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Reconciling Fitts' Law with Shannon's Information Theory

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Julien Gori* Olivier Rioul** Yves Guiard***

*ENS Cachan **Telecom ParisTech ***CNRS LTCI

Paris, France

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Presentation outline

Annus Mirabilis : 1948

Claude Shannon's *A Mathematical Theory of Communication*

- Information
- Uncertainty
- Communication system
- Capacity

Two Telling Quotes

Information is quantifiable and measurable ! A tremendous impact on psychologists :

> *We now call them experiments on the capacity of people to transmit information.*
>
> *(*G. A. Miller, **1956**, The Magical Number Seven, Plus or Minus Two*)*

> *Presented with a shiny new tool kit [information theory] and a somewhat esoteric new vocabulary to go with it, more than a few psychologists reacted with an excess of enthusiasm.*
>
> *(*F. Attneave, **1959**, Applications of Information Theory to Psychology*)*

A Strong Reaction from Shannon and Colleagues

The first paper has the generic title « Information Theory, Photosynthesis and Religion » *(* [Elias, 1958] *)*

[. . .] the basic results of the subject are aimed in a very specific direction, a direction that is not necessarily relevant to such fields as psychology, economics, and other social sciences. *(* [Shannon, 1956] *)*

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The Channel Capacity

Maximum amount of information transmittable over noisy communication link (channel)

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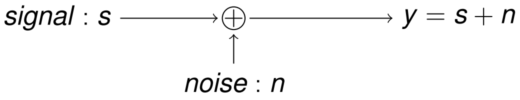
Maximum amount of information transmittable over noisy communication link (channel)

Additive **W**hite **G**aussian **N**oise channel



The Channel Capacity

Maximum amount of information transmittable over noisy communication link (channel)

Additive **W**hite **G**aussian **N**oise channel

$$\textit{signal} : s \longrightarrow \oplus \longrightarrow y = s + n$$

$$\uparrow$$

$$\textit{noise} : n$$

Shannon's famous Theorem 17 (1948)

$C = \frac{1}{2}\log\left(1 + \frac{S}{N}\right) = \frac{1}{2}\log\left(1 + SNR\right)$ bits per channel use

- $N = \mathbb{E}(n^2), \quad S = \mathbb{E}(s^2)$

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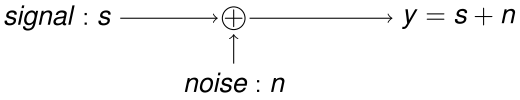


Shannon's famous Theorem 17 (1948)

$C = \frac{1}{2}\log\left(1 + \frac{S}{N}\right) = \frac{1}{2}\log\left(1 + SNR\right)$ bits per channel use

- $N = \mathbb{E}(n^2), \quad S = \mathbb{E}(s^2)$
- Any achievable rate (=reliable communication) $R \leq C$

Whatever Happened to Information Theory in Psychology ?

- Information theory discredited in psychology

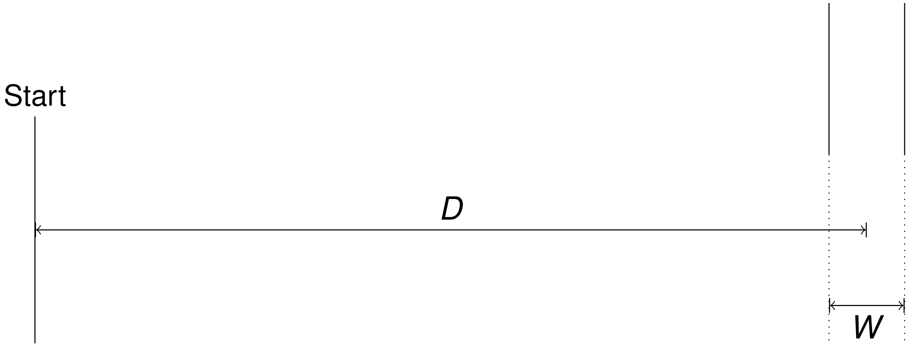
  One rarely sees Shannon's information theory in contemporary psychology articles
  *(*R. Luce, **2003**, Whatever Happened to Information Theory in Psychology ?*)*

- There is one notable exception : **Fitts' Law** , since 1954, and more generally the speed-accuracy trade-off for rapid aimed movement [Soukoreff and MacKenzie, 2009].
- Part of ISO 9241-9. Used for device assessment and movement time prediction in HCI.

Presentation outline

The Paradigm

Aiming at a target of size W from a distance D



How Do D and W Affect Target Acquisition Time ?

- Fitts' definition of an **I**ndex of **D**ifficulty (ID), by analogy with Shannon's Theorem 17 :

$$ID = \log_2\left(\frac{2D}{W}\right)\ (bits)$$

- (**M**ovement **T**ime) $MT = a + b \cdot ID$ through linear regression $\rightarrow$ Speed-accuracy trade-off
- a and b determined through experimentation.

Other Formulations for ID

- Fitts' original formulation, [Fitts, 1953]

$$ID = \log_2 \left(\frac{2D}{W} \right)$$

- Welford's formulation [Welford, 1960]

$$ID = \log_2 \left(0.5 + \frac{D}{W} \right)$$

- MacKenzie's formulation [MacKenzie, 1989]

$$ID = \log_2 \left(1 + \frac{D}{W} \right)$$

$a(\frac{D}{W})^b$

$a + b\sqrt{A}$

$a + b\log(\frac{A}{W})$

$-a + b(c + D)\log(\frac{2A}{W})$

Many more formulations !

MacKenzie's Formulation

- an analogy with Shannon's capacity :

$$ID = \log_2\left(1 + \frac{D}{W}\right) \qquad\qquad C = \frac{1}{2}\log_2\left(1 + \frac{S}{N}\right)$$

- D, W target distance and size
- S, N powers of signal and noise
- is $\frac{D}{W}$ an amplitude SNR ? What is the communication model ? What are the input, output and noise ?
- What about the $\frac{1}{2}$ factor ?

Mackenzie's Formulation (cont'd)

Capacity for a system

→ Communication model

$$\textit{signal} : s \longrightarrow \underset{\underset{\textit{noise}\,:\,n}{\uparrow}}{\oplus} \longrightarrow y = s + n$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{N} \right)$$

Achievable rate (vanishing error probability)

MacKenzie formulation

→ Speed-accuracy trade-off

$$ID = \log_2 \left(1 + \frac{D}{W} \right)$$

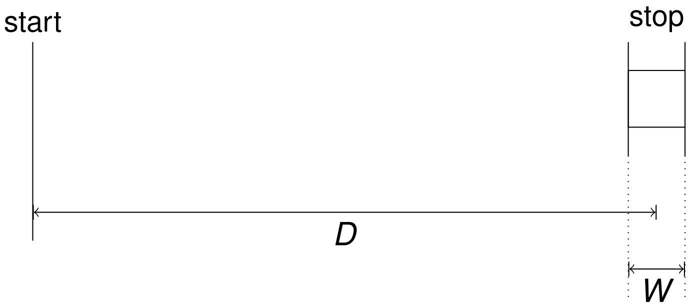
Presentation outline

Geometric Framework

Idea : aiming $=$ choosing !

Geometric Framework

Idea : aiming $=$ choosing !



aiming at a target is equivalent to choosing one target among N

Geometric Framework

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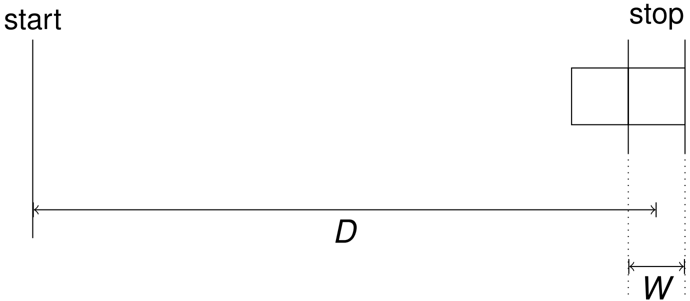

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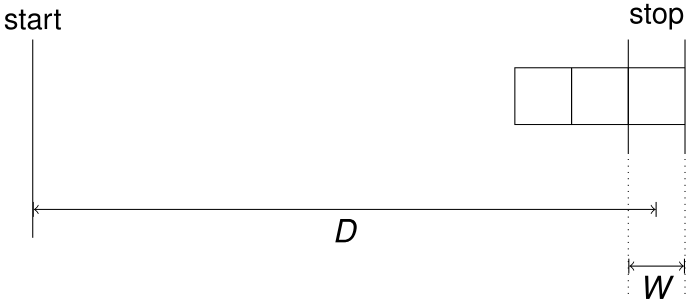

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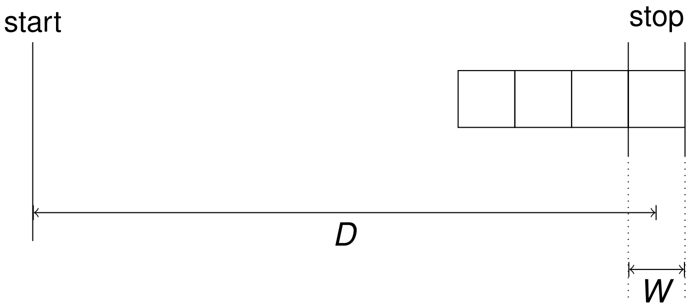

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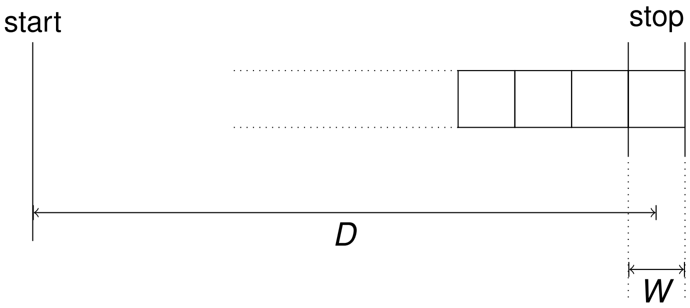

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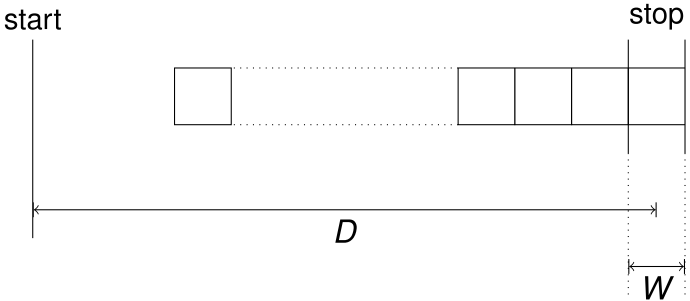

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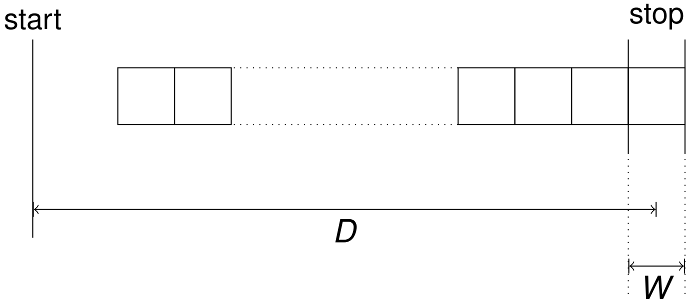

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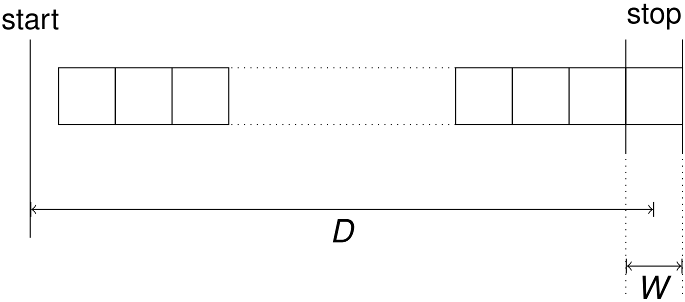

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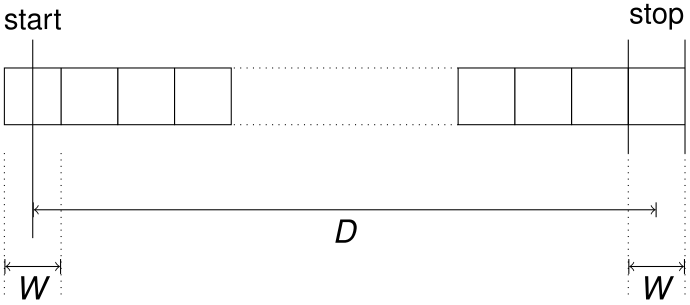

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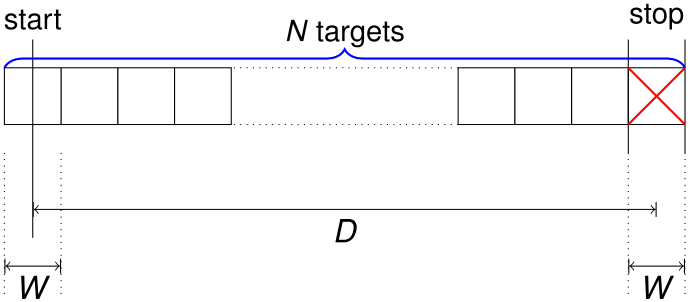

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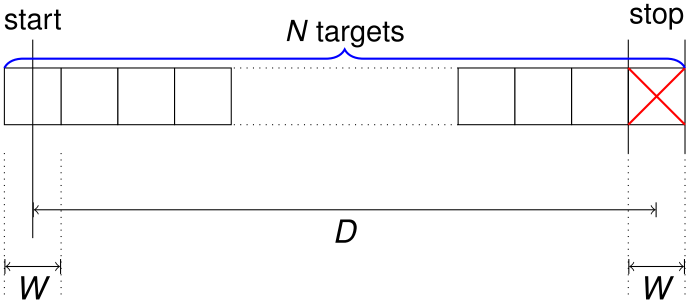
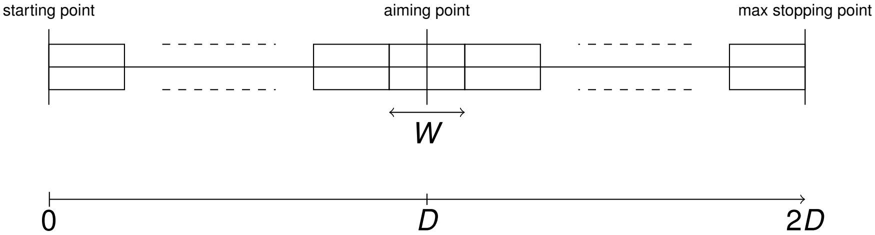


aiming at a target is equivalent to choosing one target among N

Rederiving the Fitts Formulation

- An analogy with Hick's law (Fitts 1953)
- An analogy with Shannon's Capacity (Fitts 1954)
- the movement terminates somewhere in between 0 and $2D$:



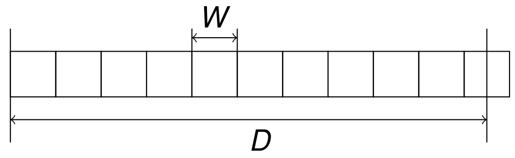
Rederiving the Fitts Formulation (cont'd)

- Number of targets : $n = \frac{2D}{W},\ n \in \mathbb{N}$
- Corresponding entropy assuming a uniform distribution :

$$H = \log_2(n) = \log_2 \frac{2D}{W} = ID_F$$

Rederiving the Welford Formulation

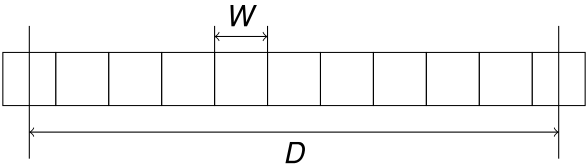
To put it in another way, he is called to choose a distance W out of a total distance extending from his starting point to the far edge of the target. (A. T. Welford, 1960)



- choose a distance W out of a total $\frac{W}{2} + D$
- number of possible targets : $n = \frac{D - \frac{W}{2}}{W} + 1 = \frac{1}{2} + \frac{D}{W}, \ n \in \mathbb{N}$

$$H = \log n = \log\left(\frac{1}{2} + \frac{D}{W}\right) = ID_W$$

Rederiving the MacKenzie Formulation



- number of possible targets : $n = 1 + \frac{D}{W},\ \textit{if}\ \frac{D}{W} \in \mathbb{N}$
- Entropy :

$$H = \log\left(1 + \frac{D}{W}\right) = ID_{McK}$$

Where Are We ?

Capacity for a system

→ Problem stated

→ Communication model

$$signal : s \longrightarrow \oplus \longrightarrow y = s + n$$

$$\uparrow$$

$$noise : n$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{N}\right)$$

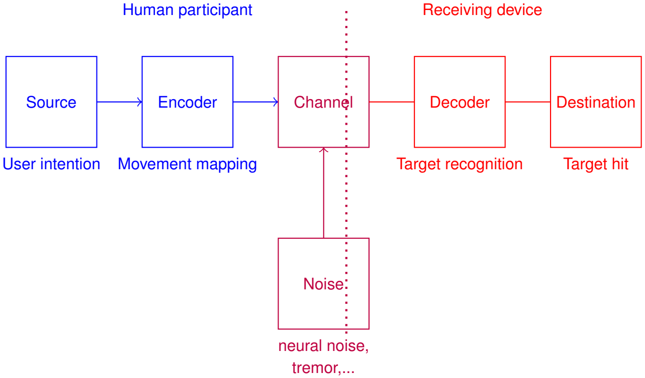
MacKenzie formulation

→ Speed-accuracy trade-off :
$ID = \log_2 \left(1 + \frac{D}{W}\right)$

What model ?

Presentation outline

The Human-motor System as a Communication System



Adapted from [Zhai et al., 2012]

Communication Model

Message

choosing a target = aiming at its center
Set of messages : $\{-\frac{D}{2}, -\frac{D}{2}+W, \ldots, \frac{D}{2}-W, \frac{D}{2}\}$
Messages uniformly distributed

Noise

ensuring reliable communication (= error-free) $\rightarrow$ the noise has absolute amplitude less than $\frac{W}{2}$
Uniform distribution

Output

choosing a target = aiming at its center = hitting the target

Gaussian versus Uniform Channel

Shannon Capacity for gaussian noise

- Signal power limited to S
- Noise power limited to N
- Gaussian distribution for noise

$$\mathit{signal} : s \longrightarrow \oplus \longrightarrow y = s + n$$
$$\uparrow$$
$$\mathit{noise} : n$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{N} \right)$$

MacKenzie formulation

- Signal amplitude limited to $\frac{D}{2}$
- Noise amplitude limited to $\frac{W}{2}$
- Uniform distribution for noise

$$|s| \leq \frac{D}{2} \longrightarrow \oplus \longrightarrow s + n$$
$$\uparrow$$
$$|n| \leq \frac{W}{2}$$

$$C' = ?$$

Capacity Formula for the Uniform Channel [Rioul and Magossi, 2014]

Theorem 1 :

$$C' = \log_2\left(1 + \frac{D}{W}\right)$$

Proof :

- $C' = \max_{x,|x|\leq\frac{D}{2}} I(x,y)$
- $I(x,y) = h(y) - h(y|x) = h(y) - h(n+x|x) = h(y) - h(n) = h(y) - \log_2(W)$
- Thus maximizing the mutual information between X and Y is equivalent to maximizing $h(Y)$
- $|y| \leq |x| + |n| \leq \frac{D+W}{2}$
- For a continuous RV under amplitude constraint, the uniform density maximizes differential entropy
- x discrete with uniform density gives y uniform
- $C' = h(y) - \log_2(W) = \log_2(D+W) - \log_2(W) = \log_2(1+\frac{D}{W})$

Gaussian versus Uniform Channel

Shannon's Capacity Formula

- Signal power limited to S
- Noise power limited to N
- Gaussian distribution for noise

$$\mathit{signal} : s \longrightarrow \oplus \longrightarrow y = s + n \qquad \uparrow \ \mathit{noise} : n$$

$$C = \frac{1}{2} \log_2 \left(1 + \frac{S}{N} \right)$$

MacKenzie formulation

- Signal amplitude limited to $\frac{D}{2}$
- Noise amplitude limited to $\frac{W}{2}$
- Uniform distribution for noise

$$|s| \leq \frac{D}{2} \longrightarrow \oplus \longrightarrow s + n \qquad \uparrow \ |n| \leq \frac{W}{2}$$

$$C' = \log_2 (1 + \frac{D}{W})$$

More than an Analogy

Theorem 2 :

$$C = C'$$

Proof :

- $C = \frac{1}{2}\log(1 + SNR)$
- uniform noise and uniform output
- Y : power of $y \propto (D + W)^2$
- N : power of $n \propto W^2$
- $C = \frac{1}{2}\log\left(\frac{S+N}{N}\right) = \frac{1}{2}\log\left(\frac{power\ \ of\ \ y=s+n}{power\ \ of\ \ n}\right) = \frac{1}{2}\log\left(\frac{(D+W)^2}{W^2}\right) = \log\left(\frac{D+W}{W}\right) = C'$

A true identity !

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Pending Issues

- A more realistic model
- With feedback ?
- What is the interpretation of throughput ?
- Can we take non-zero error into account ?

Thank You !

Any questions ?

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