# Contents

# Chapter 2

# Regular variation and tail measures

## 2.1 Regular variation of a real function

**Definition 2.1.1** (Regular variation). *A function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is* regularly varying *(RV) if $\exists \rho \in \mathbb{R}$ such that*

$$\forall x > 0, \lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho.$$

*The parameter $\rho$ is called the* regular variation index. *We write '$U \in RV(\rho)$', meaning $U$ is RV with regular variation index $\rho$. If $\rho = 0$, $U$ is called* slowly varying.

**Example 2.1** (Fréchet survival function)**:**
$U(x) = 1 - \Phi_\alpha(x) = 1 - e^{-x^{-\alpha}}$ is $RV(-\alpha)$.

**Example 2.2** (Generalized Pareto)**:**
$U(x) = (1 + \gamma x)^{-1/\gamma}$ is $RV(-1/\gamma)$.

**Example 2.3** (Canonical: Pareto tail)**:**
$U(x) = x^{-\alpha}$, $x > 1$, is $RV(-\alpha)$.

**Example 2.4** (slow variation)**:**
$U(x) = \log(1 + x)$ is slowly varying. If $\lim_{t\to\infty} f(t) = \ell \in \mathbb{R}$, then $f$ is slowly varying, the converse is false.

**Remark 2.1.2.** *Remind from last chapter that the max-domain of attraction condition* (MDA) *is equivalent to condition* (1.19) *concerning the tail regularity, which is*

$$\frac{1 - F(t + \sigma(t)x)}{1 - F(t)} \xrightarrow[t \nearrow x_\star]{} (1 + \gamma x)_+^{-1/\gamma}.$$

*This 'resembles' a RV condition. It will be shown that it* is *equivalent to regular variation of $U = 1 - F$ in the case $\gamma > 0$.*

**Remark 2.1.3** (Equivalent formulation of RV). *$U$ is $RV(\rho)$ $\iff$ $\exists L$ a slowly varying function such that $U(x) = x^\rho L(x)$.*
*(Proof: exercise)*

**Proposition 2.1.4** (A sufficient condition for RV)
*If $\exists h : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$, measurable such that $\forall x > 0, \lim_{t\to\infty} \frac{U(tx)}{U(t)} = h(x)$ then $U$ is RV.*

*Proof.* (sketch of): Show that such $h$ satisfies the Hamel equation $h(xy) = h(x)h(y)$. ■

**Proposition 2.1.5** (Another sufficient condition)
*If $U$ is non decreasing and if $\exists(a_n)_{n\geq 0} \in \mathbb{R}$ s.t. $a_n \to +\infty$, and a function $h : \mathbb{R}^{+*} \to \mathbb{R}^{+*}$ such that $\forall x > 0, \lim_n nU(a_n x) = h(x)$ then $U$ is RV.*

*Proof.* Put $n(t) = \inf\{n \geq 0 : a_n \geq t\}$. Then

$$\frac{U(a_{n(t)-1}x)}{U(a_{n(t)})} \leq \frac{U(tx)}{U(t)} \leq \frac{U(a_{n(t)}x)}{U(a_{n(t)-1})},$$

and both sides of the sandwich converge to $h(x)$. Using Proposition 2.1.4 concludes. ■

**Exercise 2.1** (Reciprocal for Proposition 2.1.5):
Let $F$ be a cdf and asssume that $(1-F) \in RV(-\alpha)$, for some $\alpha < 0$. Define $U(t) = 1/(1-F)(t)$ and let $a_n = U^{\leftarrow}(n)$. Show that

$$n(1 - F(a_n x)) \to x^{-\alpha}, \text{ for } x > 0, \text{ as } t \to \infty. \tag{2.1}$$

*hint:* consider the ratio $\frac{1-F(a_n x)}{1-F(a_n)}$, and derive the limit of $U(U^{\leftarrow}(n))/n$.

**Exercise 2.2** (regular variation and Fréchet domain of attraction):
Let $F$ be a *c.d.f.* The goal is to show the following: '$(1-F)$ is regularly varying with index $-\alpha < 0$ if and only if

$$\exists(a_n)_{n\geq 0} > 0 : F^n(a_n \bullet\ ) \to \Phi_\alpha, \qquad \text{where } \Phi_\alpha(x) = e^{-x^{-\alpha}}, x > 0, \tag{2.2}$$

and to characterize the possible sequences $a_n$, up to tail equivalence.

1. Show that (2.2) $\Rightarrow \forall x > 0, F(x) < 1,\ F(a_n x) \to 1$, and $a_n \to \infty$.
2. Prove that (2.2)$\Rightarrow 1 - F$ is $RV(-\alpha)$.
3. Switching to the inverse function, show that (2.2)$\Rightarrow a_n \sim \left(\frac{1}{1-F}\right)^{\leftarrow}(n)$ as $n \to \infty$.
4. Check that if $(1 - F)$ $n(1 - F(a_n x)) \to x^{-\alpha}$ for some sequence $a_n$, then (2.2) holds true. Check that convergence also holds for any sequence $\tilde{a}_n \sim a_n$. Conclude.

## 2.2 Karamata theorem and consequences

*Idea:* For integration purposes (of the kind $\int_x^\infty U(t)\mathrm{d}t$ or $\int_0^x U(t)\mathrm{d}t$), If $U$ is $RV(\rho)$ then it behaves as $t \mapsto t^\rho$ would, as $x \to \infty$. More precisely,

- if $U(t) = t^\rho$ and $\rho < -1$, $\int_x^\infty U(t)\mathrm{d}t = -(\rho+1)^{-1}x^{\rho+1} = -(\rho+1)^{-1}xU(x)$.
- if $U(t) = t^\rho$ and $\rho > -1$, $\int_0^x U(t)\mathrm{d}t = (\rho+1)^{-1}x^{\rho+1} = (\rho+1)^{-1}xU(x)$.

Karamata's theorem says that the same is true as $x \to \infty$ when $U \in RV(\rho)$.

**Theorem 2.2.1** (Karamata)
*Let $U : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a $RV(\rho)$ function, s.t. $\int_0^x U < \infty \forall x > 0$.*

1. *If* $\rho \geq -1$ *then* $x \mapsto \int_0^x U$ *is* $RV(\rho+1)$ *and*

$$\lim_{x\to\infty} \frac{xU(x)}{\int_0^x U} = \rho + 1. \tag{2.3}$$

*Conversely if* (2.3) *then* $U \in RV(\rho)$.

2. *If* $\rho < -1$ *of if* $\rho = 1$ *and* $\int_1^\infty U < \infty$ *then* $x \mapsto \int_x^\infty U$ *is* $RV(\rho+1)$ *and*

$$\lim_{x\to\infty} \frac{xU(x)}{\int_x^\infty U} = -\rho - 1. \tag{2.4}$$

*Conversely if* (2.4) *then* $U \in RV(\rho)$.

*Proof.* See Resnick (1987), p. 17 or Resnick (2007), p. 25. ■

**Corollary 2.2.2** (Karamata representation)
*A function* $L : \mathbb{R}^+ \to \mathbb{R}^+$ *is slowly varying if and only if*

- $\exists c : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_\infty c(x) = c \in (0,\infty)$, *and*
- $\exists \epsilon : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_\infty \epsilon(x) = 0$,

*such that*

$$L(x) = c(x) \exp\left( \int_1^x \frac{\epsilon(t)}{t} \mathrm{d}t \right). \tag{2.5}$$

*Proof.* The proof of the sufficiency of (2.5) is an easy exercise. For the converse, let $L \in RV(0)$. From Karamata theorem, we have

$$b(x) := \frac{xL(x)}{\int_0^x L} \to 1 \quad \text{as } x \to \infty.$$

By definition of $b$ we may write

$$L(x) \quad = \quad \frac{b(x)}{x}\int_0^x L \quad = \quad b(x)\exp\{\log \frac{\int_0^x L}{x}\} \tag{2.6}$$

But also

$$\begin{aligned}
\log \frac{\int_0^x L}{x} &= \int_1^x \frac{\mathrm{d}}{\mathrm{d}t}\Big[\log\Big(\int_0^t L\Big) - \log t\Big]\mathrm{d}t + D \quad (D : \text{a constant}) \\
&= \int_1^x \left( \frac{L(t)}{\int_0^t L} - \frac{1}{t} \right)\mathrm{d}t + D \quad (D : \text{a constant}) \\
&= \int_1^x \frac{b(t)-1}{t}\mathrm{d}t + D
\end{aligned}$$

Setting $\epsilon(t) = b(t) - 1$, we have $\epsilon(t) \to 0$ and the latter display combined with (2.6) yields

$$L(x) = \underbrace{b(x)e^D}_{:=c(x)\to e^D>0} \exp\Big\{ \int_1^x \frac{\epsilon(t)}{t}\mathrm{d}t \Big\},$$

which concludes the proof.

■

**Corollary 2.2.3** (Karamata representation of RV functions)
$U \in RV(\rho) \iff U(x) = c(x) \exp \int_1^x \alpha(t)/t\mathrm{d}t$, *for some functions* $c(x) \to c > 0$ *and* $\alpha(t) \to \rho$.

*Proof.*

$$\begin{aligned} U \in RV(\rho) &\iff U(x) = L(x)x^\rho \\ &\iff U(x) = c(x)\exp\Big(\int_1^x \epsilon(t)/t\mathrm{d}t\Big)\ exp(\rho \log x) \quad \text{(Corollary 2.2.2)} \\ &\iff U(x) = c(x)\exp\Big(\int_1^x [\epsilon(t)+\rho]/t\mathrm{d}t\Big). \end{aligned}$$

■

The Karamata representation will prove useful at the end of this chapter, for proving the consistency of the Hill estimator (an estimator for the regular variation index).

## 2.3 Vague convergence of Radon measures

Most of the material of this section is borrowed from Resnick (1987), Chapter 3, which contains detailed proofs.

### 2.3.1 The space of Radon measures

In this course, the 'extreme events' will take place in a 'nice' space such as $(0, \infty)$ or $(0, \infty]$. Later on, for multivariate extremes, a very convenient space will be $\mathbf{E} = [0, \infty]^d \setminus \{\mathbf{0}\}$ where $\mathbf{0} = (0, \dots, 0)$. The reason why we include $+\infty$ (via the Alexandroff's compactification) is that it makes the intervals $[x, \infty]$, for $x > 0$ compact.

**Remark 2.3.1** (Alexandroff's space ). *The space* $[0, \infty]$ *is defined as* $[0, \infty) \cup \{+\infty\}$, *where* $+\infty$ *is an arbitrary element which is greater than any element of* $[0, \infty)$. *The order* $\leq$ *on* $[0, \infty)$ *is thus extended to* $[0, \infty]$. *The topology on* $[0, \infty]$, i.e. *the family of open sets then consists of*

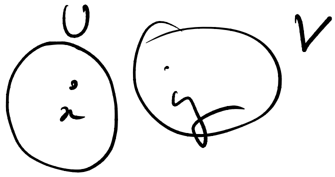
- *All sets* $V \subset [0, \infty)$ *which are open sets for the usual topology (Euclidean) on* $\mathbb{R}$.
- *All sets* $V \subset [0, \infty]$ *such that* $+\infty \in V$ *and* $V^c$ *is compact in* $[0, \infty)$.

*After having compactified* $[0, \infty)$ *at infinity, it is convenient to 'uncompactify' it by removing* 0. *We obtain the space* $\mathrm{E} = (0, \infty]$. *The idea behind is that we want* $tx \to \infty$ *as* $t \to \infty$ *for all* $x \in \mathrm{E}$.

**Exercise 2.3:**
Prove that the sets $[a, \infty]$, for $a > 0$ are compact in $\mathrm{E} = (0, \infty]$.

More generally, in the remainder of this course, we consider a space $\mathbf{E}$ which is locally compact, second countable, Hausdorff (LCSCH). Locally compact means that each point in $\mathbf{E}$ has a compact neighborhood. Second countable means that the topology on $\mathbf{E}$ has a countable base. Finally, Hausdorff means that for any pair $x \neq y \in \mathbf{E}$, there exists disjoint open sets $U, V$ such that $x \in U$ and $y \in V$. In the sequel $\mathrm{E}$ is endowed with the Borel $\sigma$-field $\mathcal{E}$.

**Definition 2.3.2** (Radon measures).

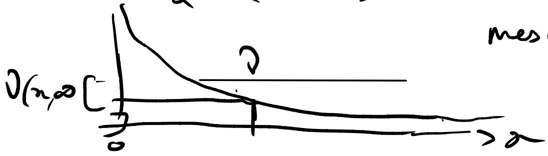
hausdorff


$U \cap V = \emptyset$

rem : sur $[0,\infty]\setminus\{0\}$

(ex) typiquement : $\nu_\alpha$ $(\alpha>0)$ : $\nu_\alpha([x,\infty]) = x^{-\alpha}$

mesure de Radon sur $]0,\infty]$




- *A measure* $\mu : (\mathrm{E},\mathcal{E}) \to [0,\infty]$ *is called a* Radon measure *if for all compact set* $K \subset \mathrm{E}$, $\mu(K) < \infty$. → on peut montrer que alors $\mu$ est régulière
- *We denote* $\mathrm{M}(\mathrm{E})$ *the set of all Radon measures on* $\mathrm{E}$.
- *In particular,* $\mathrm{M}(\mathrm{E})$ *contains* $\mathrm{M}_p(\mathrm{E})$ *the set of* Radon point measures, i.e. *measures of the kind* $\mu = \sum_{i\in D} \delta_{x_i}$*; where* $D$ *is countable, and* $(x_i)_{i\in D} \in \mathrm{E}$ *has no accumulation point.*

$x_i$ $x_\infty$ interdit

### 2.3.2 Vague topology on M(E)

Let $(\mu_n)_{n\in\mathbb{N}}, \mu \in \mathrm{M}(\mathrm{E})$. The sequence $(\mu_n)$ *converges vaguely* to $\mu$ if for all function $f \in \mathcal{C}_K$ (continuous with compact support), $\int_{\mathrm{E}} f \mathrm{d}\mu_n \to \int_{\mathrm{E}} f \mathrm{d}\mu$. We denote $\mu_n \xrightarrow{v} \mu$. In the sequel we denote $\mu(f) := \int_{\mathrm{E}} f \mathrm{d}\mu$. The topology associated to this notion of convergence is called the *vague topology* on $\mathrm{M}(\mathrm{E})$, denoted by $\mathcal{V}$. It is the topology generated by the evaluation maps $T_f : \mu \mapsto \mu(f)$, for $f \in \mathcal{C}_K$. A basis for $\mathcal{V}$ is the family of open sets

$$\Big\{V = \{\mu \in \mathrm{M}(\mathrm{E}) : a_i < \mu(f_i) < b_i, \quad \forall 1 \le i \le k\}, k \in \mathbb{N}, a_i < b_i \in \mathbb{R}, f_i \in \mathcal{C}_K\Big\}.$$

↳ $= \bigcap_{i=1}^{k} V_i$  $V_i = \{\mu : a_i < \mu(f_i) < b_i\}$

It can be shown that $(\mathrm{M}(\mathrm{E}),\mathcal{V})$ is a Polish space (separable, completely metrizable). Separable means that it contains a dense sequence; completely metrizable means that one can construct a distance on $\mathrm{M}(\mathrm{E})$ which is compatible with the topology, and for which $\mathrm{M}(\mathrm{E})$ becomes a complete space.

Similarly to the case of weak convergence, we have a 'Portmanteau theorem'

**Theorem 2.3.3**
*The following are equivalent:*

*(i)* $\mu_n \xrightarrow{v} \mu$.

*(ii)* $\mu_n(B) \to \mu(B)$ *for all set* $B$ *such that* $\bar{B}$ *is compact and* $\mu(\partial B) = 0$.

*(iii) For all compact* $K \subset \mathrm{E}$, $\limsup \mu_n(K) \le \mu(K)$ *and for all open set* $G \subset \mathrm{E}$, $\liminf \mu_n(G) \ge \mu(G)$.

### 2.3.3 Regular variation and vague convergence of tail measures

In this section $\mathrm{E} = (0,\infty]$.

**Theorem 2.3.4**
*Let* $F$ *be a* c.d.f. *and* $X \sim F$*. The following are equivalent*

⚠ en exo on voit (i) + $b_n = 0$ de MDA ⇒ (ii)
en fait (i) ⇒ (ii)

*(i)* $F$ *belongs to the max-domain of attraction of* $\Phi_\alpha$ *(Fréchet distribution)*

*(ii)* $1 - F \in RV(-\alpha)$

*(iii)* $\exists (a_n)_{n\ge0} : n(1 - F(a_n x)) \xrightarrow[n\to\infty]{} x^{-\alpha}$.

*(iv)* $\mu_n(\ \bullet\ ) := n\mathbb{P}(\frac{X}{a_n} \in (\ \bullet\ )) \xrightarrow{v} \nu_\alpha(\ \bullet\ )$*, where* $\nu_\alpha[x,\infty) = x^{-\alpha}, x > 0$.

↳ $\mu_n[x,\infty[ = n(1-F(a_n x))$

(convergence vague dans $E = ]0,+\infty]$)

*Proof.* $(ii) \iff (iii)$ and $(ii) \Rightarrow (i)$ have been proven in Exercise 2.2. The fact that $(i) \Rightarrow (ii)$ is shown in Resnick (1987), proposition 1.11 p. 54. The proof relies on Karamata's theorem. It remains to see why $(iii) \iff (iv)$. Assume $(iv)$. Then for $x > 0$,

$$\begin{aligned} n(1 - F(a_n x)) &= n\mathbb{P}(X/a_n \in (x, \infty)) \\ &= \mu_n(x, \infty) \longrightarrow \\ &\to \nu_\alpha(x, \infty) = x^{-\alpha} \quad \text{by Theorem 2.3.3 (ii),} \end{aligned}$$

(portemanteau)

$\partial(x,\infty) = \{x\} \cup \{+\infty\}$ : de mesure nulle pour la mesure limite.

which proves $(iii)$. Now assume $(iii)$. On order to show that $(iv)$ holds, we need to show that for $f \in \mathcal{C}_K$, $\mu_n(f) \to \nu_\alpha(f)$. Let $f \in \mathcal{C}_K$. Let $S = supp(f) = cl\{x > 0 : f(x) > 0\}$, where $cl(A) = \bar{A}$ denotes the closure of a set $A$. Necessarily $0 \notin S$ otherwise $S$ would not be closed in $(0, \infty]$. Thus $S \subset [\delta, \infty]$ for some $\delta > 0$. Introduce the probability measures $P_n$ on $[\delta, \infty]$ defined by

( $\mu_n = \frac{\mu_n(\cdot)}{\mu_n[\delta,+\infty]}$ )

on se débarasse de 0

$$P_n(A) = \mu_n(A)/\mu_n[\delta, \infty], \qquad A \subset [\delta, \infty],$$

0 δ +∞

(which is well defined because $[\delta, \infty]$ is compact, thus $\mu_n[\delta, \infty] < \infty$).

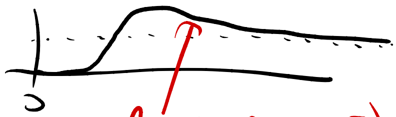
Using $(iii)$, for all $x > \delta$, $P_n[x, \infty] \to (x/\delta)^{-\alpha}$. Thus, using the Portmanteau theorem for probability measures, $P_n$ converges weakly to $P = \delta^\alpha \nu(\,\cdot\,)$. Now, $f$ has compact support in $[\delta, \infty]$ implies that $f$ is continuous and bounded on $[\delta, \infty]$. Thus, $P_n(f) \to P(f)$, which yields $\mu_n(f) \to \mu(f)$. ■

$P_n(x,\infty) = \frac{n(1-F(a_n x))}{n(1-F(a_n\delta))} \to \frac{x^{-\alpha}}{\delta^{-\alpha}}$

donc $P_n \to P$ avec $P(x,\infty) = \frac{x^{-\alpha}}{\delta^{-\alpha}} = \delta^\alpha \nu_\alpha(x,\infty)$ où $\nu_\alpha(x,\infty) = x^{-\alpha}$

$P_n(f) \to P(f) \iff \frac{\mu_n(f)}{\mu_n(\delta,\infty)} \to \nu(f)\delta^\alpha \iff \cancel{\delta^\alpha}\, \mu_n(f) \to \nu(f)\cancel{\delta^\alpha}$

$\boxed{\text{Rem}}$ $f$ à support compact $C^0$ sur $(0,\infty]\setminus\{0\} \iff$
- $f$ s'annule au voisinage de 0
- $f(x_n) \to f(+\infty)$ lorsque $x_n \to +\infty$ (i.e. $\lim_{+\infty} f = C$ existe)
- $f$ est continue sur $\mathbb{R}_+$



$f \in \mathcal{C}_K(]0,+\infty])$

### 2.3.4 Exercises

The following exercises are borrowed from Resnick (1987), chapter 3.4

**Exercise 2.4:**
Show that the following transformations are continuous:

1. $$T_1 : \mathrm{M}(E) \times \mathrm{M}(E) \to \mathrm{M}(E)$$
$$(\mu_1, \mu_2) \to \mu_1 + \mu_2.$$

2. $$T_1 : \mathrm{M}(E) \times (0, \infty) \to \mathrm{M}(E)$$
$$(\mu, \lambda) \to \lambda\mu.$$

**Exercise 2.5:**
Let $(x_n)_{n\in\mathbb{N}}, x$ in $E$ and $c_n \cancel{<} 0, c > 0$. Show that in $\mathrm{M}(E)$,

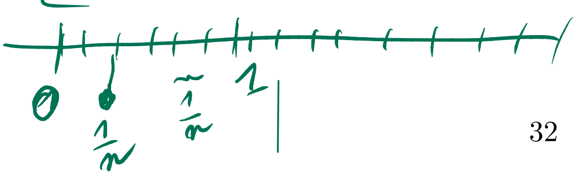
$c_n \geq 0$

$$\mu_n := c_n \delta_{x_n} \xrightarrow{v} c\delta_x$$

if and only if $x_n \xrightarrow[n\to\infty]{} x$ and $c_n \xrightarrow[n\to\infty]{} c$.

**Exercise 2.6:**
Let $m_n = \sum_{i\in\mathbb{N}^*} n^{-1}\delta_{\left(\frac{i}{n}\right)}$ and let $m$ be the lebesgue measure on $(0, \infty)$. Show that $m_n \xrightarrow{v} m$.

masse totale = 1



↳ on regardera des $\nu_n = n \sum_{i=1}^{n} \delta_{(X_i/a_n)}(\cdot)$ sur $(0,+\infty]$
aléatoire
aléatoire

## 2.4 Weak convergence of tail empirical measures

### 2.4.1 Random measures

Recall $(\mathrm{M(E)}, \mathcal{V})$ is a topological space. Thus it has a Borel $\sigma$-field $\mathcal{M}(\mathrm{E})$. It can be shown by monotone class arguments that $\mathcal{M}(\mathrm{E})$ is generated by the evaluation maps $T_f : \mu \mapsto \mu(f)$, for $f \in \mathcal{C}_K(\mathrm{E})$, or by the $T_F : \mu \mapsto \mu(F)$, for $F \subset E$ closed. Thus

↳ i.e: $\mathcal{M}(E)$ est la + petite tribu qui contienne les $T_f^{-1}(A)$ $A \subset \mathbb{R}^+$ mesurable (qui rende les $T_f$ mesurable)

$$\mathcal{M}(\mathrm{E}) = \sigma\Big\{T_f\,, f \in \mathcal{C}_K\Big\} \;=\; \sigma\Big\{T_F, F \subset \mathrm{E},\ \text{closed}\Big\}.$$

rappel : tribu trace
$\mathcal{F}$ tribu sur $E$ et $A \subset E$
$\mathcal{F}_A = \{B \cap A,\ B \in \mathcal{F}\}$

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a random measure $\xi$ is thus a measurable mapping $(\Omega, \mathcal{A}) \to (\mathrm{M(E)}, \mathcal{M}(\mathrm{E}))$. A point process is a special case of such mapping, taking its value in $(\mathrm{M}_p(\mathrm{E}), \mathcal{M}_p(\mathrm{E}))$,where $\mathcal{M}_p(\mathrm{E})$ is the trace $\sigma$-field of $\mathcal{M}$ on $\mathrm{M}_p$. The distribution of a random measure $\xi$ is entirely determined by the 'finite dimensional distributions', *i.e.* by the laws of the random vectors $(\xi(f_1), \dots, \xi(f_k))$, where $k \in \mathbb{N}$ and $f_i \in \mathcal{C}_K, i \le k$.

A convenient tool for characterizing the law of random measures and their convergence in distribution is the Laplace transform, defined next.

**Definition 2.4.1** (Lapace transorm of a random measure)**.** *The Laplace transform of a random measure $\xi$ is the functional*

$$\begin{aligned} \mathcal{L}_\xi :\quad & \mathcal{C}_K \to \mathbb{R} \\ & f \mapsto \mathcal{L}_\xi(f) = \mathbb{E}\left(e^{-\xi(f)}\right) \quad = \int_\Omega e^{-\int_{\mathrm{E}} f(x)\xi(\omega, \mathrm{d}x)}\, \mathrm{d}\mathbb{P}(\omega) \end{aligned}$$

Since the law of a random vector $X \in \mathbb{R}^k$ is determined by its (usual) Laplace transform $t \mapsto \mathbb{E}\left(e^{-\langle t, X\rangle}\right)$, it is easy to see that the Laplace transform of a random measure also determines uniquely its distribution. In fact more is true: pointwise convergence of Laplace transforms of a sequence $(\xi_n)$ determines weak convergence, as stated next.

$\xi$ : mesure aléatoire $\Rightarrow$ sa loi est déterminée par la loi des valeurs $(\xi(f_1), \dots \xi(f_k))$ cf page suivante

### 2.4.2 Weak convergence in M(E)

**Proposition 2.4.2** (Characterization of weak convergence)
*Let $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of random measures on* E. *The following statements are equivalent*

*(i)* $\xi_n \xrightarrow{w} \xi$, i.e.$\forall \phi$ *bounded continuous* $\mathrm{M(E)} \to \mathbb{R}$, $\mathbb{E}(\phi(\xi_n)) \to \mathbb{E}(\phi(\xi))$.

*(ii)* $\forall k \in \mathbb{N}, \forall (f_1, \dots, f_k) \in \mathcal{C}_K$, $(\xi_n(f_1), \dots, \xi_n(f_k)) \xrightarrow{w} (\xi(f_1), \dots, \xi(f_k))$.

*(iii)* $\forall f \in \mathcal{C}_K$, $\mathcal{L}_{\xi_n}(f) \to \mathcal{L}_\xi(f)$ *(pointwise convergence of the Laplace transforms)*

*Proof.*

- $(ii) \iff (iii)$ comes from standard properties of weak convergence in $\mathbb{R}^k$. Assume $(ii)$ and fix $f \in \mathcal{C}_K$. Then letting $X_n = \xi_n(f)$ and $X = \xi(f)$, we have

$$\mathcal{L}_{\xi_n}(f) = \mathbb{E}\left(e^{-\xi_n(f)}\right) = \mathcal{L}_{X_n}(1) \to \mathcal{L}_X(1), \quad = \mathcal{L}_\xi(f)$$

$= E[e^{-1 \cdot X_n}]$

where the latter convergence comes from the fact that pointwise convergence of the Laplace transform of random variables is equivalent to their weak convergence. This proves $(iii)$.

(E, ℰ) ← mesures (convergence vague) $\boxed{M(E)}$, $\mathcal{V}$, topologie vague ← une loi de proba (convergence en loi) si $X = (\xi(f_1) \dots \xi(f_n))$ :

la loi de $X$ est déterminée par sa transformée de Laplace [probas classiques],

ie : $\mathcal{L}_X(t) = \mathbb{E}\left[e^{-\langle t, X\rangle}\right]$ $\quad (X_j \geq 0) \ \forall j$

$t \in \mathbb{R}^k$

$$= \mathbb{E}\left[e^{-\sum_{j=1}^{k} t_j X_j}\right]$$

$$= \mathbb{E}\left[e^{-\sum_{j=1}^{k} t_j \xi(f_j)}\right]$$

$$= \mathbb{E}\left[e^{-\xi\Big(\underbrace{\sum_{j=1}^{k} t_j f_j}_{\tilde{f}}\Big)}\right]$$

$$= \mathcal{L}_\xi(\tilde{f}) \qquad \tilde{f} = \text{somme de } f_j \in \mathcal{C}_K \in \mathcal{C}_K$$

d'où : $\mathcal{L}_\xi$ détermine la loi des $(\xi(f_1) \dots \xi(f_n))$

donc la loi de $\xi$.

CV de la tr. de Laplace

Conversely, assume $(iii)$ and notice that the Laplace transform of the random vector $X_n = (\xi_n(f_1), \dots, \xi_n(f_k))$ is, for $t \in \mathbb{R}^k$,

$$\mathcal{L}_{X_n}(t) = \mathbb{E}\left(e^{-\langle t, X_n\rangle}\right) = \mathbb{E}\left(e^{-\sum_i t_i \xi_n(f_i)}\right) = \mathbb{E}\left(e^{-\xi_n(\overbrace{\textstyle\sum_i t_i f_i}^{=\tilde f \in \mathcal{C}_K})}\right) = \mathcal{L}_{\xi_n}\Big(\overbrace{\sum_i t_i f_i}^{\tilde f}\Big).$$

Since $\sum_i t_i f_i \in \mathcal{C}_k$, the right-hand-side converges to $\mathcal{L}_\xi(\sum t_i f_i) = \mathcal{L}_X(t)$, where $X = (\xi(f_1), \dots, \xi(f_k))$ and the proof of $(iii) \Rightarrow (ii)$ is complete.

rappel : "continuous mapping theorem" : $X_n \xrightarrow{w} X$ (en loi) des variables aléatoires $\in E$ si $\phi : E \to F$ est $\mathcal{C}^0$, alors $Y_n := \phi(X_n) \xrightarrow{w} Y = \phi(X)$

ici $\phi = T$, d'où $\phi(X_n) = (\xi_n(f_1) \dots \xi_n(f_k)) \xrightarrow{w} (\xi(f_1) \dots \xi(f_k)) = \phi(\xi)$

- $(i) \Rightarrow (ii)$ is a direct application of the continuous mapping theorem applied to the mapping $T : \mu \mapsto (\mu(f_1), \dots, \mu(f_k))$ which is continuous by definition of the vague topology.

  $\mathcal{P}(E) \to \mathbb{R}^k$

  ↳ is : devrait vérifier que $T^{-1}(\prod_{i=1}^k ]a_i, b_i[)$ est ouvert or $T^{-1}(\prod ]a_i, b_i[) = \{\mu : \forall i \in \{1..k\} : a_i < \mu(f_i) < b_i\}$ ouvert dans $\mathcal{V}$

- $(ii) \Rightarrow (i)$ :

Assume $(ii)$. We need to show that $(\xi_n)_{n\in\mathbb{N}}$ is (a) relatively compact ($i.e.$that its closure is compact for the weak topology of weak convergence in $\mathrm{M}(E)$), and (b) the limits of any two converging subsequence coincide in distribution.

(b) is an easy exercise: it is enough to show that for two possible limits $\xi^1, \xi^2$,

$$\mathbb{P}\left(a_i < \xi^1(f_i) < b_i, 1 \le i \le k, a_i < b_i\right) = \mathbb{P}\left(a_i < \xi^2(f_i) < b_i, 1 \le i \le k, a_i < b_i\right).$$

$= \lim P(\xi_{\varphi(n)} \in T^{-1} \prod_{i=1}^k (]a_i, b_i[))$

(a) requires more care. Since $\mathrm{M}(\mathrm{E})$ is a separable, metric space, the Prohorov's theorem applies (tightness implies relative compactness). It is thus enough to show that $(\xi_n)$ is tight. To do this, use Lemma 3.20 p.153 in Resnick (1987): a sufficient condition is that $\xi_n(f)_{n\in\mathbb{N}}$ be tight, for all fixed $f \in \mathcal{C}_K$. Now the latter condition is satisfied because $\xi_n(f)$ converges weakly in $\mathbb{R}$. ∎

rappel : $X_n : \Omega \to (E, \mathcal{E})$
$X_n$ tight : $\forall \varepsilon\ \exists K$ compact : $\sup_n \mathbb{P}(X_n \in K^c) < \varepsilon$

### 2.4.3 Tail measure and tail empirical measure

In this section $\mathrm{E} = (0, \infty]$. Recall from Theorem 2.3.4 that for a $c.d.f.$ $F$ and $X \sim F$, the following equivalence:

- $1 - F$ is $RV(-\alpha)$, ($i.e.$ $\exists (a_n)_{n\ge 0} : n(1 - F(a_n x) \to x^{-\alpha})$
  $\Longleftrightarrow$
- $\mu_n \xrightarrow{v} \nu_\alpha$, where $\mu_n(A) = n\mathbb{P}(X/a_n \in A), A \subset (0, \infty]$ and $\nu_\alpha[x, \infty] = x^{-\alpha}, x > 0$.

We now define the empirical version of $\mu_n$, and we shall see that this empirical version (a random measure) converges in distribution to $\nu_\alpha$ as well, under the same assumptions.

**Definition 2.4.3** (tail empirical measure)**.** *Let $F$ be a* c.d.f. *on $\mathbb{R}^+$ and $X, (X_i)_{i\in\mathbb{N}} \overset{i.i.d.}{\sim} F$. Let $(a_n)_{n\in\mathbb{N}} > 0$ a sequence of positive numbers.*

*Consider a sequence of integers $k(n)_{n\in\mathbb{N}} \in \mathbb{N}$, such that $k(n) \xrightarrow[n\to\infty]{} \infty$ and $\frac{k(n)}{n} \xrightarrow[n\to\infty]{} 0$. Write $k$ instead of $k(n)$ for convenience. The* tail empirical measure *associated to $F$ and the sequence $(a_n)$ is the random point measure*

$$\nu_{n,k} = \frac{1}{k} \sum_{i=1}^{\cancel{k}\, n} \delta_{\left\{\frac{X_i}{a_{\lfloor n/k \rfloor}}\right\}}$$

$\in M_p(E)$

⚠ pas une vraie statistique car $(a_n)$ dépend loi F et n'est pas observée

rem : on a "vu" que on peut prendre $a_n = \overleftarrow{F}(1 - \frac{1}{n})$

c.a.d. $n(1-F(a_n x)) \to x^{-\alpha}$

**Proposition 2.4.4** (weak CV of the tail empirical measure)
*If $F \in RV(-\alpha)$ and $(a_n)_{n\geq 0} > 0$ is such that $\mu_n \xrightarrow{v} \nu_\alpha$, then the tail empirical measures converge weakly in* $\mathrm{M}_+(\mathrm{E})$,

$$\nu_{n,k} \xrightarrow{w} \nu_\alpha.$$

où $\nu_\alpha[x,+\infty] = x^{-\alpha}$

Proposition 2.4.4 means that the tail empirical measure is a consistent estimator for the tail measure.

*Proof.* (sketch of) According to Proposition 2.4.2, we need to show that $\mathcal{L}_{\nu_{k,\alpha}}(f) \to \mathcal{L}_{\delta_{\nu_\alpha}}(f) = e^{-\nu_\alpha(f)}$, for $f \in \mathcal{C}_K$.

... preuve derrière ■

**Exercise 2.7:**
Let $\{X_{k,n}, 1 \leq k \leq n, n \geq 1\}$ be random elements of $E$ such that for each $n$, the $(X_{k,n}, k \leq n)$ are i.i.d. Let $(a_n)_{n\geq 0} > 0$ be a sequence such that $a_n \xrightarrow[n\to\infty]{} \infty$ and let $\mu \in \mathrm{M}(E)$. Define $\xi_n = \frac{1}{a_n}\sum_{k=1}^n \delta_{X_{k,n}}$ (a random measure) and $\mu_n = \frac{n}{a_n}\mathbb{P}\left(X_{1,n} \in \bullet\right)$. Show that

$$\mu_n \xrightarrow{v} \mu \iff \xi_n \xrightarrow{w} \mu \text{ in } \mathrm{M}(E).$$

### 2.4.4 Statistical application: Hill estimator

The Hill estimator is a classical estimator of the tail index $\alpha$. Many other estimators exist (Pickand's estimator, CFG estimator, ...). In this course we limit ourselves to studying the consistency of the Hill estimator. Notice that sharper results exist such as asymptotic normality or concentration inequalities under additional regularity assumptions on the tails.

The idea behind the estimator is the following: Notice first that

$$\int_1^\infty \nu_\alpha[x,\infty]x^{-1}\mathrm{d}x = 1/\alpha.$$

$= \int_1^\infty x^{-\alpha-1}\,dx = \left(\frac{x^{-\alpha}}{-\alpha}\right)_1^\infty = \frac{1}{\alpha}$.

The Hill estimator aims at approaching the quantity $1/\alpha$. The heuristic is to successively replace $\nu_\alpha$ with $\mu_n$, then $a_n$ by a quantile, then $\mu_n$ with its empirical version $\nu_{k,n}$, as follows

$$\nu_\alpha(x,\infty] \approx n(1-F(a_n x)) = n\mathbb{P}\left(X/a_n > x\right)$$
$$\approx \nu_{k,n}[x,\infty] = \frac{1}{k}\sum_{i=1}^n \mathbb{1}\left\{\frac{X_i}{a_{\lfloor n/k\rfloor}} > x\right\}$$

Now take $a_n = (1/(1-F))^{\leftarrow}(n) = F^{\leftarrow}(1-1/n)$ (see exercise 2.2 for the reason of this choice), and replace $F^{\leftarrow}(1-k/n)$ with its empirical version, which is the the $k^{th}$ largest order statistic $X_{(k)}$, so that $a_{\lfloor n/k\rfloor} \approx \cancel{X_{(n-k)}}$ $X_{(k)}$. We get

← on remplace $a_{\frac{n}{k}}$ par $X_{(k)}$

$\hat{\nu}_{n,k}[x,+\infty[$

$X_{(1)} \geq X_{(2)} \geq \dots X_{(k)}$

$X_{(k)}$ = la $k^e$ + grande statistique.

$$\frac{1}{\alpha} = \int_1^\infty x^{-1}\nu_\alpha(x,\infty]$$
$$\approx \int_1^\infty x^{-1}\frac{1}{k}\sum_{i=1}^n \mathbb{1}\left\{\frac{X_i}{X_{(k)}} > x\right\}\mathrm{d}x$$
$$= \frac{1}{k}\sum_{i=1}^n \int_1^\infty x^{-1}\mathbb{1}\left\{x < \frac{X_i}{X_{(k)}}\right\}\mathrm{d}x$$
$$\boxed{= \frac{1}{k}\sum_{i=1}^k \log\frac{X_{(i)}}{X_{(k)}}}$$

$= \int_1^{X_i/X_{(k)}} x^{-1}\,dx = \log\frac{X_i}{X_{(k)}}$

Les estimateur de Hill de $\frac{1}{\alpha}$



rem: condition MDA: $\frac{M_n - b_n}{a_n} \xrightarrow{loi} Z \sim G(x) = \exp\{-(1+\gamma x)^{-\frac{1}{\gamma}}\}$
si $1-G \in RV(\frac{1}{\gamma})$ d'où $\frac{1}{\gamma} = \alpha$

Hill estimateur de $\gamma$ (Gamma)

prop. 2.4.4

On veut montrer $\mathcal{V}_{nk} := \sum_{1}^{n} \delta_{\frac{X_i}{a_{[n/k]}}} \xrightarrow{w} \mathcal{V}_\alpha$

isdmq $\mathcal{L}_{n,k}(f) \left( = \mathcal{L}_{\mathcal{V}_{nk}}(f) = E\left(e^{-\mathcal{V}_{nk}(f)}\right)\right) \longrightarrow \mathcal{L}_{\mathcal{V}_\alpha}(f) = e^{-\mathcal{V}_\alpha(f)} = e^{-\int f \cdot d\mathcal{V}_\alpha}$

où $\mathcal{V}_\alpha(x,\infty) = x^{-\alpha}$

$\frac{d\mathcal{V}_\alpha}{dx} = \alpha x^{-\alpha-1}$ (consideré)

or $\mathcal{L}_{nk}(f) = E\left(e^{-\frac{1}{k}\sum_{1}^{n}\delta_{\frac{X_i}{a_{n/k}}}(f)}\right)$

$= E\left(\underbrace{e^{-\frac{1}{k}\sum_{i=1}^{n} f\left(\frac{X_i}{a_{n/k}}\right)}}_{\left(\prod_{i=1}^{n}\left(\quad\right)\right)}\right)$ → iid

$\overset{\text{indep.}}{=} \prod_{i=1}^{n} E\left(e^{-\frac{1}{k}f\left(\frac{X_1}{a_{n/k}}\right)}\right) = \left\{E\left(e^{-\frac{1}{k}f\left(\frac{X_1}{a_{\frac{n}{k}}}\right)}\right)\right\}^n$

on prend $X_n : \Omega \to \mathbb{R}^+$

$= \left\{\int_0^{+\infty} e^{-\frac{1}{k}f\left(\frac{x}{a_{n/k}}\right)} dP_X(x)\right\} = I_n \quad P_X = \text{loi de } X.$

soit $P_t$ = loi de $\frac{X}{a_{[t]}}$

$P_{n/k}$ = loi de $\frac{X}{a_{[n/k]}}$

alors $I_n = \left\{\int_0^{\infty} e^{-\frac{1}{k}f(y)} dP_{\frac{n}{k}}(y)\right\}^n$

rappelons nous : $n(1-F(a_n x)) \longrightarrow x^{-\alpha}$

ici $\mu_n = n P\left(\frac{X}{a_n} \in \cdot\right) \xrightarrow{v} \mathcal{V}^\alpha$

$\left(1+\frac{x}{n}\right)^n \underset{n\to\infty}{\longrightarrow} e^x$

ie $n P_{\frac{n}{k}} \xrightarrow{v} \mathcal{V}^\alpha$

$I_n = \left\{1 - \frac{\int_0^\infty \left(1-e^{-\frac{1}{k}f(y)}\right) \overbrace{n\, dP_{\frac{n}{k}}(y)}^{k \cdot \frac{n}{k} dP_{\frac{n}{k}} = k \cdot \mu_{\frac{n}{k}}}}{n}\right\}^n \longleftarrow$ propre

"idée" : $\int 1 - e^{-\frac{1}{k}f(y)}\, k\, d\mu_{\frac{n}{k}}(y) \approx \int \cancel{\tfrac{1}{k}} f(y)\, \cancel{k}\, d\mu_{\frac{n}{k}}(y) \longrightarrow \int f d\mathcal{V}_\alpha$

ensuite $\left\{1 - \frac{\int f d\mathcal{V}_\alpha}{n}\right\}^n \longrightarrow e^{-\int f d\mathcal{V}_\alpha}$.

fin de la preuve : (2.4.4)

or $\left| \int_0^{\infty} 1 - e^{-\frac{f(x)}{k}} \, n \, dP_{\frac{n}{k}}(x) - \int_0^{\infty} f(x) \frac{n}{k} dP_{\frac{n}{k}}(x) \right|$

$f \in \mathcal{C}_K([0,\infty[ \setminus \{0\})$ : $supp(f) \subset [\delta, +\infty[$

et $f(x)$ = borné

$$e^{-\frac{f(x)}{k}} = 1 - \frac{f(x)}{k} + o\left(\frac{1}{k}\right)$$

Les uniformément en $x$.

$$\int_\delta^{\infty} \left( \frac{f(x)}{k} + o\left(\frac{1}{k}\right) - \frac{f(x)}{k} \right) n \, dP_{\frac{n}{k}}(x)$$

$$= \int_\delta^{\infty} o\left(\frac{1}{k}\right) \frac{n}{k} dP_{\frac{n}{k}}(x) \leq \varepsilon \int_\delta^{\infty} \frac{n}{k} dP_{\frac{n}{k}}(x)$$

$$= \varepsilon \, \mu_{\frac{n}{k}}([\delta, \infty[)$$

$$\longrightarrow \varepsilon \, \delta^{-\alpha}$$

$\varepsilon$ aussi petit qu'on veut

d'où

$$\left| \int_0^{\infty} 1 - e^{-\frac{f(x)}{k}} \, n \, dP_{\frac{n}{k}} - \int_0^{\infty} f(x) \frac{n}{k} dP_{\frac{n}{k}}(x) \right| \to 0$$

donc $I_n = \left\{ 1 - \frac{\int_0^{\infty} f(x) \frac{n}{k} dP_{\frac{n}{k}}(x) + o(1)}{n} \right\}^n$

or $\int f(x) \frac{n}{k} dP_{\frac{n}{k}}(x) \longrightarrow \int f \, d\nu_\alpha$

$$I_n = \left\{ 1 - \frac{\int_0^{\infty} f \, d\nu_\alpha + o(1)}{n} \right\}^n$$

[prop 7.4.4]

d'où $t_n \longrightarrow \underbrace{\exp\left\{-\int_0^\infty f \, d\vartheta_\alpha\right\}}_{= \mathcal{L}_{\vartheta_\alpha}(f)}$.

ce qu'on voulait

$\mathcal{L}_{\vartheta_{n,k}}(f)$

*N.B* Here the order statistics are ranked in decreasing order, $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$.

**Proposition 2.4.5** (Hill estimator)
*Let $X, (X_i) \overset{i.i.d.}{\sim} F$, where $1 - F \in RV(-\alpha)$, for some $\alpha > 0$. Let $k = k(n) \xrightarrow[n\to\infty]{} \infty$ such that $k/n \xrightarrow[n\to\infty]{} 0$. The Hill estimator, defined by*

$$\widehat{1/\alpha}_n = \frac{1}{k}\sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k)}}$$

*is a consistent estimator of $1/\alpha$,* i.e. *it converges in probability to $1/\alpha$.*

rem : ici $\frac{1}{\alpha}$ : constante
CV en proba $\Longleftrightarrow$ CV en loi.

*Proof.* The proof follows the lines from Resnick (2007). Remind that $a(t) = a_{\lfloor t \rfloor}$. To alleviate notations, we denote

$$\nu_n = \frac{1}{k}\sum_{i=1}^{k} \delta_{\left\{\frac{X_i}{a(n/k)}\right\}} \quad (= \nu_{n,k})$$

$$\hat{\nu}_n = \frac{1}{k}\sum_{i=1}^{k} \delta_{\left\{\frac{X_i}{X_{(k)}}\right\}}$$

According to the arguments leading to the statement, $\widehat{1/\alpha}_n = \int_1^\infty x^{-1}\hat{\nu}_n[x,\infty]\mathrm{d}x$ and $1/\alpha = \int_1^\infty x^{-1}\nu_\alpha[x,\infty]\mathrm{d}x$. We need to show that

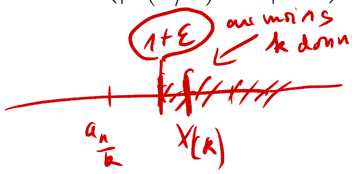
$$\int_1^\infty x^{-1}\hat{\nu}_n[x,\infty]\mathrm{d}x \xrightarrow[n\to\infty]{P} \int_1^\infty x^{-1}\nu_\alpha[x,\infty]\mathrm{d}x. \tag{2.7}$$

**1. Behavior of the order statistics**
We show that

$$\frac{X_{(k)}}{a(n/k)} \xrightarrow{P} 1. \tag{2.8}$$

Indeed for $\epsilon > 0$,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{X_{(k)}}{a(n/k)} - 1\right| > \epsilon\right) &= \mathbb{P}\left(\frac{X_{(k)}}{a(n/k)} > 1+\epsilon\right) + \mathbb{P}\left(\frac{X_{(k)}}{a(n/k)} < 1-\epsilon\right) \\
&= \mathbb{P}\left(\frac{1}{k}\sum_{i=1}^{n} \delta_{\frac{X_i}{a(n/k)}}(1+\epsilon,\infty\right) > 1] + \mathbb{P}\left(\frac{1}{k}\sum_{i=1}^{n} \delta_{\frac{X_i}{a(n/k)}}(1-\epsilon,\infty\right) < 1] \\
&= \mathbb{P}(\nu_n(1+\epsilon,\infty] > 1) + \mathbb{P}\left(\nu_n(1-\epsilon,\infty) < 1\right]
\end{aligned}$$




Now, Proposition 2.4.4 implies that $\nu_n(1+\epsilon,\infty] \xrightarrow{P} (1+\epsilon)^{-\alpha} < 1$ and $\nu_n(1-\epsilon,\infty] \xrightarrow{P} (1-\epsilon)^{-\alpha} < 1$. Whence, the latter display converges to zero and (2.8) is proved.

**2. Convergence of $\hat{\nu}_n$ in probability in $\mathrm{M}_+(0,\infty]$**
Notice first that

$$\hat{\nu}_n(\,\cdot\,) = \nu_n\left(\frac{X_{(k)}}{a(n/k)}\,\cdot\right).$$

car $\hat{\nu}_n[x,\infty] = \frac{1}{k}\sum_1^n 1\{\frac{X_i}{X_{(k)}} > x\} = \frac{1}{k}\sum 1\{\frac{X_i}{a_{n/k}} > \frac{X_{(k)}}{a_{n/k}}x\}$ $= \nu_n[x,\infty]$

idée : $\hat{\nu}_n$ = on divise par un $X_{(k)}$ aléatoire
$\Leftrightarrow$ évaluer $\nu_n$ sur un ensemble aléatoire

Consider the operator

$$\begin{aligned}
T : \mathrm{M}(0,\infty] \times \mathbb{R}_+^* &\to \mathrm{M}(0,\infty] \\
(\mu, x) &\mapsto \mu(x\ \cdot).
\end{aligned}$$



$T_x(A) = \mu(xA)$
$xA = \{xy \text{ où } y \in A\}$.

$= \nu_n\left(\frac{X_{(k)}}{a_{n/k}}\,\cdot\right) = T\left(\nu_n, \frac{X_{(k)}}{a_{n}}\right)$ donc (continuité en $(\nu_\alpha, 1)$) $T\left(\nu_n, \frac{X_{(k)}}{a_{n(k)}}\right) \xrightarrow{P} T(\nu_\alpha, 1) = \nu_\alpha$

$\nu_n \xrightarrow{P} \nu_\alpha$, $\frac{X_{(k)}}{a_n} \xrightarrow{P} 1$

Then $\hat{\nu}_n = T\left(\nu_n, \frac{X_{(k)}}{a(n/k)}\right)$. It can be shown (SeeResnick (2007), p. 83) that $T$ is continuous at $(\nu_\alpha, x)$ for $x > 0$ (see Resnick (2007) p. 84). Then (2.8) combined with the continuous mapping theorem yields

$$\hat{\nu}_n \xrightarrow{P} \nu_\alpha \qquad \text{in } \mathrm{M}(0,\infty]. \tag{2.9}$$

**3. Convergence of $\int_1^\infty x^{-1}\hat{\nu}_n[x,\infty]\mathrm{d}x$** We are ready to prove (2.7). For $M > 0$, (2.7) is equivalent to

$$\underbrace{\int_1^M x^{-1}\hat{\nu}_n[x,\infty]\mathrm{d}x}_{A_{M,n}} + \underbrace{\int_M^\infty x^{-1}\hat{\nu}_n[x,\infty]\mathrm{d}x}_{B_{M,n}} \xrightarrow{P} \underbrace{\int_1^M x^{-1}\nu_\alpha[x,\infty]\mathrm{d}x}_{A_M} + \underbrace{\int_M^\infty x^{-1}\nu_\alpha[x,\infty]\mathrm{d}x}_{B_M} \tag{2.10}$$

- For any fixed $M > 0$, the mapping $\mu \mapsto \int_1^M x^{-1}\mu[x,\infty]\mathrm{d}x$ is continuous on $\mathrm{M}(0,\infty]$. To see this, notice that the integrand is a decreasing function of $x$, so that the integral can be framed between to Riemann sums. In addition, $\mu_n \xrightarrow{v} \mu$ implies that for fixed $x > 0$ which is not an atom of $\mu$, $\mu_n[x,\infty] \to \mu[x,\infty]$.

- The continuous mapping theorem combined with (2.9) thus implies that $A_{M,n} \xrightarrow{P} A_M$, for any fixed $M$.

- Since $\lim_{M\to\infty} B_M = 0$, it is enough to show that for any $\epsilon > 0, \exists M_0 > 1$ such that $\forall M \geq M_0$ ,

$$\lim_n \mathbb{P}(B_{M,n} > \epsilon) \leq \delta. \tag{2.11}$$

Let $M > 1$ and $\eta > 0$. We have

$$\mathbb{P}(B_{M,n} > \epsilon) = \underbrace{\mathbb{P}\left(B_{M,n} > \epsilon, \left|\frac{X_{(k)}}{a(n/k)} - 1\right| > \eta\right)}_{p^1_{n,M}} + \underbrace{\mathbb{P}\left(B_{M,n} > \epsilon, \left|\frac{X_{(k)}}{a(n/k)} - 1\right| \leq \eta\right)}_{p^2_{n,M}}.$$

From (2.8), $p^1_{n,M} \leq \mathbb{P}\left(\left|\frac{X_{(k)}}{a(n/k)} - 1\right| > \eta\right) \to 0$ as $n \to \infty$. Also ,

$$\begin{aligned}
p^2_{n,M} &= \mathbb{P}\left(\int_M^\infty x^{-1}\nu_n\left[\frac{X_{(k)}}{a(n/k)}x,\infty\right]\mathrm{d}x > \epsilon\,,\ \left|\frac{X_{(k)}}{a(n/k)} - 1\right| \leq \eta\right)\\
&\leq \mathbb{P}\left(\int_M^\infty x^{-1}\nu_n\left[(1-\eta)x,\infty\right]\mathrm{d}x > \epsilon\right)\\
&= \mathbb{P}\left(\int_{M(1-\eta)}^\infty y^{-1}\nu_n\left[y,\infty\right]\mathrm{d}y > \epsilon\right)\\
&\overset{\text{Markov}}{\leq} \frac{1}{\epsilon}\ \mathbb{E}\left(\int_{M(1-\eta)}^\infty x^{-1}\nu_n\left[x,\infty\right]\mathrm{d}x\right)\\
&= \frac{1}{\epsilon}\int_{M(1-\eta)}^\infty x^{-1}\frac{n}{k}(1-F)\left(a(n/k)x\right)\mathrm{d}x\\
&= \frac{1}{\epsilon}\,\frac{n}{k}\int_{M(1-\eta)a(n/k)}^\infty \underbrace{x^{-1}(1-F)(x)}_{U(x)}\mathrm{d}x
\end{aligned}$$

The function $U$ in the latter integrand is $RV(-\alpha-1)$, so Karamata theorem implies that $\int_T^\infty U \sim TU(T)/\alpha$, with $TU(T) = (1-F)(T)$, *i.e.*

$$\begin{aligned}
\frac{1}{\epsilon}\,\frac{n}{k}\int_{M(1-\eta)a(n/k)}^{\infty} x^{-1}(1-F)\,(x)\,\mathrm{d}x \sim_{n\to\infty} &\;\; \frac{1}{\epsilon}\,\frac{n}{k}(1-F)\Big(M(1-\eta)a(n/k)\Big) \\
\xrightarrow[n\to\infty]{} &\;\; \frac{1}{\epsilon}\,\nu_\alpha\Big[M(1-\eta),\infty\Big] \\
= &\;\; \frac{1}{\epsilon}\,(M(1-\eta))^{-\alpha}
\end{aligned}$$

Choosing $M_0$ large enough so that the latter quantity is less than $\delta/2$ for $M = M_0$ shows (2.11) and concludes the proof.

■