Sequential Estimation of a Gaussian Random Walk First-Passage Time from Noisy or Delayed Observations

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Outline

The Tracking Stopping Time problem (Niesen and T. 2009)

- $X = \{X_t\}_{t\geq 0}$ and stopping time τ on X
- Statistician has access to X through $Y = \{Y_t\}_{t\geq 0}$,
- and wishes to find a stopping time η close to τ, e.g., so that

$$\mathbb{E}|\eta - \tau|^p$$

is minimized.

Applies in monitoring, forecasting, communication, finance,...

Generalizes the Bayesian change-point problem

- θ positive integers valued random variable
- $\{Y_t\}$ with $Y_t \sim P_0$ if $t < \theta$ and $Y_t \sim P_1$ if $t \geq \theta$
- Goal: find η close to θ.

Equivalent to TST problem:

- $\{X_t\}$ with $X_t = 0$ if $t < \theta$ and $X_t = 1$ if $t \geq \theta$.
- $\tau = \inf\{t \geq 1 : X_t = 1\}$ (i.e., $\tau = \theta$)
- Goal: find η close to τ.

C-P and TST: any difference?

for $k > t$

C-P: $\mathbb{P}(\theta = k | Y^t = y^t, \theta > t) = \mathbb{P}(\theta = k | \theta > t)$

TST: $\mathbb{P}(\tau = k | Y^t = y^t, \tau > t) \neq \mathbb{P}(\tau = k | \tau > t)$ in general

i.e., for the TST problem the first t samples may be useful for predicting τ, even if $\{\tau > t\}$.

Unfortunately

the C-P point problem formulation is known to be hard (mostly asymptotic results).

Bottom line:

- The TST problem is a generalization of a hard problem.
- Any hope?

Tracking a first-passage time through noisy observations

Given

$$X: \quad X_0 = 0 \qquad X_t = \sum_{i=1}^{t} V_i + st \quad t \geq 1$$

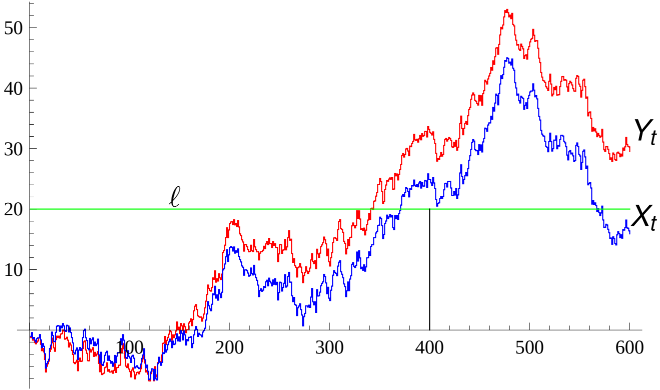
$$Y: \quad Y_0 = 0 \qquad Y_t = X_t + \epsilon \sum_{i=1}^{t} W_i \quad t \geq 1$$

$s \geq 0$, $\epsilon \geq 0$, and $\{V_i\}$ and $\{W_i\}$ i.i.d. $\mathcal{N}(0,1)$

$$\tau = \inf\{t \geq 0 : X_t \geq \ell\} \quad \ell > 0$$

Find

$$\inf_{\eta} \mathbb{E}|\eta - \tau|^p \quad p \geq 1$$


50
40
30
20
10
100
200
300
400
500
600
ℓ
Y_t
X_t

Theorem: $s = 0$

For $\ell > 0$ and $p \geq 1/2$

$$\mathbb{E}|\eta - \tau|^p = \infty$$

for any $\eta = \eta(Y_0^\infty)$.

Theorem: $s > 0$

For $p \geq 1$

$$\inf_{\eta} \mathbb{E}|\eta - \tau|^p \simeq \frac{\mathbb{E}|N|^p}{s^p} \left(\frac{\ell \epsilon^2}{s(1+\epsilon^2)} \right)^{p/2} \simeq \left(\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau|^p \right)$$

as $\ell \to \infty$, with $N \sim \mathcal{N}(0, 1)$.

$\Rightarrow$ Causality doesn't come at the expense of delay (asympt.).

Optimality and simplicity

$$\eta^* = \bar{t} + \frac{(\ell - e(Y_{\bar{t}}))_+}{s}$$

where $\bar{t} = \ell/s - (\ell/s)^q$, $q \in (1/2, 1)$, and where

$$e(Y_t) = a \cdot Y_t + b \cdot t$$

is the MMSE estimate of X_t given $\{Y_t\}$.

η^* depends on the single observation $Y_{\bar{t}}$

Tracking a first-passage time through delayed observations

Given

$$X: \quad X_0 = 0 \qquad X_t = \sum_{i=1}^{t} V_i + st \quad t \geq 1$$

$$Y: \quad Y_0^d = 0 \qquad Y_t = X_{t-d} \quad t \geq d$$

$$\tau = \inf\{t \geq 0 : X_t \geq \ell\} \quad \ell > 0$$

Find

$$\inf_{\eta} \mathbb{E}|\eta - \tau|^p \quad p \geq 1$$

Theorem: $s = 0$

For $p \geq 1/2$

$$\inf_{\eta} \mathbb{E}|\eta - \tau|^p = d^p$$

with $N \sim \mathcal{N}(0, 1)$.

Optimal tracker: $\eta = \inf\{t \geq 0 : Y_t \geq \ell\}$

Theorem: $s > 0$

For $p \geq 1$

$$\inf_{\eta} \mathbb{E}|\eta - \tau|^p \simeq \left(\frac{d}{s^2}\right)^{p/2} \mathbb{E}|N|^p \quad (d \to \infty, \ell \geq s \cdot d)$$

with $N \sim \mathcal{N}(0,1)$.

Optimal tracker: $\eta = \inf\{t \geq 0 : Y_t \geq \ell - s \cdot d\}$

Lower bound: intuition ($s > 0$)

$$X: \quad X_0 = 0 \qquad X_t = \sum_{i=1}^{t} V_i + st \qquad t \geq 1$$

$$Y: \quad Y_0 = 0 \qquad Y_t = X_t + \epsilon \sum_{i=1}^{t} W_i \qquad t \geq 1$$

$$\text{Find} \qquad \inf_{\eta} \mathbb{E}|\eta - \tau|^p \quad (\star)$$

Main idea

reduce ($\star$) to

$$\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - X_t|^p, \quad t \approx \ell/s$$

1. On $\{\tau \geq t\}$, $\tau \simeq t + (\ell - X_t)/s$, hence

$$\inf_{\eta} \mathbb{E}|\eta - \tau| \geq \inf_{\eta(Y_0^\infty)} \mathbb{E}[|\eta - \tau|; \tau \geq t]$$
$$\simeq \inf_{\eta(Y_0^\infty)} \mathbb{E}[|\eta - t - (\ell - X_t)/s|; \tau \geq t]$$
$$= \frac{1}{s} \inf_{\eta(Y_0^\infty)} \mathbb{E}[|\eta + X_t|; \tau \geq t]$$

2. Gaussianity: $X_t \overset{\text{d}}{=} e(Y_t) + c \cdot t^{1/2} N$

$$e(Y_t) \triangleq a \cdot Y_t + b \cdot t, \quad N \sim \mathcal{N}(0,1) \perp\!\!\!\perp \{Y_t\}$$

3.

$$\inf_{\eta(Y_0^\infty)} \mathbb{E}[|\eta + X_t|; \tau \geq t] \gtrsim c \cdot t^{1/2} \mathbb{E}|N| P(\tau \geq t) \qquad \eta = -e(Y_t)$$

4. Optimize over $t \Rightarrow t \lesssim \ell/s$

Upper bound: intuition

Since

$$\tau \simeq t + \frac{\ell - X_t}{s} \qquad t < \ell/s$$

a natural candidate

$$\eta = t + \frac{\ell - e(Y_t)}{s}\,.$$

Then,

$$\begin{aligned}\eta - \tau &\simeq \frac{X_t - e(Y_t)}{s} \\ &= \frac{c \cdot t^{1/2}}{s} N\,.\end{aligned}$$

$\eta - \tau$ has optimal distribution with $t \lessapprox \ell/s$.

Summary

- The TST problem formulation naturally appears in
  - detection
  - prediction
  - communication
  - quality control
  - information econometrics
  - ...
- Contribution: correlated gaussian random walks, first-passage time, moment loss function
- Open problems:
  - Light-tailed loss functions
  - General stopping times which “concentrate”

Example: monitoring

- X_t: distance of an object from a barrier at time t
- τ: first time when $X_t = 0$
- Y_t: noisy measurements of X_t
- η: alarm time based on Y should be close to τ

Example: forecasting

- X_t: fatigue up to day t of a big manufacturing machine
- τ: first day t when X_t crosses critical fatigue threshold
- Machine replacement period: 10 days
- η: first day when new machine is operational

Wanted η close to τ because

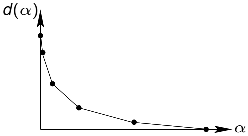
- $\{\eta > \tau\}$: interrupted manufacturing process
- $\{\eta < \tau\}$: storage costs

$\Rightarrow$ TST problem with $Y_t = X_{t-10}$ if $t > 10$ and $X_t = 0$ else.

A first step: algorithmic solution (Niesen and T, 2009)

Given finite alphabet process (X, Y) and $\tau \leq b$, the algorithm outputs

- $d(\alpha) = \min_{\tau : \mathbb{P}(\eta < \tau) \leq \alpha} \mathbb{E}(\eta - \tau)_+ \quad \alpha \in [0, 1]$



- the corresponding optimal stopping times.

Under conditions on (X, Y) and τ, the algorithm is poly(b).