Detection of a Stopping Time Through Noisy Observations

Aslan Tchamkerten

Telecom ParisTech

Joint work with U. Niesen (MIT)

IWSM 2009
Outline

1. Problem formulation and examples
2. Main results
3. Conclusion
The Tracking Stopping Time problem

Process \{ (X_i, Y_i) \}_{i \geq 1} whose law is known to Alice and Bob.

Alice
- chooses a stopping time \( S \) with respect to \{ \( X_1, X_2, \ldots \) \}

Bob
- observes only \( Y_1, Y_2, \ldots \)
- knows the rule of \( S \)

Bob’s goal
Find \( T \) so that \( \mathbb{E}(T - S)^+ \) is minimized while \( \mathbb{P}(T < S) \leq \alpha \).
Example: monitoring

- $X_n$: distance at time $n$ of an object from a barrier
- $S$: first time $n$ when $X_n = 0$
- $Y_n$: noisy measurements of $X_n$

What time $T$ should an alarm be raised?

- $\mathbb{E}(T - S)^+ = \text{“reaction time”}$
- $\mathbb{P}(T < S) = \text{false-alarm probability}$
Example: forecasting

- $X_n$: fatigue up to day $n$ of a big manufacturing machine
- $S$: first day $n$ for which $X_n$ crosses critical fatigue threshold
- Machine replacement period: 10 days
- $T$: first day new machine is operational

Wanted $T$ close to $S$ because
- $T > S$: interrupted manufacturing process
- $T < S$: storage costs

$\Rightarrow$ TST problem with $Y_n = X_{n-10}$ if $n > 10$ and $X_n = 0$ else.
Example: change-point problem

- $\theta$ positive integers valued random variable
- $\{Y_i\}_{i \geq 1}$ i.i.d. with $Y_i \sim P_0$ if $i < \theta$ and $Y_i \sim P_1$ if $i \geq \theta$
- Goal: find $T$ such that $\mathbb{E}(T - \theta)^+$ minimized while $\mathbb{P}(T < \theta) \leq \alpha$

Equivalent to:

- $\{X_i\}_{i \geq 1}$ with $X_i = 0$ if $i < \theta$ and $X_i = 1$ if $i \geq \theta$.
- $S = \inf\{i \geq 1 : X_i = 1\}$ (i.e., $S = \theta$)
- Goal: find $T$ such that $\mathbb{E}(T - S)^+$ minimized while $\mathbb{P}(T < S) \leq \alpha$
Change-point or tracking a stopping time?

C-P and TST are not equivalent:

for \( k > n \)

C-P: \( \mathbb{P}(\theta = k | Y^n = y^n, \theta > n) = \mathbb{P}(\theta = k | \theta > n) \)

TST: \( \mathbb{P}(S = k | Y^n = y^n, S > n) \neq \mathbb{P}(S = k | S > n) \) in general

i.e., for the C-P problem,

if \( \{\theta > n\} \), the first \( n \) samples \( Y^n \) are useless for predicting \( \theta \).
Goal

Given the law of \(\{(X_i, Y_i)\}_{i \geq 1}\) and a s.t. \(S\) on \(\{X_i\}_{i \geq 1}\) find

- the curve \(d(\alpha) = \min_{T: \mathbb{P}(T < S) \leq \alpha} \mathbb{E}(T - S)^+\)

- the corresponding optimal stopping times.
Here
- discrete time, finite alphabet processes
- \( S \leq \kappa \)

**Stopping time \( \equiv \) full tree**

Ex: \( X_i \in \{0, 1\} \), \( S = \begin{cases} 1 & \text{if } X_1 = 1 \\ 3 & \text{else} \end{cases} \)
What we can easily deduce

Epigraph of \( d(\alpha) = \min_{T: \mathbb{P}(T < S) \leq \alpha} \mathbb{E}(T - S)^+ \)

- Decreasing
- Convex
- Finitely many extreme points: deterministic stopping times

\[ \Rightarrow d(\alpha) \text{ is piecewise linear and convex.} \]
Main result

Algorithm for constructing \( \{ T_m \}_{m=0}^M \) using key property:

\[
T_0 \geq T_1 \geq T_2 \geq \ldots \geq T_M
\]
Construction of \( \{ T_m \}_{m=0}^{M} \)

Tree pruning algorithm: \( T_m \rightarrow T_{m+1} \)

- cost associated to each intermediate nodes of \( T_m \)
- find nodes in \( T_m \) with maximum cost
- get \( T_{m+1} \) by cutting the branches above those nodes
Construction of $\{ T_m \}_{m=0}^M$

Tree pruning algorithm: $T_m \rightarrow T_{m+1}$

- cost associated to each intermediate nodes of $T_m$
- find nodes in $T_m$ with maximum cost
- get $T_{m+1}$ by cutting the branches above those nodes
Construction of $\{T_m\}_{m=0}^{M}$

Tree pruning algorithm: $T_m \rightarrow T_{m+1}$

- cost associated to each intermediate nodes of $T_m$
- find nodes in $T_m$ with maximum cost
- get $T_{m+1}$ by cutting the branches above those nodes
Construction of $\{T_m\}^M_{m=0}$

Tree pruning algorithm: $T_m \rightarrow T_{m+1}$

- cost associated to each intermediate nodes of $T_m$
- find nodes in $T_m$ with maximum cost
- get $T_{m+1}$ by cutting the branches above those nodes
Construction of $\{T_m\}_{m=0}^M$

Tree pruning algorithm: $T_m \rightarrow T_{m+1}$

- cost associated to each intermediate nodes of $T_m$
- find nodes in $T_m$ with maximum cost
- get $T_{m+1}$ by cutting the branches above those nodes
Construction of \( \{ T_m \}_{m=0}^M \)

Tree pruning algorithm: \( T_m \rightarrow T_{m+1} \)

- cost associated to each intermediate nodes of \( T_m \)
- find nodes in \( T_m \) with maximum cost
- get \( T_{m+1} \) by cutting the branches above those nodes
Construction of \( \{ T_m \}_{m=0}^M \)

Tree pruning algorithm: \( T_m \rightarrow T_{m+1} \)

- cost associated to each intermediate nodes of \( T_m \)
- find nodes in \( T_m \) with maximum cost
- get \( T_{m+1} \) by cutting the branches above those nodes
Cost function

Given a tree $T_m$ define for all of its nodes $y$

$$a(y) = \mathbb{E} (|y| - S^+) \bigg| Y = y \bigg) \ P(Y = y)$$
$$b(y) = P (|y| < S \bigg| Y = y \bigg) \ P(Y = y) .$$

For each intermediate node $y$ in $T_m$

$$r(y) \triangleq \frac{a(T_y) - a(y)}{b(y) - b(T_y)} .$$

To get $T_{m+1}$ cut branches of $T_m$ above maximal cost intermediate nodes.
- Algorithm complexity: $\exp(O(\kappa))$
- Exhaustive search: $\exp(\exp(\Omega(\kappa)))$
Easy TST instances
Easy TST instances

$S$ is permutation invariant if

$$\mathbb{P}(S \leq n|X^n = x^n) = \mathbb{P}(S \leq n|X^n = \pi(x^n))$$

for any $\pi$ and $n \geq 1$.

Theorem 1
If

- the $(X_i, Y_i)$'s are i.i.d.
- $S$ is permutation invariant

then $T_1$ is obtained in $\text{poly}(\kappa)$. 

![Diagram](https://via.placeholder.com/150)
Easy TST instances

$S$ is permutation invariant if

$$\mathbb{P}(S \leq n | X^n = x^n) = \mathbb{P}(S \leq n | X^n = \pi(x^n))$$

for any $\pi$ and $n \geq 1$.

**Theorem 2**

If

- the $(X_i, Y_i)$’s are i.i.d.
- $S$ and $\{T_m\}_{m=0}^M$ permutation invariant

then the algorithm is poly($\kappa$).
Easy TST instances (cont.)

Example 1:
Easy TST instances (cont.)

Example 1:

- $\{(X_i, Y_i)\}_{i \geq 1}$ i.i.d. with $X_i \in \{0, 1\}$
- $S \triangleq \inf\{i \geq 1 : X_i = 1 \text{ or } i = \kappa\}$.

Note that (for $n < \kappa$)

- $\mathbb{P}(S = n) = p(1 - p)^{n-1}$, where $p \triangleq \mathbb{P}(X = 1)$

$$
\mathbb{P}(Y_i = y_i | S = n) = \begin{cases} 
\mathbb{P}(Y_i = y_i | X_i = 0) & \text{if } i < n, \\
\mathbb{P}(Y_i = y_i | X_i = 1) & \text{if } i = n, \\
\mathbb{P}(Y_i = y_i) & \text{if } i > n.
\end{cases}
$$
Easy TST instances (cont.)

Example 1:

- \{ (X_i, Y_i) \}_{i \geq 1} \text{ i.i.d. with } X_i \in \{0, 1\}
- S \triangleq \inf \{ i \geq 1 : X_i = 1 \text{ or } i = \kappa \}.

Note that (for \( n < \kappa \))

- \( \mathbb{P}(S = n) = p(1 - p)^{n-1} \), where \( p \triangleq \mathbb{P}(X = 1) \)

\[
\mathbb{P}(Y_i = y_i | S = n) = \begin{cases} 
\mathbb{P}(Y_i = y_i | X_i = 0) & \text{if } i < n, \\
\mathbb{P}(Y_i = y_i | X_i = 1) & \text{if } i = n, \\
\mathbb{P}(Y_i = y_i) & \text{if } i > n.
\end{cases}
\]

\( \Rightarrow \) change-point problem.
Example 2: a pure TST problem

- $\{(X_i, Y_i)\}_{i \geq 1}$ be i.i.d. with $X_i \in \{0, 1\}$
- $S \triangleq \inf\{i \geq 1 : \sum_{j=1}^{i} X_j = 2\}$. 
Summary

A change-point problem generalization appearing in:
- detection
- prediction
- communication (e.g., feedback communication)
- quality control
- information econometrics

Contribution:
- Algorithmic solution (discrete time, finite alphabet)
- Criterion for low complexity solution
A lot remains to be done:

- Algorithmically: explicit conditions for polynomial algorithm
- Analytically: exact results may be difficult to get. Given \((X, Y)\) and \(S\) good lower bound on \(\mathbb{E}(T - S)^2\) as a function of the correlation between \(X\) and \(Y\) and ‘variability’ of \(S\)?