Preliminary version

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Invited talks

The local-global principle for real analysis

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The logical foundations of real analysis were developed in the end of 19th century and beginning of 20th century by mathematicians such as Bolzano, Cauchy, Weierstrass, Dedekind, Cantor, Heine, Borel, Cousin and Lebesgue. Motivated by teaching, they departed from the geometric intuition of the "real line" by establishing rigorous proofs based on "completeness" axioms that characterize the real number continuum. Today, the foundation is still recognized as satisfactory, and all classical textbooks define $\mathbb{R}$ as any ordered field satisfying one of the many equivalent completeness axioms, such as the l.u.b. property, Dedekind cuts, Cantor's property of nested intervals, monotone convergence, Bolzano-Weierstrass, Borel-Lebesgue, Cousin's partitions, continuous induction, etc.

It is somewhat striking that all these equivalent axioms look so diverse. Also, several classical proofs of the basic theorems in real analysis seem difficult and subtle for the beginner (e.g., proofs of the extreme value theorem or Heine's theorem using Bolzano-Weierstrass). This calls for a need of a unifying principle from which all other axioms *and* basic continuity theorems could be easily derived. We discuss yet another equivalent axiom in two equivalent versions of the form:

Local-Global: Any local and additive property is global;

Global-Local: Any global and subtractive property has a limit point.

The earliest reference that explicitly describes this principle is Guyou's little-known French textbook [1]. It was later re-discovered independently many times in many disguises in some American circles [2, 3, 4, 5, 6]. The modern presentation we give is both enjoyable and successful for obtaining easy proofs of all useful theorems of real analysis, including those for continuity and differentiability of functions of the real variable. As a perspective it can be extended to a new presentation of the integral (just as powerful as the Kurzweil-Henstock integral—which is more powerful then Lebesgue's integral) where the so-called "fundamental theorem of calculus" becomes the actual definition for a generalized notion of the antiderivative (primitive) function.

This work was initiated from a joint collaboration with José Carlos Magossi, Faculdade de Tecnologia, Universidade Estadual de Campinas, Brazil.

References

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