Note on “A Remez Exchange Algorithm for Orthonormal Wavelets”

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Abstract


1 Introduction

Let us recall the two optimization problems described in the paper and introduce useful notations. The pass-band is denoted by $BP = [0, \omega_p]$. For variable $y = \cos^2 \omega$, we have $y \in I \iff \omega \in BP$. For any continuous function $f(\omega)$ depending on variable $\omega$ we write $f(y)$ the dependency of $f$ on variable $y$. The ambiguity should be easily resolved from the context. The following functional norm is used:

$$\|f\| = \max_{\omega \in BP} |f(\omega)| = \max_{y \in I} |f(y)|$$

1.1 Problem # 1

This is the initial optimization problem presented in section II.A of the paper: Given $L$, $0 < K \leq L/2$, and transition bandwidth $BP$, find the best trigonometric polynomial

$$P(\omega) = 1 + \sum_{n=1}^{L/2} a_n \cos(2n-1)\omega$$

($L/2$ variables $a_n$) such that $\delta$ (the tolerance in the pass-band) is minimized subject to the constraints of magnitude specification

$$\|2 - \delta - P\| \leq \delta$$

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and $K$th order flatness (equations (10)-(11) in the paper). The latter constraints leaves

$$N = L/2 - K$$

independent variables $a_n$.

Since this is a classical linear program whose set of constraints is not empty (maximally flat solution $P_K(\omega)$ satisfies the constraints for some $\delta > 0$) optimal solutions $\hat{P}(\omega)$ exist. Note: $\hat{P}(\omega)$ is a priori not unique. Any optimal solution $\hat{P}(\omega)$ has pass-band tolerance equal to

$$\delta = \min_{P(\omega)} \delta$$

which is unique by definition.

### 1.2 Problem # 2

Reformulation of problem # 1 is done in section III of the paper. The main difference with problem # 1 is that $\delta$ is fixed to some value. Given this value of $\delta$, problem # 2 is to find the best $P(\omega)$ which minimizes $\|2 - \delta - P\|$ subject to the $K$ flatness constraints.

As explained in the paper, this is equivalent to the following.

$$\min_{R(y)} \|E\|$$

where $R(y)$ is a polynomial of degree $N - 1$ and

$$E(y) = W(y)(D(y) - R(y)).$$

Here

$$D(y) = \frac{D_0(y) - \delta}{W(y)}$$

where $D_0(y)$ and $W(y) \geq 0$ are continuous functions of $y$ and do not depend on $\delta$.

From section IV of the paper, to each value of $\delta$ corresponds

- A unique optimal solution $P^*(\omega)$ or $R^*(y)$ characterized by $N + 1$ alternations $y_1 < y_2 < \cdots < y_{N+1}$ in $I$:

$$E(y_i) = \pm (-1)^i \|E\|$$

- A number

$$\delta^* = \|E^*\| = \min_{R(y)} \|E\| = \min_{P(\omega) flat} \|2 - \delta - P\|$$

Note that implicitly there is an application $^*$ that maps $\delta > 0$ to $\delta^* > 0$.

Keep in mind notations $\hat{P}$, $\delta$, $P^*$, $E^*$, $\delta^*$. They will be constantly used in this note.
2 Preliminaries

The issue of the paper is to solve problem # 1 using a Remez exchange algorithm for problem # 2. In this section we state the precise connections between these two problems.

Lemma 1 For problem # 2 we always have
\[ \delta \leq \delta^* \]

Proof: Because of the flatness constraints we always have \( P(\omega = 0) = 2 \), hence
\[ \delta^* = \| 2 - \delta - P^* \| \geq | 2 - \delta - P^*(0) | = \delta \]

\( \square \)

Proposition 1 \( \bar{P}(\omega) \), the optimal solution for problem # 1, is unique. It is also the optimal solution for problem # 2 where \( \delta \) is set to \( \bar{\delta} \), and we have \( \bar{\delta} = \delta^* \).

Proof: We have \( \bar{\delta} \leq \delta^* \) from the preceding lemma. But \( \delta^* = \min_{\bar{P}(\omega)} \| 2 - \delta - \bar{P} \| \) and the latter quantity is \( \leq \bar{\delta} \) because of the constraints of problem # 1 for \( \bar{P}(\omega) \). Hence \( \delta^* = \bar{\delta} \) and \( \| 2 - \delta - \bar{P} \| = \min_{\bar{P}(\omega)} \| 2 - \delta - \bar{P} \| \) for any optimal solution \( \bar{P} \) of problem # 1. This means that \( \bar{P}(\omega) \) is the optimal solution to problem # 2 for \( \delta = \bar{\delta} \), and is therefore unique. \( \square \)

Proposition 2 Let \( P^*(\omega) \) be the solution to problem # 2 for some \( \delta \). Then it satisfies the constraints of problem # 1 if and only if \( \delta = \delta^* \). In particular, \( \delta \geq \bar{\delta} \).

Proof: If \( P^* \) satisfies the constraints of problem # 1 then \( \delta \geq \bar{\delta} \) and
\[ \delta^* = \min_{\bar{P}(\omega)} \| 2 - \delta - \bar{P} \| \leq \| 2 - \delta - \bar{P} \| \]
\[ \leq \| 2 - \bar{\delta} - \bar{P} \| + \bar{\delta} - \delta \]
\[ \leq \delta^* + \bar{\delta} - \bar{\delta} \]

but \( \bar{\delta} = \bar{\delta} \) from the preceding lemma so we end up with \( \delta^* \leq \delta \). But from the first lemma \( \delta^* \geq \delta \), so \( \delta^* = \delta \).

Conversely, assume \( \delta = \delta^* \), i.e., \( \delta = \| 2 - \delta - P^* \| \). Then because of this equality \( P^* \) satisfies the constraints of problem # 1.

This result shows that there is hope in solving problem # 1 using problem # 2 provided the optimal solution is such that \( \delta = \delta^* \).
3 Modified Remez algorithm

Let us summarize the proposed algorithm as it was described in section VI of the paper.

At the \( n \)th iteration, we are given \( N + 1 \) critical points \( y^n_i, i = 0, \ldots, N \). Then equation
\[
E_n(y^n_i) = \pm (-1)^i \delta_n
\]
where
\[
E_n(y) = W(y)(D_n(y) - R(y))
\]
and
\[
D_n(y)W(y) = D_0(y) - \delta_n
\]
suffices to determine \( \delta_n \) and \( R(y) = R_n(y) \) uniquely.

Note that this does not mean that \( R_n(y) \) is the optimal solution to problem \# 2 for \( \delta = \delta_n \), since \( \delta_n < \| E_n(y) \| \) in general.

From here a multiple exchange procedure gives the next critical points \( y^{n+1}_i \) in such a way that for all \( i \),
\[
E_n(y^{n+1}_i) \geq \delta_n
\]
and there exists \( i_0 \) such that
\[
|E_n(y^{n+1}_{i_0})| = \| E_n \|.
\]

From here another iteration starts.

The purpose of this note is to show that

1. Convergence holds to \( R_\infty(y) = \lim_{n \to \infty} R_n(y) \), corresponding to \( P_\infty(\omega) \) whose tolerance in the pass-band is \( \delta_\infty = \lim_{n \to \infty} \delta_n \).

2. At convergence, we have \( \delta_\infty = \delta^*_\infty = \bar{\delta} \) hence the obtained solution \( P_\infty(\omega) \) is indeed the optimal solution \( \bar{P}(\omega) \) of initial problem \# 1.

4 Analysis of convergence

**Lemma 2** Let \( R_n^*(y) \) be the optimal solution of problem \# 2 for \( \delta = \delta_n \). If \( \delta_n = \delta^*_n \) then \( R_n(y) = R_n^*(y) \).

**Proof:** We have seen that \( R_n(y) \) is determined by equation \( E_n(y^n_i) = \pm (-1)^i \delta_n \), where \( \delta_n = |E_n(y^n_i)| \). Now, this is exactly the alternation theorem for the “discrete” problem
\[
\min_{R(y)} \max_{y \in \{y^n_i\}} |E_n(y)|
\]
Indeed Chebyshev’s alternation theorem still applies for \( I = \{y^n_i\} \) where all \( y^n_i \) are alternations! Therefore, \( R_n(y) \) is the unique solution to the discrete problem.
and we have
\[
\delta_n = \min \max_{R(y)} \max_i |E_n(y_i^n)| \\
\leq \max_i |E_n^*(y_i^n)| \\
\leq \|E_n^*\| = \delta_n^*.
\]

Since \(\delta_n = \delta_n^*\) it follows that \(\min_{R(y)} \max_i |E_n(y_i^n)| = \max_i |E_n^*(y_i^n)|\). This means that \(R_n^*(y)\) is also the optimal solution to the discrete problem, hence by uniqueness \(R_n(y) = R_n^*(y)\).

**Proposition 3** As long as we did not converge, \(\delta_n < \tilde{\delta}\), hence \(\delta_n\) is bounded for all \(n\). Moreover \(\delta_n < \|E_n\|\).

**Proof:** Suppose \(\delta_n \geq \tilde{\delta}\). From proposition II.3, this implies \(\delta_n = \delta_n^*\). Then by the preceding lemma we would have \(R_n(y) = R_n^*(y)\) and therefore \(\delta_n = \delta_n^* = \|E_n^*\| = \|E_n\|\). But then in the multiple exchange procedure we would find \(y_{n+1} = y_n\): the critical points, hence \(\delta_n\) and \(R_n(y)\) are stationary, which means that we have converged (in a finite number of steps).

Since from the preceding discussion \(\delta_n < \delta_n^*\), and \(\delta_n^* = \|E_n^*\| \leq \|E_n\|\) it follows that \(\delta_n < \|E_n\|\).

**Lemma 3** There exist \(N + 1\) numbers \(\lambda_i \geq 0\) satisfying \(\sum_i \lambda_i = 1\) such that
\[
\delta_n = \sum_{i=0}^N \lambda_i |E_n(y_i^n)|
\]
where \(E_n(y) = W(y)(D_n(y) - R(y))\), for any polynomial \(R(y)\) of degree \(\leq N - 1\).

**Proof:** This follows, of course, from the definition of \(\delta_n\) if \(R(y) = R_n(y)\). This lemma states that \(R_n(y)\) can in fact be replaced by any polynomial \(R(y)\) of degree \(\leq N - 1\). Since by definition, \(E_n(y_i^n)\) has alternating signs, if suffices to choose \(\lambda_i\) such that \(\sum_i (-1)^i \lambda_i W(y_i^n) R(y_i) = 0\) for any \(R(y)\).

Using Lagrangian interpolation formula we have \(R(y_i^n) = \sum_{i=0}^N L_i(y_i^n) R(y_i^n)\) for any \(R(y)\) of degree \(\leq N - 1\), where \(L_i(y) = \prod_{j \neq i}^{N} \frac{y - y_j}{y_i^n - y_j}\). Set \(a_i = \prod_{j \neq i}^{N} (y_j^n - y_i^n)\) where index \(j\) goes from 0 to \(N\). Then \(L_i(y_i^n) = -a_0/a_i\) and we have \(\sum_{i=0}^N (1/a_i) R(y_i) = 0\) for any \(R(y)\). A solution is given by \(\lambda_i = |\mu_i|/(\sum |\mu_i|)\) where \(1/\mu_i = a_i W(y_i^n)\) has the same sign as \(\pm (-1)^i\) since \(a_i\’s\) have alternating signs and \(W(y) \geq 0\).

Remark. The equation used for \(\delta_n\) in the paper follows from this derivation by setting \(R(y) = 0\).
Proposition 4  As long as we did not converge, $\delta_n < \delta_{n+1}$. From the lemma IV. 2, it follows that $\delta_n$, a bounded increasing sequence, converges as $n \to \infty$.

Proof: Use the preceding lemma for the expression of $\delta_{n+1}$, where we set $R(y) = R_n(y)$. We obtain
\[
\delta_{n+1} = \sum_i \pm (-1)^i \lambda_i W(y_i^{n+1})(D_{n+1}(y_i^{n+1}) - R_n(y_i^{n+1}))
\]
Since $W(y)D_{n+1}(y) = W(y)D_n(y) + \delta_n - \delta_{n+1}$, we obtain
\[
\delta_{n+1} = \sum_i \lambda_i |E_n(y_i^{n+1})| + (\delta_n - \delta_{n+1}) \sum_i \varepsilon_i \lambda_i
\]
where $\varepsilon_i = \pm (-1)^i$.

After multiple exchange described in section III, we have
\[
\delta_{n+1} \geq \delta_n + \lambda_i (\|E_n\| - \delta_n) + (\delta_n - \delta_{n+1}) \sum_i \varepsilon_i \lambda_i
\]
It follows that
\[
\delta_{n+1} - \delta_n \geq \alpha_{n+1}(\|E_n\| - \delta_n)
\]
where $\alpha_{n+1} = \lambda_{i_0}/\sum_i (1 - \varepsilon_i) \lambda_i > 0$. By proposition IV.2, $\|E_n\| > \delta_n$, so the proof is complete.

5 Finding the solution to the initial problem

Let $\delta_\infty = \lim_{n \to \infty} \delta_n$. From propositions IV.2 and 4 this limit exists, is positive and is $\leq \delta$. In the sequel we prove that $\delta_\infty = \delta$, and the final result will follow.

Lemma 4  Critical points $y_i^n$ always stay within a certain distance to each other as $n \to \infty$. That is, $\inf |y_{i+1}^n - y_i^n| > 0$.

Proof: Otherwise there would be a converging subsequence of $y_i^n$, which we denote by $y_{m}^n$, whose limit is $\tilde{y}_i$, such that $\tilde{y}_i = \tilde{y}_{i+1}$. Hence for $i = 0, \ldots, N$, there are at most $N$ distinct values in the set $\{\tilde{y}_i\}$. Therefore, there exists $R(y)$, a polynomial of degree $\leq N - 1$, such that $R(\tilde{y}_i) = D_\infty(\tilde{y}_i)$, i.e., $\tilde{E}_\infty(\tilde{y}_i) = 0$.

Then given arbitrarily small $\varepsilon$ and for $m$ large enough:

- $E_m(y_{m}^n) = \pm (-1)^i \delta_m$ where $\delta_m > 2\varepsilon$ (since $\delta_\infty > 0$).
- $|\tilde{E}_\infty(y_{m}^n)| < \varepsilon$ since $\tilde{E}_\infty(y)$ is continuous and $\tilde{E}_\infty(\tilde{y}_i) = 0$.
- $|\delta_\infty - \delta_m| < \varepsilon$ since $\delta_m \to \delta_\infty$. 

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Therefore,

\[ R_m(y^m_i) - \hat{R}(y^m_i) = \frac{1}{W(y^m_i)}(\hat{E}_\infty(y^m_i) - E_m(y^m_i) + \delta_\infty - \delta_m) \]

has same sign as \( \pm(-1)^i \). The polynomial \( R_m(y) - \hat{R}(y) \) oscillates at \( N + 1 \) distinct points \( y^m_i \), hence it has \( N \) distinct zeroes. Since it is of degree \( N - 1 \), we must have \( \hat{R}(y) = R_m(y) \). But this is impossible since it implies that \( \delta_m = |E_m(y^m_i)| = |\hat{E}_m(y^m_i)| < \varepsilon \), hence \( \delta_m \to \delta_\infty = 0 \) whereas \( \delta_m \) is strictly increasing. \( \Box \)

**Theorem 1** We have \( \delta_\infty = \delta \) and \( P_n(\omega) \) in the modified Remez exchange algorithm converges to \( \bar{P}(\omega) \), the optimal solution to initial problem \# 1.

**Proof:** From the proof of proposition IV.4 we have \( \|E_n\| - \delta_n \leq \frac{1}{\alpha_n+1}(\delta_{n+1} - \delta_n) \) where from the expression giving \( \alpha_{n+1} \) and from the preceding lemma, there exists \( \alpha > 0 \) such that \( \alpha_{n+1} \geq \alpha > 0 \). This shows that \( \|E_n\| - \delta_n \) tends to zero, hence \( \|E_n\| = \|2 - \delta_n - P_n(\omega)\| \to \delta_\infty \).

Now let \( P_m(\omega) \) be a converging subsequence of \( P_n(\omega) \), whose limit is denoted by \( P_\infty(\omega) \). From the preceding discussion we have \( \|2 - \delta_\infty - P_\infty(\omega)\| = \delta_\infty \), hence \( \delta_\infty = \min_{P(\omega)} \|2 - \delta_\infty - P(\omega)\| \leq \delta_\infty \). Therefore, by lemma II.1, \( \delta_\infty = \delta_\infty^* \). This implies \( \delta_\infty \geq \delta \) by proposition II.3. But since \( \delta_\infty \leq \bar{\delta} \), we obtain \( \delta_\infty = \delta \).

Moreover \( \delta_\infty = \delta_\infty^* \) can be rewritten as \( \|2 - \delta - P_\infty\| = \min_{P(\omega)} \|2 - \delta - P\| \) which shows that \( P_\infty(\omega) \) is the optimal solution of problem \# 2 for \( \delta = \bar{\delta} \), hence by proposition II.2, \( P_\infty(\omega) = \bar{P}(\omega) \). Thus, we have shown that any converging subsequence of \( P_n(\omega) \) converges to \( \bar{P}(\omega) \).

Now if \( P_n(\omega) \) did not converge, there would exist a subsequence \( P_m(\omega) \) such that \( \|\bar{P}(\omega) - P_m(\omega)\| \geq \varepsilon > 0 \) for any \( m \) large enough. But from \( P_m(\omega) \) we could extract a converging subsequence whose limit would be \( \bar{P}(\omega) \), and we would have a contradiction: \( \|\bar{P}(\omega) - \bar{P}(\omega)\| = 0 \geq \varepsilon > 0 \). Therefore the whole sequence \( P_n(\omega) \) converges to \( \bar{P}(\omega) \). \( \Box \)