Hölder Regularity of Subdivision Schemes and Wavelets

O. Rioul

Abstract. We study special functions obtained as limits of dyadic and p/q-adic subdivision schemes in one dimension. While the former are used for designing compactly supported wavelet bases, the latter are a flexible generalization which has already found application in digital signal processing. For $q > 1$, however, we obtain an infinite set of different limit functions instead of shifted copies of a single one, and a direct application of former ideas becomes impossible. However, our "discrete" approach allows us to extend the results of Daubechies and Lagarias (on Hölder regularity estimates based on infinite products of matrices) to p/q-adic schemes. We obtain easily implementable, sharp Hölder regularity estimates.

§1. Introduction

A *subdivision scheme* in one dimension can be defined as follows: consider a finite sequence $\{g_n\}$ ($g_n \neq 0$ for a finite number of values of n), referred to as the *subdivision mask* in the sequel, and the discrete operator acting on finite sequences

$$\mathcal{G} : u_n \to v_n = \sum_k u_k g_{qn-pk}, \tag{1}$$

where p and q are positive integers.

Definition 1. *A p/q-adic subdivision scheme is a collection of sequences g_n^j, computed using the recursion $g_n^{j+1} = \mathcal{G}(g_n^j)$, where $\mathcal{G}$ is given by (1).*

We assume that p and q are coprime, otherwise replace g_n by g_{dn} where $d = \gcd(p, q)$. Note that a subdivision scheme is uniquely defined given a subdivision mask g_n and an initial sequence g_n^1. For $p/q = 2$, we recover the classical binary subdivision schemes, which have long been used in computer-aided geometric design [6] and yield compactly supported wavelets under additional conditions on g_n [4,5].

Wavelets, Images and Surface Fitting
P. J. Laurent, A. Le Méhauté, and L. L. Schumaker (eds.), pp. 437–444.

ISBN 1-56881-040-7.

Being a natural extension to binary subdivision schemes, p/q-adic subdivision schemes may find application in computer-aided geometric design. However, the motivation for this study comes from signal decomposition problems: for $q > 1$, subdivision schemes are strongly related to filter banks with rational sampling changes, as studied by Kovačević and Vetterli [7]. Some of the material presented here and the derivation of "rational" pseudo-wavelets were investigated by Blu and the author in [1,2]. For many signal processing applications, p/q-adic schemes ($1 < p/q < 2$) form a more flexible tool than "pure" dyadic wavelets as they allow signal decomposition on fractions $\log_2 p/q$ of an octave using orthonormal bases [1,2,7].

In this paper we study special functions obtained as limits of p/q-adic subdivision schemes using the "'discrete approach" of [8]. We first define uniform convergence of the $\{g_n^j\}$'s.

Definition 2. *A p/q-adic subdivision scheme $\{g_n^j\}$ converges uniformly to the (compactly supported) function $\phi(x)$ if*

$$\lim_{j\to\infty} \sup_x |\phi(x) - g_{n_j}^j| = 0, \tag{2}$$

where n_j is a sequence of integers such that $|n_j - (p/q)^j x|$ is bounded as j increases.

Compact supports of limit functions were estimated by Blu in [1]. For $p/q = 2$ and initial sequence $g_n^1 = g_n$, we obtain the "father wavelet" $\phi(x)$ as studied in [4,5,8]. In this case every uniform limit is a finite linear combination of translates of ϕ, since replacing g_n^1 by g_{n-k}^1 gives $\phi(x-k)$. However, this shift invariance is never satisfied in the rational case ($q > 1$), as first noticed by Cohen and Daubechies in [3]. Therefore, we consider an infinite set of limit functions instead of one:

$$\phi^s(x) = \text{ uniform limit of } g_n^{j,s}, \tag{3}$$

where $s \in \mathbb{Z}$ and $g_n^{j,s}$ is defined by

$$g_n^{j,s} = \mathcal{G}^j(\delta_{n-s}). \tag{4}$$

Here the initial sequence is an *impulsion at time* $n = s$, i.e., $\delta_{n-s} = 1$ if $n = s$, 0 otherwise. It can be easily shown that any uniform limit is a linear combination of the $\phi^s(x)$, and that we obtain a pseudo-wavelet transform [1] for which the "wavelets" at resolution $(p/q)^{-j}$ are in the form $\psi_k((p/q)^j x)$. It is known that these wavelets can never be put in the exact form $\psi((p/q)^j x - k)$ (hence the name "pseudo-wavelets"), but can be approximated in this manner under conditions on g_n (see below). Our goal is to answer the following questions:

1) Can we find conditions on the subdivision mask $\{g_n\}$ such that uniform limit functions $\phi^s(x)$ exist for all s, yielding a "pseudo-wavelet" transform?

2) What are the conditions on g_n such that the obtained pseudo-wavelets (or the limit functions $\phi^s(x)$) are "smooth"? This amounts to determining e.g. the Hölder regularity of $\phi^s(x)$.
3) For $q > 1$, can we find conditions on g_n such that "pseudo-wavelets" are almost shifted versions of each other, i.e., $\psi_k((p/q)^j x) \approx \psi((p/q)^j x - k)$ within a smal error (to be estimated). This amounts to estimating $\epsilon = \inf_{s,s',x} |\phi^s(x+s) - \phi^{s'}(x+s')|$.

This paper is organized as follows. First, we introduce some useful polynomial notation. Then, we state a necessary and sufficient condition for uniform convergence, thereby answering question 1. We then answer question 2 by deriving sharp Hölder regularity estimates, which are known to be optimal in the case $q = 1$. This is a direct extension of Daubechies and Lagarias estimates [5] to the rational case, and a small example ("pseudo B-splines") is provided. Finally, as an answer to question 3, we show that regularity is also useful for achieving shift invariance within a small error.

§2. Notations and Preliminaries

In the sequel we shall heavily use the Laurent polynomial notation:

$$G(X) = \sum_n g_n X^n \tag{5}$$

for any finite sequence g_n. Polynomial multiplication amounts to performing a discrete convolution by the sequence g_n, which we shall denote by $\mathbf{G}$.

Equation (1) can be seen as resulting from three successive operations: first, insert $p-1$ zeros between every other sample of u_n (discrete dilation $U(X^p) = \mathbf{D}_p(U(X))$). Then convolve the result with g_n (operator $\mathbf{G}$), and finally retain every qth sample of the result (discrete contraction $w_{qn} = \mathbf{C}_q(w_n)$). In other words, $\mathcal{G} = \mathbf{C}_q \mathbf{G} \mathbf{D}_p$.

Using some flow graph algebra it can be easily shown that when p and q are coprime, the iterated operator $\mathcal{G}^j$ can be put in a similar form, namely $\mathcal{G}^j = \mathbf{C}_{q^j} \mathbf{G}^j \mathbf{D}_{p^j}$, where $\mathbf{G}^j$ corresponds to polynomial multiplication by

$$\begin{aligned} G^j(X) &= G(X^{q^{j-1}}) G(X^{pq^{j-1}}) \cdots G(X^{p^{j-1}q}) G(X^{p^{j-1}}) \\ &= \prod_{i=0}^{j-1} G(X^{p^i q^{j-1-i}}). \end{aligned} \tag{6}$$

By definition (4) we obtain the fundamental relation

$$g_n^{j,s} = g^j_{q^j n - p^j s}, \tag{7}$$

which can be used to compute a subdivision scheme $g_n^{j,s}$ from the sequence g_n^j associated to polynomial $G^j(X)$. Using (6), it is easily seen that these polynomials can in turn be computed recursively in two ways:

$$G^{j+1}(X) = G(X^{q^j}) G^j(X^p) = G^j(X^q) G(X^{p^j}). \tag{8}$$

In the following we shall use the notation $U^j(X) = \prod_i U(X^{p^i q^{j-1-i}})$ for any polynomial $U(X)$. The most general recursion formula for $U^j(X)$ is

$$U^{i+l}(X) = U^l(X^{p^i})U^i(X^{q^l}). \tag{8'}$$

§3. Uniform Convergence

Uniform convergence of $\{g_n^{j,s}\}$ (Definition 2) requires an important condition to be fulfilled by g_n, which was first mentioned in [7] and proved in [1].

Proposition 1. *If $g_n^{j,s}$ converges uniformly to $\phi^s(x) \not\equiv 0$, then*

$$\sum_k g_{n-pk} = 1$$

for all $n \in \mathbb{Z}$.

This condition can be rewritten as

$$G(1) = p \text{ and } \frac{1-X^p}{1-X} \text{ divides } G(X). \tag{9}$$

Now consider $F(X) = \frac{1-X^q}{1-X^p}G(X)$. It is easy to show that

$$F^j(X) = \frac{1-X^{q^j}}{1-X^{p^j}}G^j(X). \tag{10}$$

Hence the first-order differences $d_n^{j,s} = g_n^{j,s} - g_{n-1}^{j,s}$ follow a p/q-adic scheme with subdivision mask $F(X)$ (and initial sequence associated to $(1-X)X^s$).

Theorem 2. *The p/q-adic subdivision scheme $g_n^{j,s}$ converges uniformly to continuous limit function $\phi^s(x)$ for all s if and only if (9) holds and*

$$\max_n |g_{n+1}^{j,s} - g_n^{j,s}| \to 0 \text{ as } j \to \infty. \tag{11}$$

Proof: Condition (11) is clearly necessary. Now, using (10) we have $d_n^{j,s} = f^j_{q^j n - p^j s} - f^j_{q^j n - p^j s - p^j}$, hence $f^j_{q^j n - p^j s} = \sum_{k\geq 0} d_n^{j,s+k}$ where the sum is finite. Assuming (10) holds for all s it follows that $f^j_{q^j n - p^j s}$ tends uniformly to 0. On the other hand, using recursion (8') for $F^j(X)$ we end up with

$$\max_n |d_n^{i+l,s}| \leq \max_n \sum_k |f^j_{q^j n - p^j k}| \; \max_k |d_k^{l,s}|,$$

which by induction gives

$$\max_n |d_n^{j,s}| \leq c \left(\max_n \sum_k |f^i_{n-p^j k}| \right)^{j/i}.$$

Since $f^j_{q^j n - p^j s}$ tends uniformly to 0, there exists $\alpha > 0$ such that

$$\max_n |g^{j,s}_{n+1} - g^{j,s}_n| \le c\,(p/q)^{-j\alpha}, \tag{11'}$$

which is clearly stronger than (11). Now, by adapting a proof in [8], (11') implies

$$\max_{n_j} |g^{j+1,s}_{n_j+1} - g^{j,s}_{n_j}| \le c'(p/q)^{-j\alpha},$$

where n_j is chosen as in Definition 2. From this follows that $g^{j,s}_{n_j}$ is a uniform Cauchy sequence, hence converges to $\phi^s(x)$ which can be easily shown to be continuous. ■

Theorem 2 is powerful for proving uniform convergence for all s: in fact, it is sufficient that $\max_n \sum_k |f^j_{n-q^j k}| = (p/q)^{-i\alpha} < 1$ *for some* i, to ensure that all $\phi^s(x)$ are continuous. Moreover, this gives α $(0 < \alpha < 1)$, and as can be shown from (11'), the $\phi^s(x)$ are all Lipschitz of order α.

§4. Hölder Regularity Estimates

We say that $\phi^s(x)$ is *(Hölder) regular of order* $N + \alpha$ if it is C^N and its Nth derivative is Lipschitz of order α. The preceding section has given a condition on g_n such that $\phi^s(x)$ is regular of order α, where $0 < \alpha < 1$. For higher regularity orders we apply this result to Nth order finite differences of the $g^{j,s}_n$. The first-order difference is $\Delta g^{j,s}_n = (g^{j,s}_n - g^{j,s}_{n-1})/(p/q)^{-j}$, and Nth order finite differences are obtained by applying N times the operator Δ. First, we relate finite differences to derivatives of limit functions, using the following theorem proven in [1].

Theorem 3. *If $\Delta^N g^{j,s}_n$ converges uniformly, then so do $\Delta^k g^{j,s}_n$ for $0 \le k \le N$, whose respective limit functions are the kth-order derivatives of $\phi^s(x)$. Moreover, $\phi^s(x)$ is C^N and*

$$\left(\frac{1 - X^p}{1 - X}\right)^{N+1} \textit{divides } G(X) \tag{12}$$

Condition (12) is clearly a generalization of (9). Note that the subdivision mask associated to $\Delta^k g^{j,s}_n$ is

$$\left(\frac{p}{q}\frac{1 - X^q}{1 - X^p}\right)^k G(X),$$

hence, in some sense, multiplying the subdivision mask polynomial by $\frac{p}{q}\frac{1-X^q}{1-X^p}$ amounts to differentiating the limit function [1].

From Theorem 3 and the preceding section it follows that uniform convergence of the Nth-order finite differences $\Delta^N g^{j,s}_n$ can be tackled similarly as that of the $g^{j,s}_n$. That is, we have the following

Corollary 4. *If $G(X)$ satisfies (9), (12) and*

$$\max_n |\Delta^N g_{n+1}^{j,s} - \Delta^N g_n^{j,s}| \leq c\,(p/q)^{-j\alpha} \tag{13}$$

for some $\alpha < 1$, then $\phi^s(x)$ is regular of order $N + \alpha$.

In fact we can show that this result is valid for negative values of α (a proof for the case $p/q = 2$ can be found in [8]), which means that we can obtain positive regularity orders $N + \alpha$ by estimating (13) even though the $\Delta^N g_n^{j,s}$ diverges. In particular we can still apply this criterion for the maximal value of N such that (12) holds. This yields the following Hölder regularity estimate for p/q-adic subdivision schemes:

Theorem 5 (Hölder regularity estimate). *Assume $G(X)$ satisfies (9) and (12) and set*

$$F_N(X) = \left(\frac{p}{q}\right)^N \left(\frac{1 - X^q}{1 - X^p}\right)^{N+1} G(X). \tag{14}$$

Define the Hölder regularity estimate $N + \alpha_N^j$ (for a fixed value of j) by

$$\left(\frac{p}{q}\right)^{-j\alpha_N^j} = \max_{0 \leq n < p^j} \sum_k |(f_N^j)_{n-p^j k}|, \tag{15}$$

and let $\alpha = \sup_j \alpha_N^j$. Then for any j such that $N + \alpha_N^j > 0$, $\phi^s(x)$ is Hölder regular of order $N + \alpha_N^j$, for all s. Moreover α_N^j tends to α as $j \to \infty$, and $\phi^s(x)$ is Hölder regular of order $N + \alpha - \epsilon$ for any $\epsilon > 0$.

For $p/q = 2$ we recover the estimate derived in [8]. Moreover, for any value of p/q, we can rewrite (15) using p square matrices of small order $\mathbf{F}_r$:

$$\left(\frac{p}{q}\right)^{-j\alpha_N^j} = \max_{0 \leq r_i \leq p-1} \|\prod_{i=0}^{j-1} \mathbf{F}_{r_i}\|_1 \tag{15'}$$

where $\|\cdot\|_1$ denotes the ℓ^1 matrix norm and where

$$\mathbf{F}_r = \begin{pmatrix} f_r & f_{r-q} & f_{r-2q} & \cdots \\ f_{r+p} & f_{r+p-q} & f_{r+p-2q} & \cdots \\ \vdots & & & \ddots \end{pmatrix}. \tag{16}$$

This matrix is simply a convolution matrix in which one keeps every pth line and every qth column. For $p = 2$ and $q = 1$ we recover Daubechies and Lagarias estimate [5], which is shown to be optimal under very weak conditions on g_n in [8].

In general, using the matrix form we obtain lower and upper bounds on regularity as

$$N - \log_{p/q} \max_{0 \leq r < p} \|\mathbf{F}_r\| \tag{17}$$

and

$$N - \log_{p/q} \max_{0 \leq r < p} \rho(\mathbf{F}_r), \tag{17'}$$

respectively, where $\rho(A)$ denotes the spectral radius of matrix A.

§5. An Example: "Rational B-Splines"

This example is

$$G(X) = \frac{1}{p^N}\left(\frac{1-X^p}{1-X}\right)^{N+1} \tag{18}$$

for which $F_N(X)$ (14) is obtained by replacing p by q in (18). For $p/q = 2$, we recover B-spline functions, whose Hölder regularity is $N - \epsilon$ (see e.g. [8]). Let us derive Hölder regularity orders in the case $p/q = 3/2$ and $N = 1$, 2, and 3.

First, in the case $N = 1$, we obtain three 1×1 matrices $\mathbf{F}_r$, namely (1/2), (1), and (1/2). Hence it follows from Theorem 5 with (15') that all $\phi^s(x)$ are Hölder regular of order 1 (that is, almost continuously differentiable).

In the case $N = 2$, we obtain

$$\mathbf{F}_0 = \begin{pmatrix} 1/4 & 0 \\ 1/4 & 3/4 \end{pmatrix}, \qquad \mathbf{F}_1 = \begin{pmatrix} 3/4 & 0 \\ 0 & 3/4 \end{pmatrix}, \qquad \mathbf{F}_2 = \begin{pmatrix} 3/4 & 1/4 \\ 0 & 1/4 \end{pmatrix}.$$

Here lower and upper bounds (17) are equal to $2 - \log(3/4)/\log(3/2)$ which gives $2.709\ldots$ as the regularity order of the $\phi^s(x)$'s.

For $N = 3$, we obtain similarly three 3×3 matrices

$$\begin{pmatrix} 1/8 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/8 & 3/4 \end{pmatrix}, \begin{pmatrix} 1/2 & 0 & 0 \\ 1/8 & 3/4 & 1/8 \\ 0 & 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 3/8 & 1/8 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 0 & 1/8 \end{pmatrix}$$

whose norms and spectral radii are all equal to 3/4, hence the Hölder regularity order is $3 - \log(3/4)/\log(3/2) = 3.709\ldots$.

§6. Achieving Shift Invariance of Regular Pseudo-Wavelets

As pointed out in the introduction, the major difficulty in the rational case ($q > 1$) as opposed to the dyadic case is the lack of shift invariance of $\phi^s(x) \neq \phi(x-s)$. In other words, we obtain only a pseudo-wavelet transform [1] in which wavelet bases are not shifted versions of each other at a given resolution level. However, it can be shown that shift invariance is "almost" achieved within an arbitrary small error by taking N in (12) sufficiently large (thereby increasing the regularity order of the $\phi^s(x)$'s). The following theorem is due to Th. Blu (a proof is outlined in [2]), and explains a numerical observation made in [7].

Theorem 6. *Let $G(X)$ satisfy the conditions of Theorem 5, where N is given such that (12) holds. Then, provided that the $\phi^s(x)$ are regular for all s,*

$$\lim_{N\to\infty} \sup_{x,s,s'} |\phi^s(x+s) - \phi^{s'}(x+s')| = 0. \tag{19}$$

Acknowledgements. This work was done in part with Thierry Blu of CNET Paris B.

References

1. T. Blu, Iterated filter banks with rational rate changes. Connection with discrete wavelet transforms, IEEE Trans. Signal Processing, **41**-12 (1993).
2. T. Blu and O. Rioul, Wavelet regularity of iterated filter banks with rational sampling changes, IEEE Int. Conf. Acoust., Speech, Signal Processing, Minneapolis, MN (1993), Vol. III, 213–216.
3. A. Cohen and I. Daubechies, Orthonormal bases of compactly supported wavelets. III; Better frequency resolution, SIAM J. Math. Anal., **24**-2 (1993), 420–527.
4. I. Daubechies, Orthonormal bases of compactly supported wavelets, Com. Pure Applied Math., **XLI** (1988), 909–996.
5. I. Daubechies and J. Lagarias, Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals, SIAM J. Math. Anal. **23** no. 4 (1992), 1031–1079.
6. N. Dyn, Subdivision schemes in CADG, in *Advances in Numerical Analysis. II. Wavelets, Subdivision Algorithms and Radial Functions*, W. A. Light (ed.), Oxford University Press, 1991, 36–104.
7. J. Kovačević and M. Vetterli, Perfect reconstruction filter banks with rational sampling rate changes, in *Proc. IEEE Int. Conf. Acoust., Speech, Signal Processing*, Toronto, Canada (1991), 1785–1788.
8. O. Rioul, Simple regularity criteria for subdivision schemes, SIAM J. Math. Anal, **23**-6 (1992), 1544-1576.

O. Rioul
CNET Paris B
38-40, rue du Général Leclerc
92 131 Issy-Les-Moulineaux, FRANCE
rioul@balsa.issy.cnet.fr