \oplus \check{B} = \{x + y \; / \; x \in X, y \in \check{B}\} \; (or \;\; = X \oplus B) \\ &= \bigcup_{x \in X} \check{B}_x = \{x \in \mathbb{R}^n \; / \; B_x \cap X \neq \emptyset\} \end{aligned}$$

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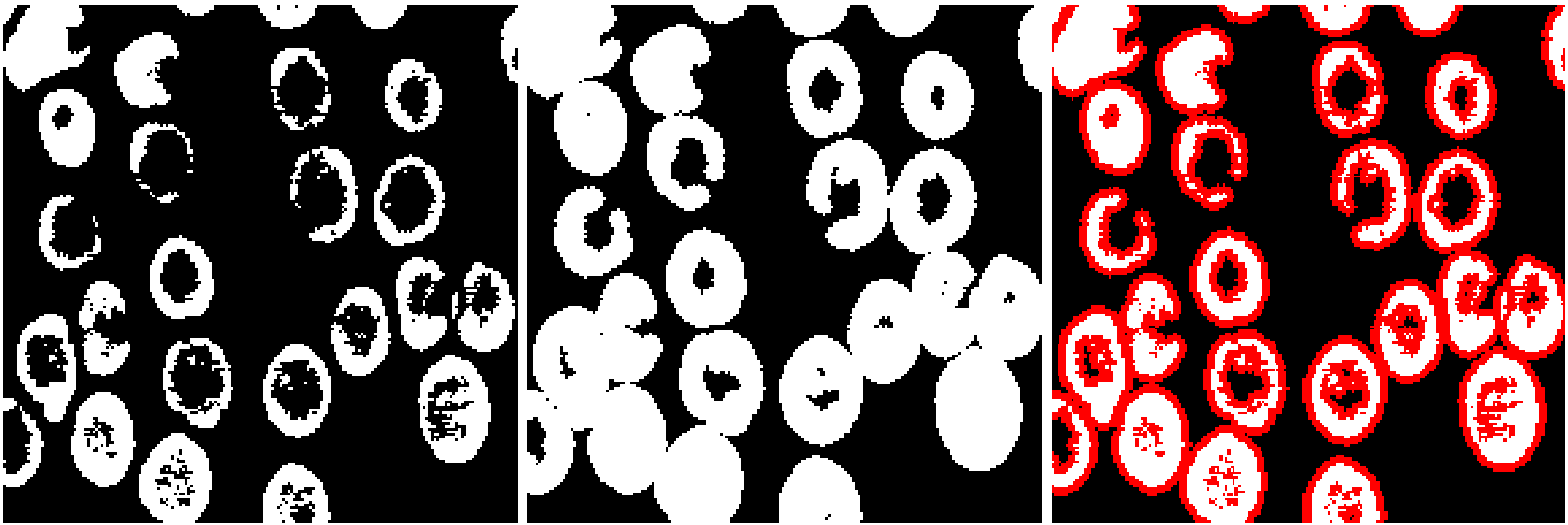
Properties of dilation:

- extensive ($X \subseteq D(X,B)$) if $O \in B$;
- increasing ($X \subseteq Y \Rightarrow D(X,B) \subseteq D(Y,B)$);
- $B \subseteq B' \Rightarrow D(X,B) \subseteq D(X,B')$;
- commutes with union, not with intersection:

$$D(X \cup Y, B) = D(X,B) \cup D(Y,B), \quad D(X \cap Y, B) \subseteq D(X,B) \cap D(Y,B);$$

- iterativity property: $D[D(X,B),B'] = D(X, B \oplus B')$.

Example of dilation



Binary erosion

$$
\begin{aligned}
E(X,B) &= \{x \in \mathbb{R}^n \ / \ B_x \subseteq X\} \\
&= \{x \ / \ \forall y \in B, x + y \in X\} = X \ominus \check{B}.
\end{aligned}
$$

Binary erosion

$$\begin{aligned} E(X,B) &= \{x \in \mathbb{R}^n \;/\; B_x \subseteq X\} \\ &= \{x \;/\; \forall y \in B, x+y \in X\} = X \ominus \check{B}. \end{aligned}$$

Properties of erosion:

- duality of erosion and dilation with respect to complementation:

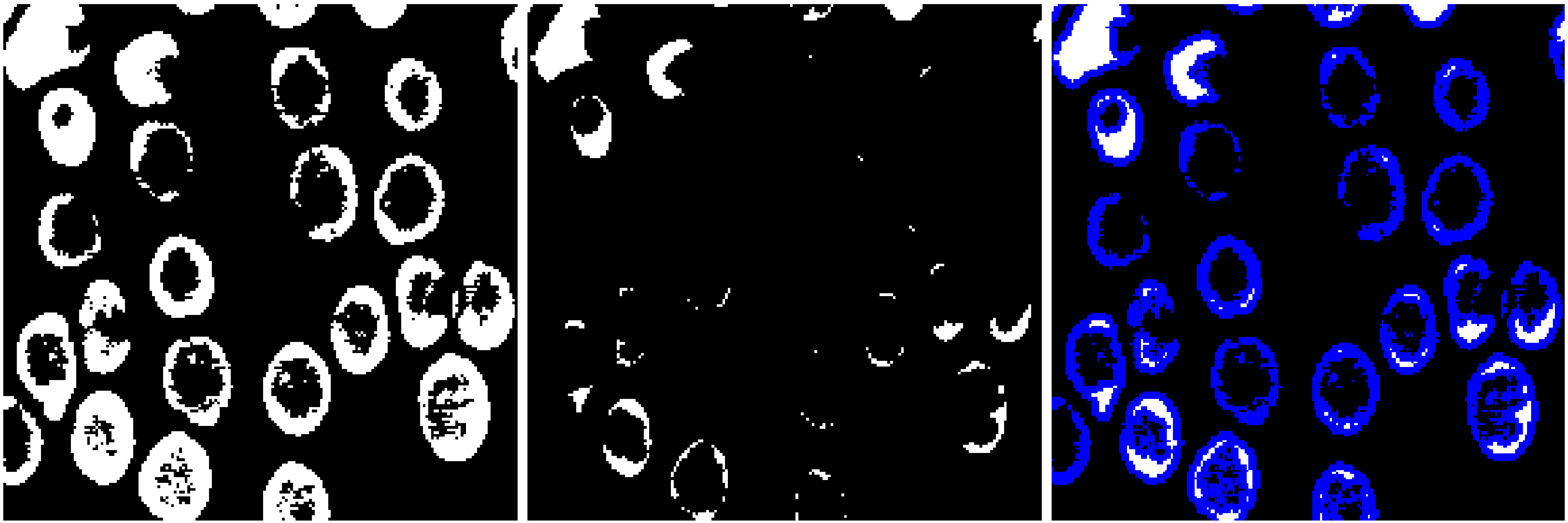
$$E(X,B) = [D(X^C, B)]^C$$

- anti-extensive ($E(X,B) \subseteq X$) if $O \in B$;
- increasing ($X \subseteq Y \Rightarrow E(X,B) \subseteq E(Y,B)$);
- $B \subseteq B' \Rightarrow E(X,B') \subseteq E(X,B)$;
- commutes with intersection, not with union:

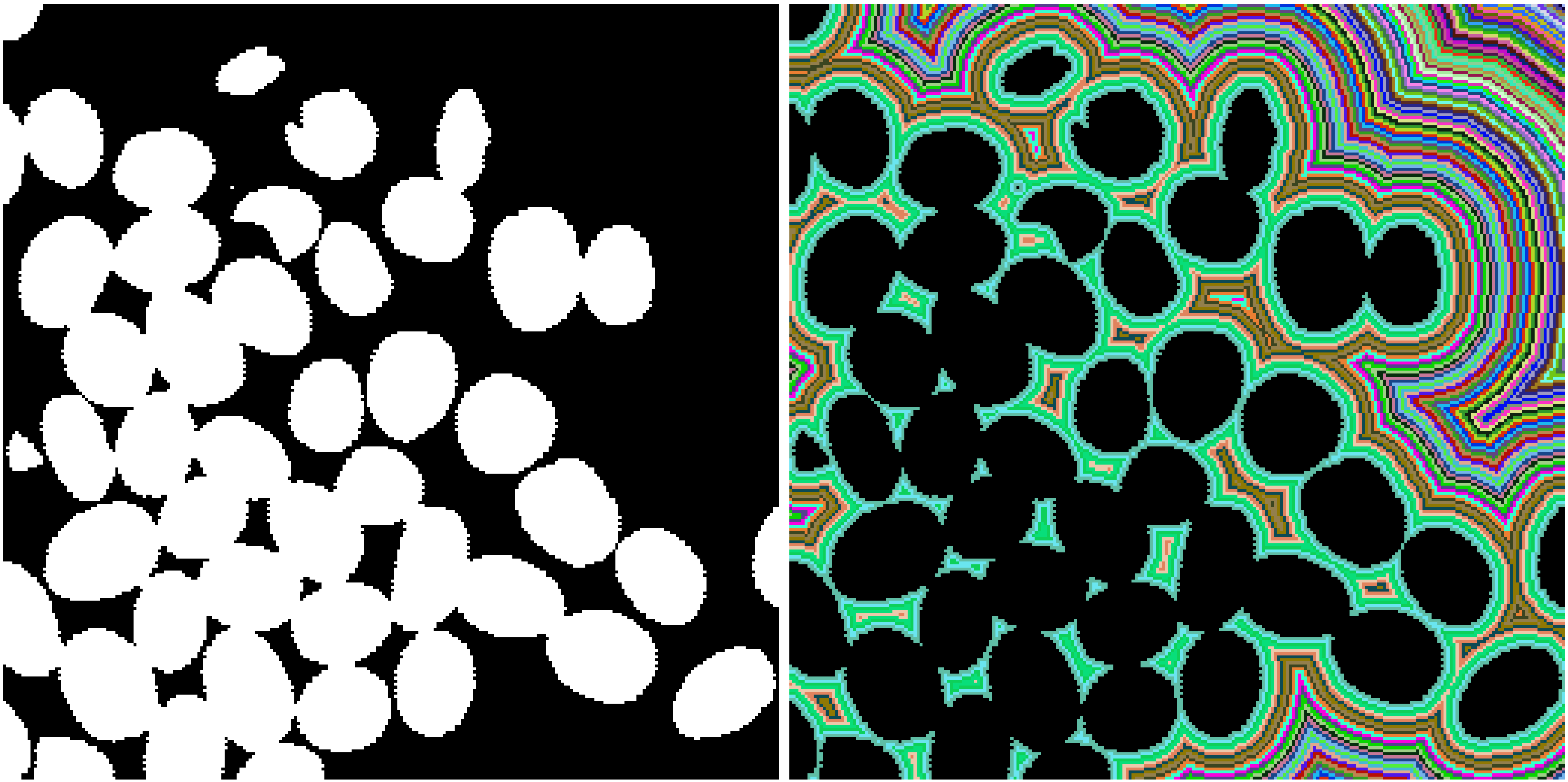
$$E[(X \cap Y), B] = E(X,B) \cap E(Y,B), \quad E[(X \cup Y), B] \supseteq E(X,B) \cup E(Y,B);$$

- iterativity property: $E[E(X,B),B'] = E(X, B \oplus B')$.
- $D[E(X,B),B'] \subseteq E[D(X,B'),B]$.

Example of erosion



Links with distances



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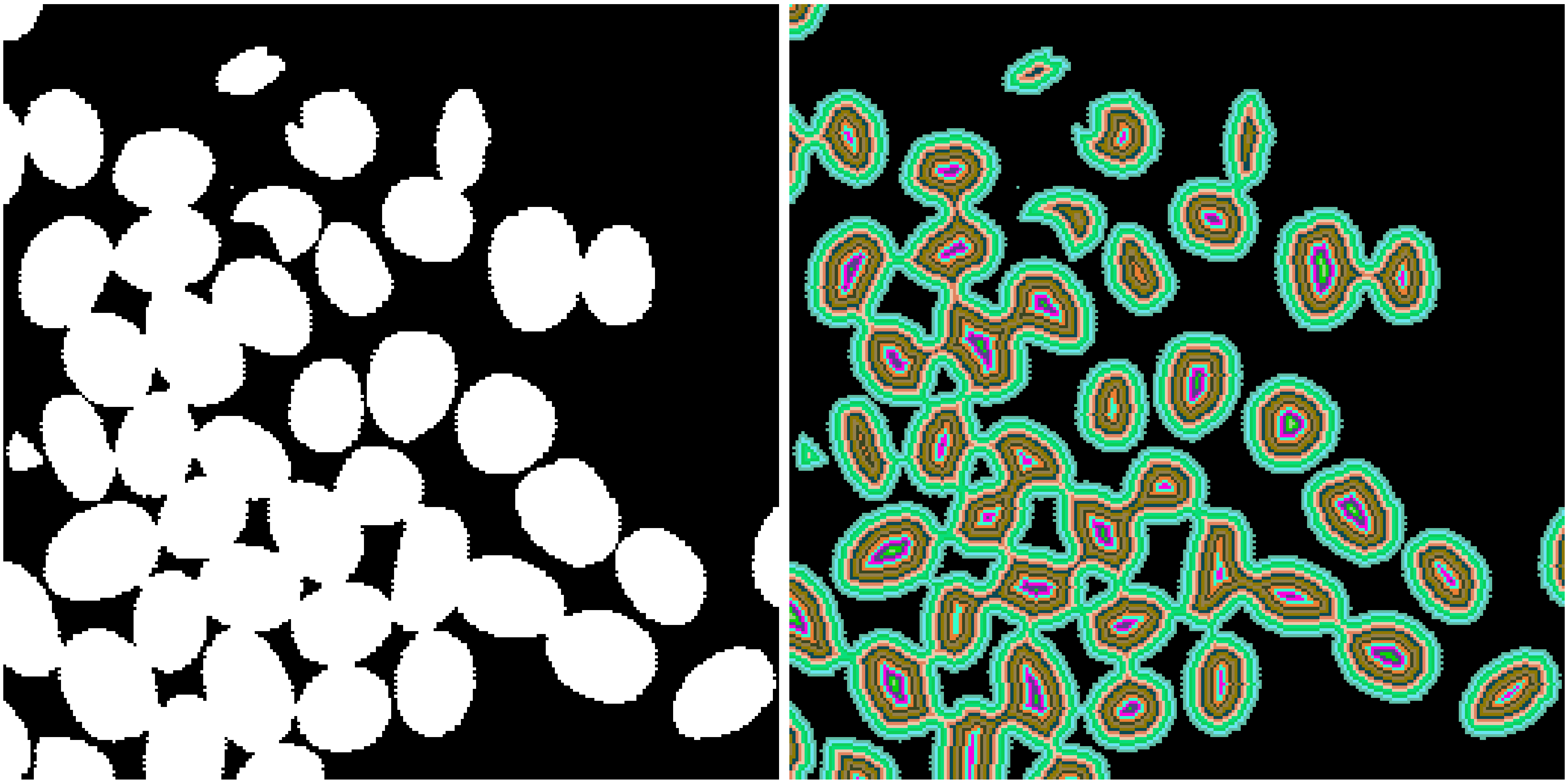

Binary opening

$$\boxed{X_B = D[E(X, B), \check{B}]}$$

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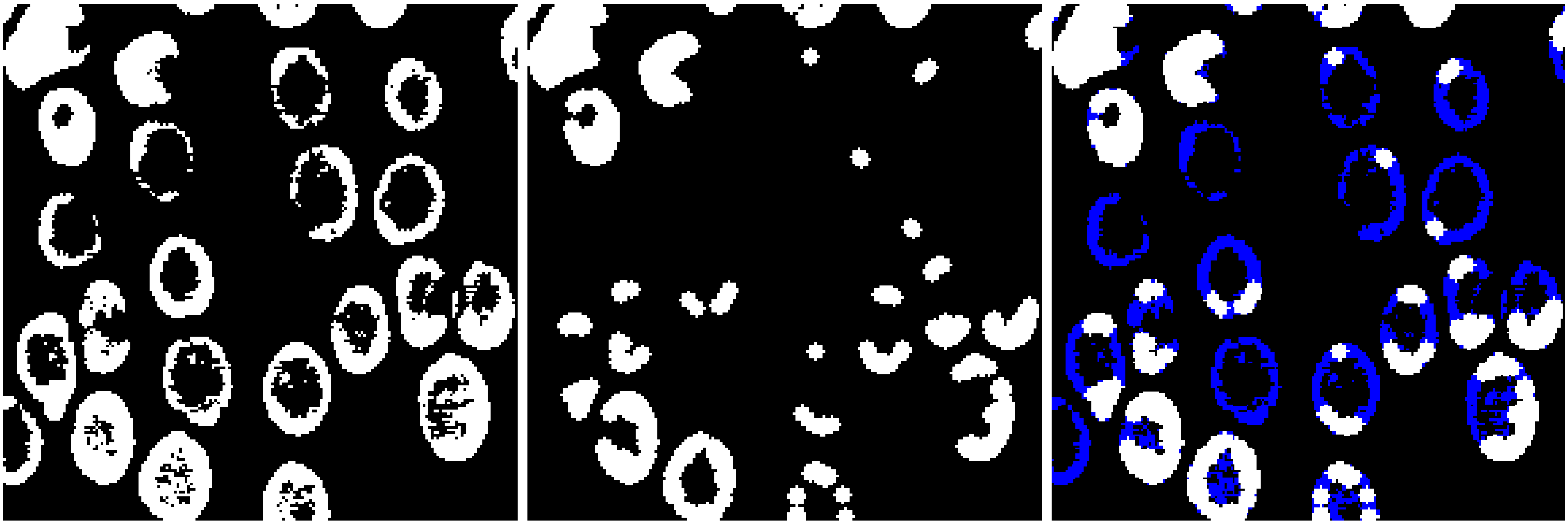
Properties of opening:

- anti-extensive ($X \supseteq X_B$);
- increasing ($X \subseteq Y \Rightarrow X_B \subseteq Y_B$);
- idempotent ($(X_B)_B = X_B$).

$\Rightarrow$ Morphological filter

- $B \subseteq B' \Rightarrow X_{B'} \subseteq X_B$;
- $(X_n)_{n'} = (X_{n'})_n = X_{\max(n,n')}$.

Example of opening



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- idempotent ($(X^B)^B = X^B$).

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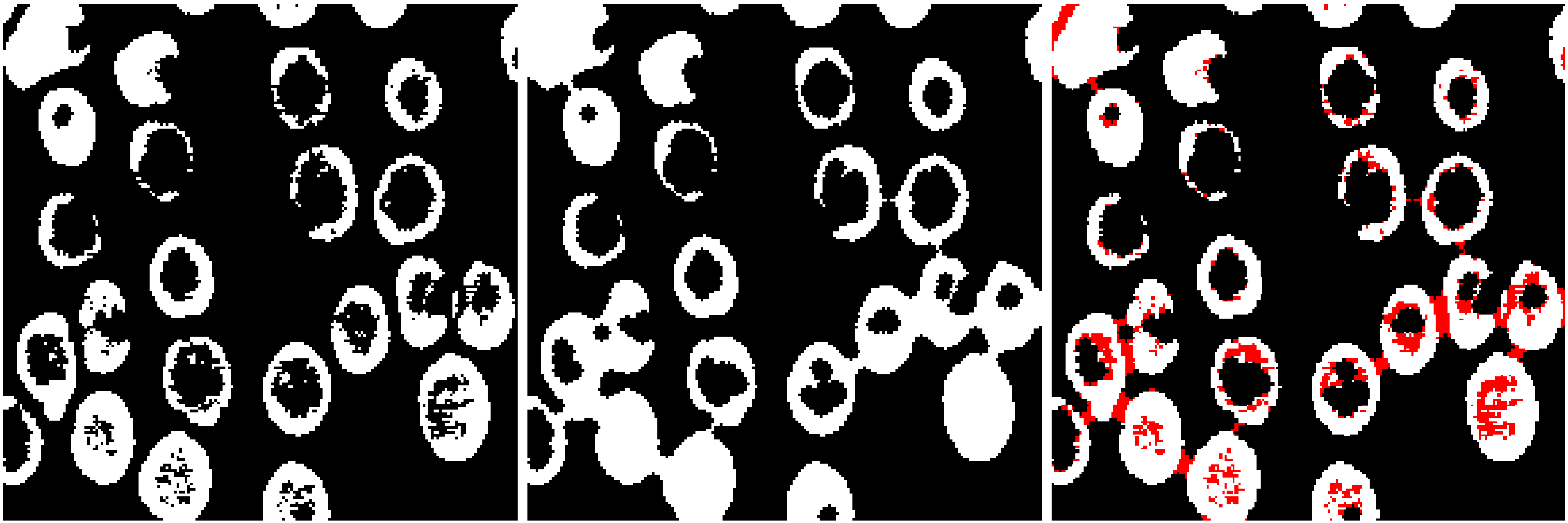
- $B \subseteq B' \Rightarrow X^B \subseteq X^{B'}$;
- $(X^n)^{n'} = (X^{n'})^n = X^{\max(n,n')}$;
- $X^B = [(X^C)_B]^C$.

Example of closing



From sets to functions

- subgraph of a function on $\mathbb{R}^n$ = subset of $\mathbb{R}^{n+1}$
- cuts of a function = sets
- functional equivalents of set operations:

$$
\begin{array}{ccc}
\cup & \longrightarrow & \sup / \vee \\
\cap & \longrightarrow & \inf / \wedge \\
\subseteq & \longrightarrow & \leq \\
\supseteq & \longrightarrow & \geq
\end{array}
$$

Dilation of a function by a flat structuring element

$$\boxed{\forall x \in \mathbb{R}^n, \quad D(f,B)(x) = \sup\{f(y) \;/\; y \in B_x\}}$$

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Properties of functional dilation:

- extensivity if $O \in B$;
- increasingness;
- $D(f \vee g, B) = D(f,B) \vee D(g,B)$;
- $D(f \wedge g, B) \leq D(f,B) \wedge D(g,B)$.

Example of functional dilation

Erosion of a function

$$\boxed{\forall x \in \mathbb{R}^n, \quad E(f,B)(x) = \inf\{f(y) \ / \ y \in B_x\}}$$

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Properties of functional erosion:

- functional dilation and erosion are dual operators;
- anti-extensivity if $O \in B$;
- increasingness;
- $E(f \vee g, B) \geq E(f,B) \vee E(g,B)$;
- $E(f \wedge g, B) = E(f,B) \wedge E(g,B)$.

Example of functional erosion

Functional opening

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Properties of functional closing:

- extensive;
- increasing;
- idempotent.

$\Rightarrow$ morphological filter

- duality between opening and closing

Example of functional closing

Some applications of erosion and dilation

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Contrast enhancement

Some applications of erosion and dilation

Morphological gradient: $D_B(x) - E_B(x)$

Alternate Sequential Filters

$$\boxed{(\ldots(((f_{B_1})^{B_1})_{B_2})^{B_2})\ldots_{B_n})^{B_n}}$$

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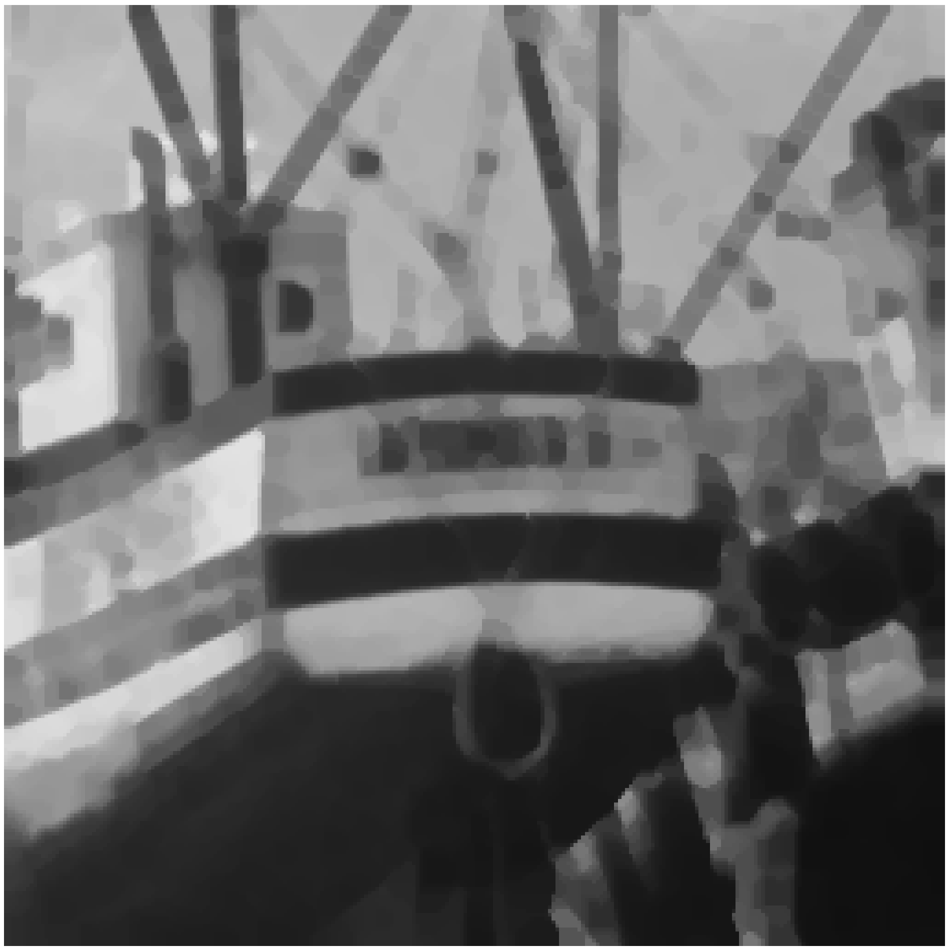

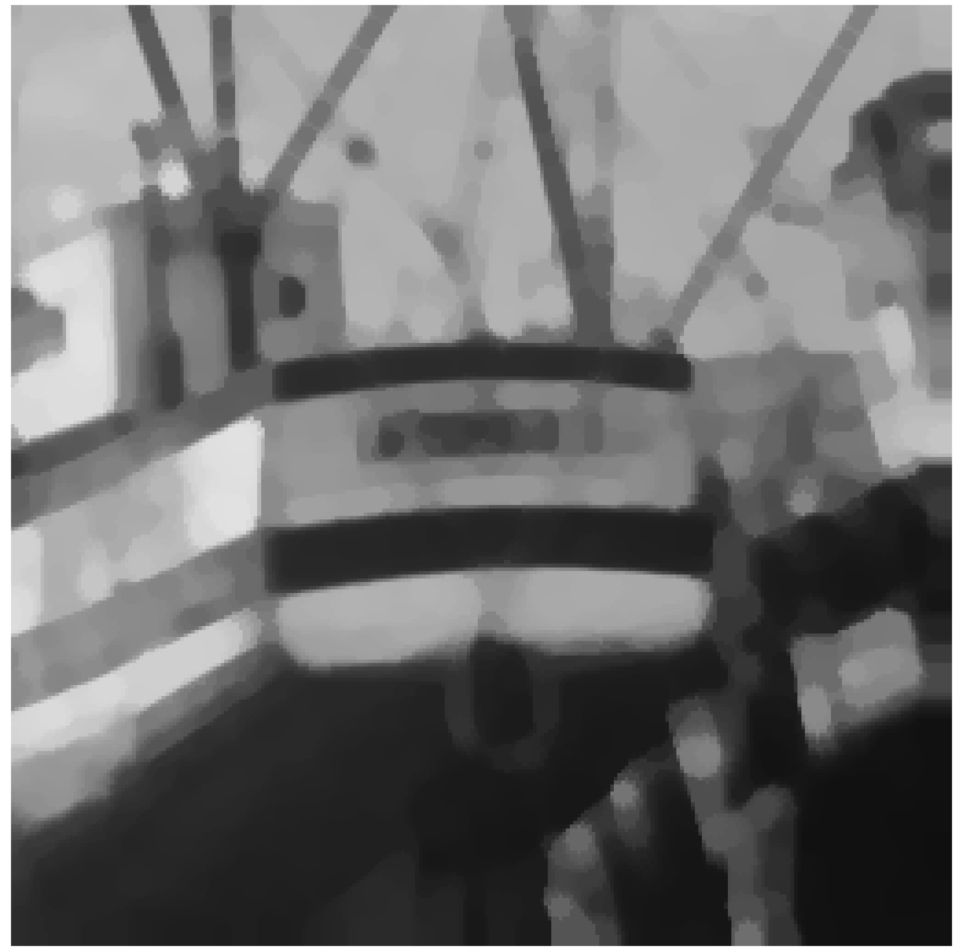

Tophat tranform

$$f - f_B$$

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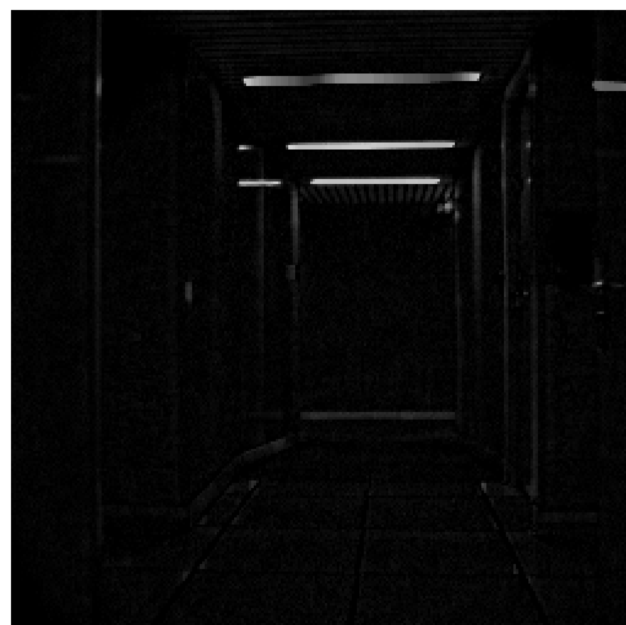

Granulometry

- $\forall X \in \mathcal{A}, \forall \lambda > 0, \;\; \phi_\lambda(X) \subseteq X$ (ϕ_λ anti-extensive);
- $\forall (X, Y) \in \mathcal{A}^2, \forall \lambda > 0, \;\; X \subseteq Y \Rightarrow \phi_\lambda(X) \subseteq \phi_\lambda(Y)$ (ϕ_λ increasing);
- $\forall X \in \mathcal{A}, \forall \lambda > 0, \forall \mu > 0 \;\; \lambda \geq \mu \Rightarrow \phi_\lambda(X) \subseteq \phi_\mu(X)$ (ϕ_λ decreasing with respect to the parameter);
- $\forall \lambda > 0, \forall \mu > 0, \;\; \phi_\lambda \circ \phi_\mu = \phi_\mu \circ \phi_\lambda = \phi_{\max(\lambda,\mu)}$.

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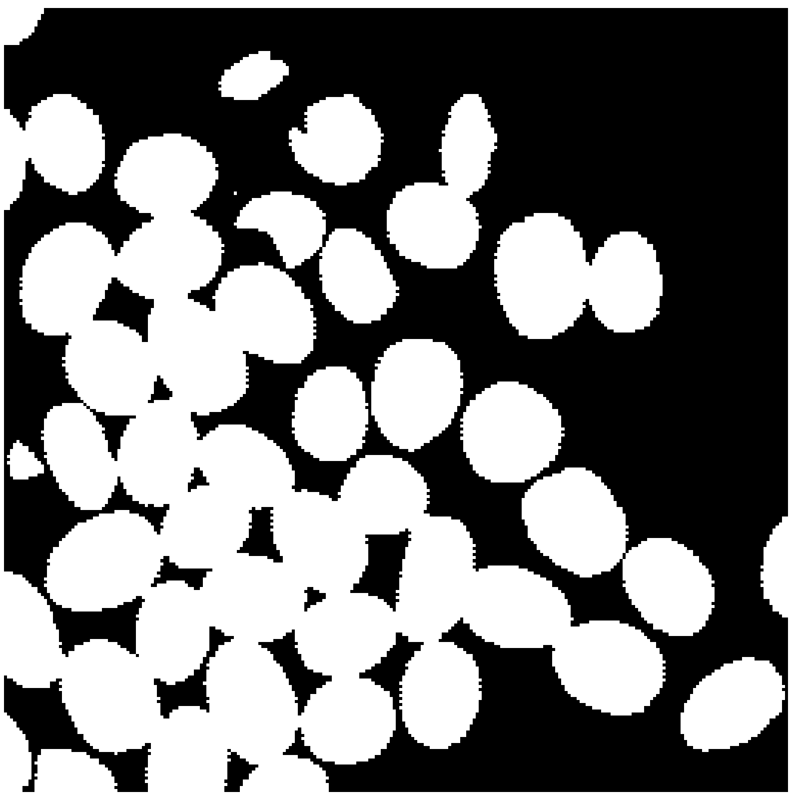

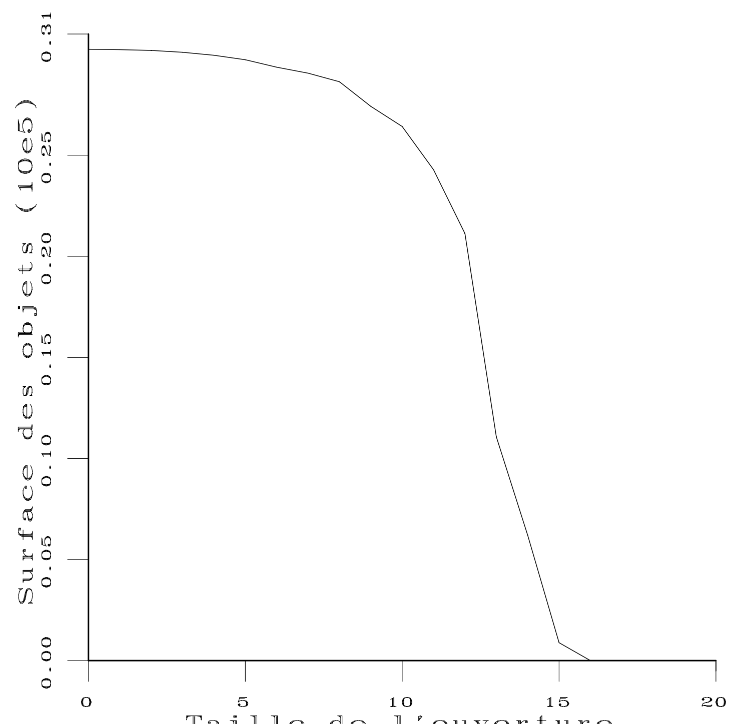

Hit-or-Miss Transformation

Structuring element: $T = (T_1, T_2)$, with $T_1 \cap T_2 = \emptyset$

HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

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HMT:

$$X \otimes T = E(X, T_1) \cap E(X^C, T_2)$$

Thinning (if $O \in T_1$):

$$X \circ T = X \setminus X \otimes T$$

Thickening (if $O \in T_2$):

$$X \odot T = X \cup X \otimes T$$

For $T' = (T_2, T_1)$:

$$X \circ T = (X^C \odot T')^C$$

Skeleton: requirements

- compact representation of objects
- thin lines
- centered in the object
- homotopic to the object
- good representation of the geometry
- invertible

Skeleton: continuous case

A: open set
$s_\rho(A)$ = set of centers of maximal balls of A of radius ρ

Skeleton:

$$r(A) = \bigcup_{\rho > 0} s_\rho(A)$$

Characterization:

$$s_\rho(A) = \bigcap_{\mu > 0} [E(A, B_\rho) \setminus [E(A, B_\rho)]_{\bar{B}_\mu}]$$

Reconstruction:

$$A = \bigcup_{\rho > 0} D(s_\rho, B_\rho)$$

Skeleton: digital case

- Direct transposition of the continuous case:

$$S(X) = \bigcup_{n \in \mathbb{N}} [E(X, B_n) \setminus E(X, B_n)_B]$$

Properties:

- centers of digital maximal balls
- reconstuction
- but poor connectivity properties

Skeleton: digital case

- Direct transposition of the continuous case:

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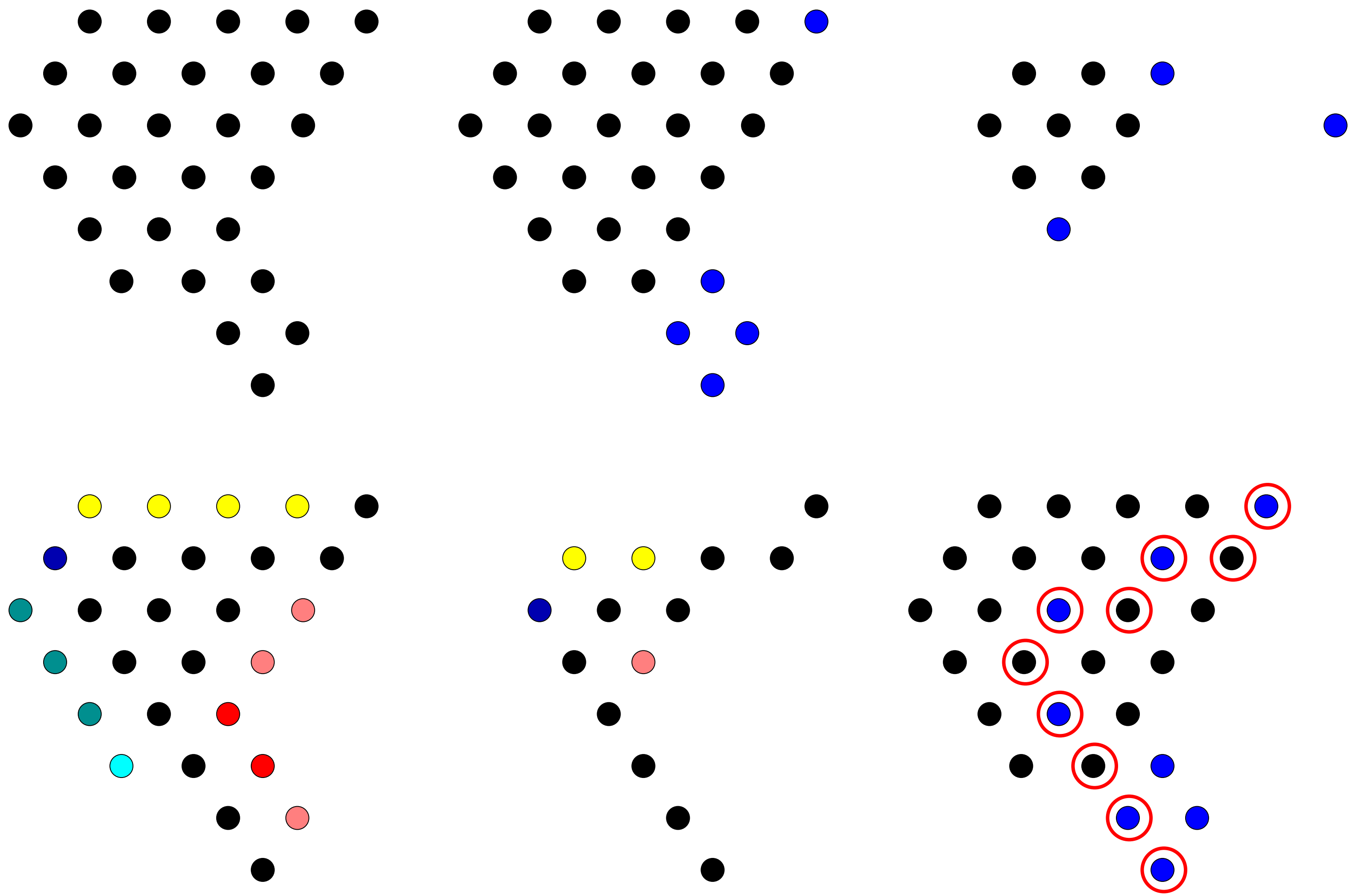
  Properties:
  - centers of digital maximal balls
  - reconstuction
  - but poor connectivity properties

- Skeleton from homotopic thinning

$$\begin{array}{ccccc} & 1 & & 1 & \\ . & & 1 & & . \\ & 0 & & 0 & \end{array}$$

  Properties:
  - perfect topology
  - no reconstruction

Centers of maximal ball vs thinning



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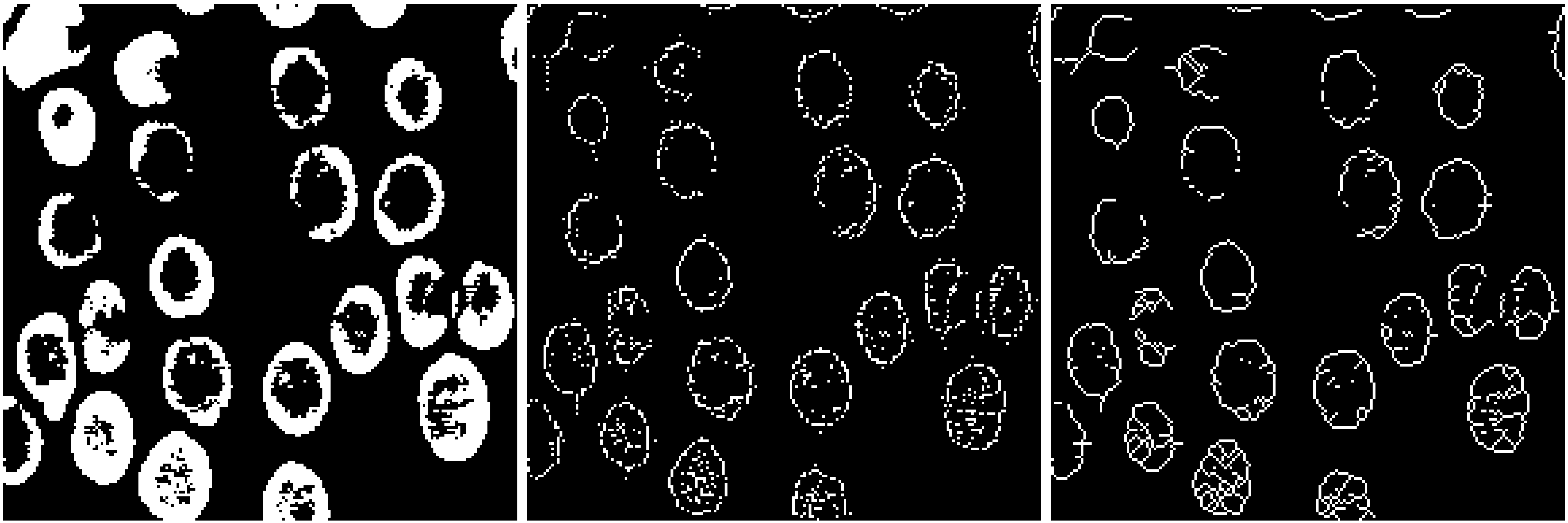


Geodesic operators and reconstruction

Geodesic ball:

$$B_X(x,r) = \{y \in X \ / \ d_X(x,y) \leq r\}$$

Geodesic dilation:

$$D_X(Y, B_r) = \{x \in \mathbb{R}^n \ / \ B_X(x,r) \cap Y \neq \emptyset\} = \{x \in \mathbb{R}^n \ / \ d_X(x,Y) \leq r\}$$

Digital case:

$$D_X(Y, B_r) = [D(Y, B_1) \cap X]^r$$

Geodesic erosion:

$$E_X(Y, B_r) = \{x \in \mathbb{R}^n \ / \ B_X(x,r) \subseteq Y\} = X \setminus D_X(X \setminus Y, B_r)$$

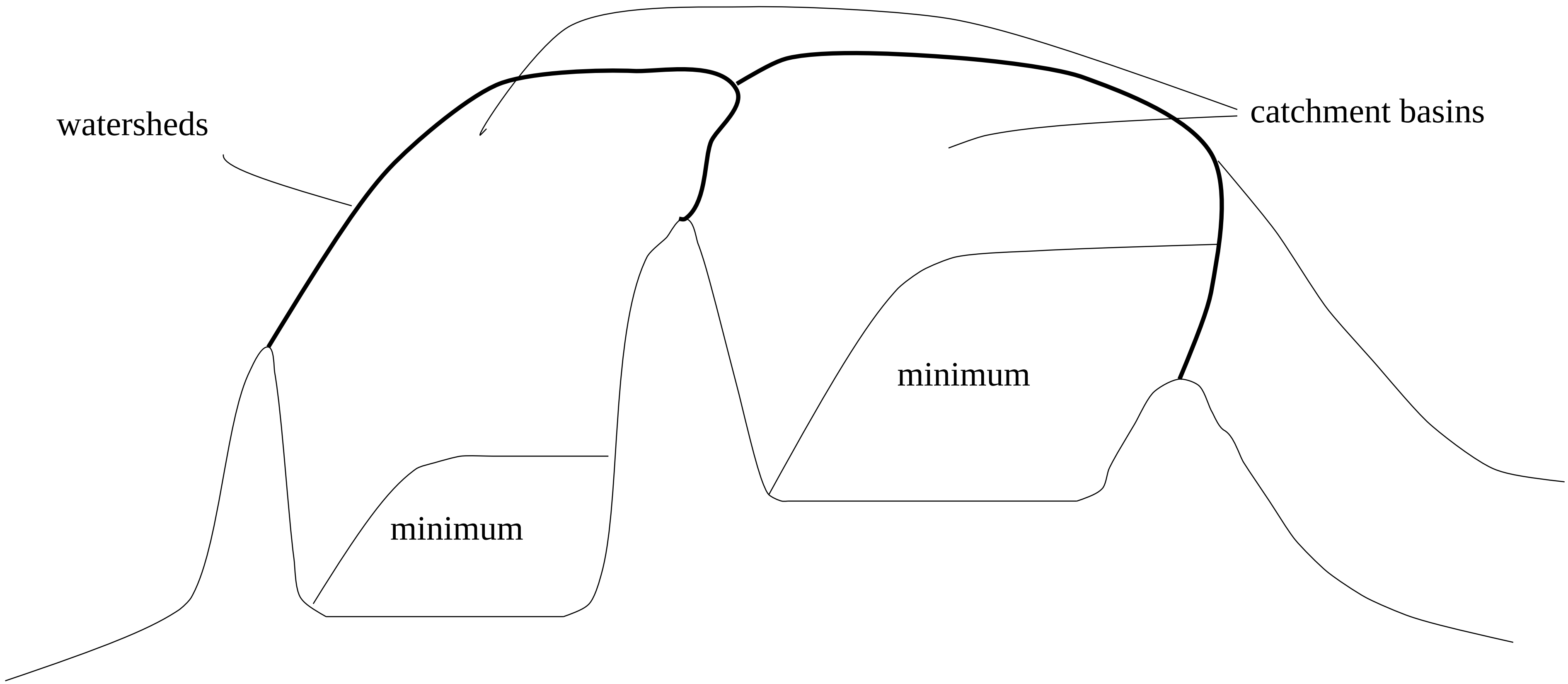
Reconstruction:

$$[D(Y, B_1) \cap X]^\infty$$

= connected components of X which intersect Y.

Extends to functions

Watersheds: a poweful segmentation tool



Watersheds and oversegmentation

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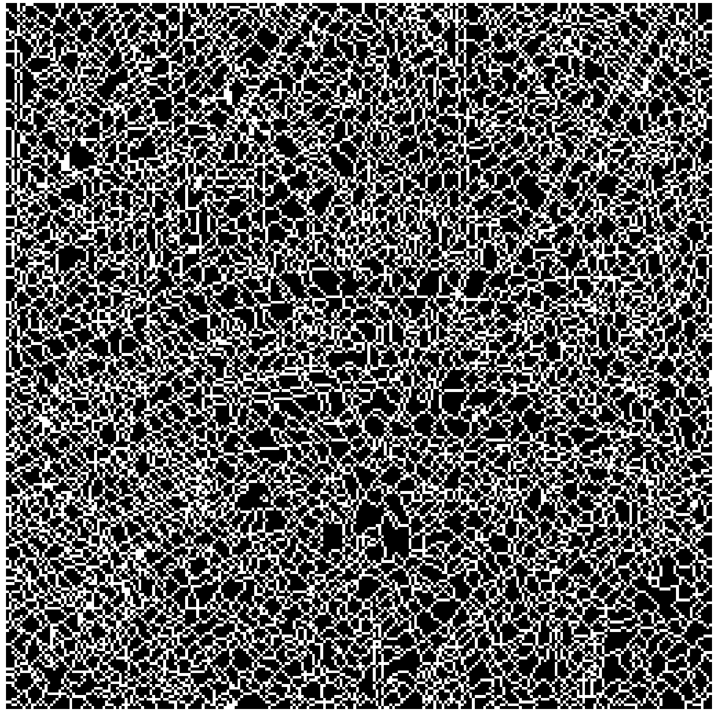
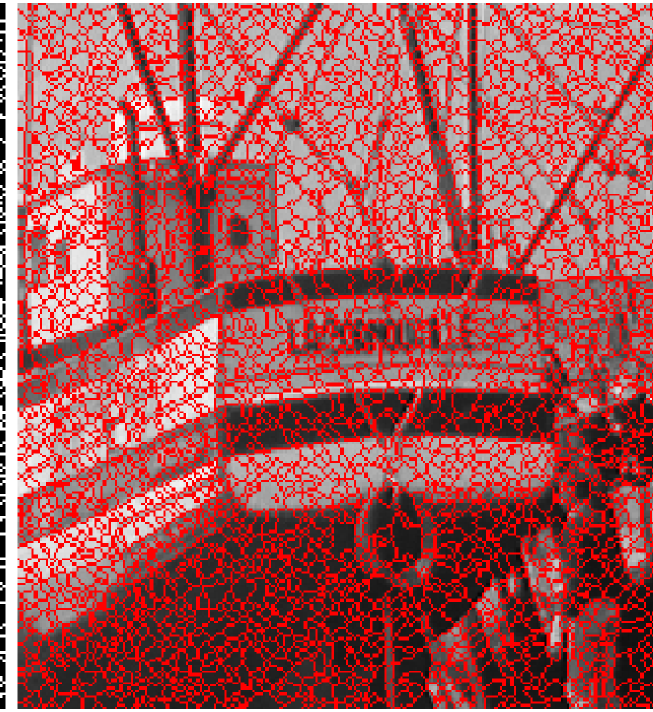
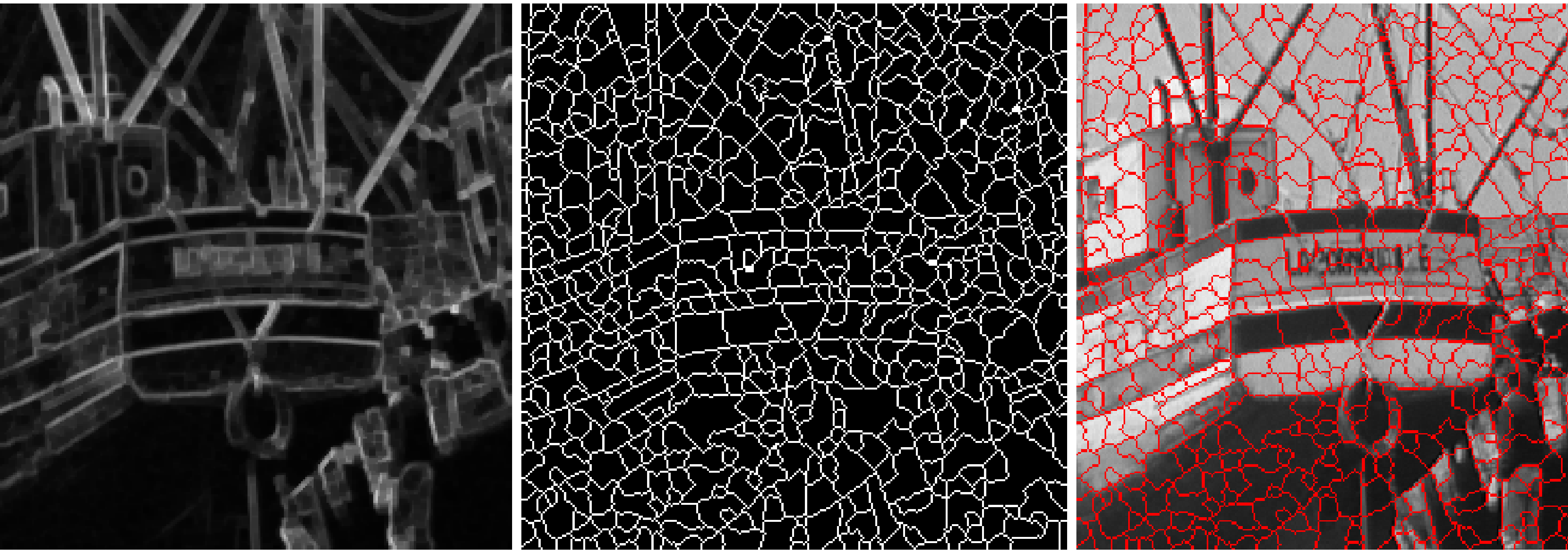
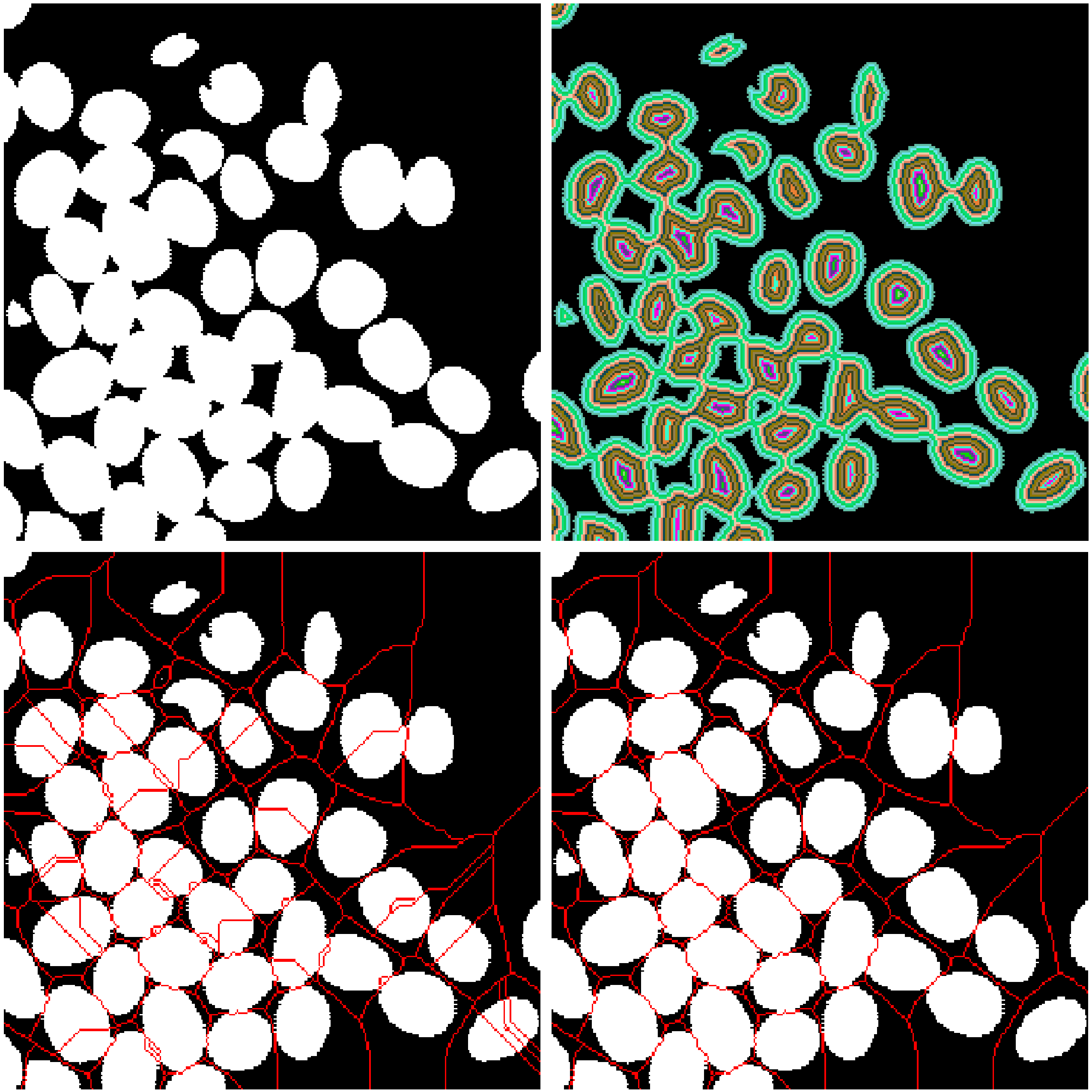
  

Watersheds and oversegmentation

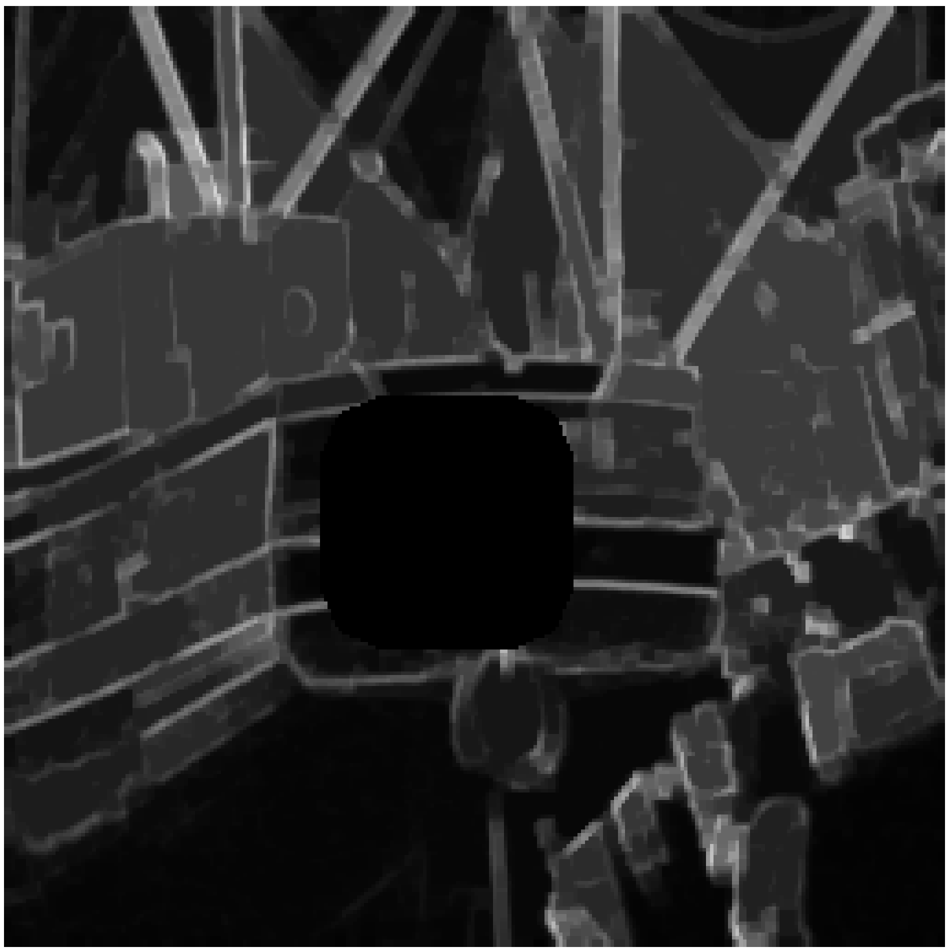


Separation of connected binary objects



Watersheds constraint by markers

f: function on which watersheds should be applied
g: marker function (selects regional minima)
Reconstruction: $E_{f \wedge g}(g, B_\infty)$ (only the selected minima)



And much more...

A few references:

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