A Discrete-Time Approach to Regularity of 1D or 2D Wavelets

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THE discrete wavelet transform (DWT), implemented as a classical octave-band filter bank, expands a signal x_n using "basis functions" of the type

$$g_{j,k}(n) = g^j(n - 2^j k), \quad k \in \mathbf{Z}, \, j = 1, 2, 3, \ldots \tag{1}$$

where j is the level of decomposition, k is a shift parameter, and $g^j(n)$ is the equivalent impulse response at level j whose z-transform is defined by iterations of the type

$$G^j(z) = G(z)G(z^2)G(z^4)\ldots G(z^{2^{j-1}}). \tag{2}$$

In two dimensions [1], any perfect reconstruction filter bank iterated on the low-pass component can be seen as a DWT whose basis functions are

$$g_{j,K}(N) = g^j(N - \mathbf{D}^j K), \quad K, N \in \mathbf{Z}^2, \, j = 1, 2, 3, \ldots \tag{3}$$

where $\mathbf{D}$ is a dilation matrix of size 2×2 with integer entries. Its determinant gives the number of subbands at each stage of the decomposition [2]. The simplest example yielding to nonseparable filters is the well-known quincunx decomposition [1] for which only two subbands are used: $|\det \mathbf{D}| = 2$. From now on, we consider the quincunx case in two dimensions.

While perfect reconstruction does not depend on the particular matrix $\mathbf{D}$ which generates the quincunx lattice $\Gamma = \mathbf{D}\mathbf{Z}^2$, the basis functions depend critically on $\mathbf{D}$; the corresponding two-dimensional $\mathcal{Z}$-transform ($\mathcal{Z} = (z, w)$) is of the form

$$G^j(\mathcal{Z}) = G(\mathcal{Z})G(\mathcal{Z}^{\mathbf{D}})G(\mathcal{Z}^{\mathbf{D}^2})\ldots G(\mathcal{Z}^{\mathbf{D}^{j-1}}) \tag{4}$$

where the monomial $\mathcal{Z}^{\mathbf{D}^j N}$, $N \in \mathbf{Z}^2$, stands for $z^p w^q$, $\mathbf{D}^j N = (p, q)$.

In this particular context, the novelty of wavelet theory comes down to the choice of filters present in the filter bank: "Wavelet" filters are *regular*. This means that the impulse responses $g^j(n)$ converge, as $j \to \infty$, to a continuous function $\varphi(t)$ (or $\varphi(x, y)$ in two dimensions), where $n - 2^j t$ ($N - \mathbf{D}^j(x, y)$ in two dimensions) remains bounded for all j. The regularity order is the number of times φ is continuously differentiable. Regularity thus requires smooth evolutions of the discrete-time basis functions; several intuitive arguments have been raised to hint that this property should be useful in image coding applications. However, understanding the role of regularity in a DWT-based compression scheme requires precise evaluation of it.

A major difficulty is that it is a mathematical notion which is expressed on the limit function rather than on the filter taps. It is therefore useful to characterize regularity in the discrete-time domain. One can show, for example, the resulting limit function is continuous if the discrete-time sequences converge *uniformly* [3]. A necessary condition for continuity, that can be proved to be sufficient in one dimension, is that finite differences of the form $|g^j(n+1)-g^j(n)|$ (1-D case), or $|g^j(N+U)-g^j(N)|$, where U is set to two independent directions of $\mathbf{Z}^2$, tends to zero as $j \to \infty$ uniformly over time or space, and that moreover the frequency response of the low-pass filter vanishes at π (1D case) or (π, π) (2-D case).

To estimate regularity more precisely, it is also useful to extend regularity orders to arbitrary, real-valued numbers. The correct way to do this is to use the Hölder definition of regularity: φ is regular of order α, $0 < \alpha < 1$, if

$$|\varphi(x+h) - \varphi(x)| < c||h||^{\alpha}, \tag{5}$$

where c is a constant independent of x and h. A similar definitions holds in two dimensions. (Higher real-valued regularity orders are defined similarly on the derivatives or partial derivatives of φ.) Now, the (natural) discrete-time characterization of regularity order α is, in 1D,

$$|g^j(n+1) - g^j(n)| < c2^{-j\alpha}. \tag{6}$$

In two dimensions, (6) becomes

$$|g^j(N+U) - g^j(N)| < c\rho^{-j\alpha} \tag{7}$$

where U is as above and ρ is the spectral radius of $\mathbf{D}$. Here the choice of $\mathbf{D}$ is important: (7) indeed implies regularity order α if $\rho > 1$. A typical choice is $\mathbf{D} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ for which $\rho = \sqrt{2}$. Other choices, which yield the same quincunx sublattice Γ, but for which $\rho \leq 1$, do not even yield to convergence toward φ.

Characterizations like (6) and (7) yields sharp estimates of regularity orders. In one dimension, we can provide regularity estimates that are, in contrast with earlier ones that can be found in the literature, easily implementable, optimal, and of general applicability [3,4]. The two-dimensional case is trickier because the factorization theorem of $\mathcal{Z}$-transforms is lost. However, similar techniques can be investigated.

In conclusion, we have shown that a discrete-time approach to regularity can be used to estimate it sharply. The determination of whether regularity is indeed useful in image compression applications then becomes possible (for example, one could integrate regularity in a filter design procedure), and remains a topic for future investigation.

References

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