Structures and Algorithms for the Orthonormal Discrete Wavelet Transform

Olivier Rioul

Centre National d'Etudes des Télécommunications - CNET/PAB/RPE
38-40 rue du Général Leclerc - 92131 Issy-Les-Moulineaux - France
Tel: + 33 1 45 29 43 37

Abstract - *Several structures for the Discrete Wavelet Transform (DWT), based on a new discrete-time description of multiresolution signal decomposition, are presented. One such structure is a self-transposed tree-iterated QMF Filter Bank in which orthonormality of the wavelets is equivalent to perfect reconstruction. Structure constraints also permit to investigate a specific reconstruction after quantization problem, namely a condition for maximized reconstruction SNR at fixed bit rate. Finally, an FFT method is used to speed up calculations.*

1. Multiresolution Approximations.

Multiresolution decompositions of signals is best described using *scaling operators* that transforms the signal to similar versions at half or twice its scale. The most general expressions are

Down-Scaling: $$x[k] \longrightarrow \sum_k x[k]\, g[n-2k] \quad (1)$$

Up-Scaling: $$c[n] \longrightarrow \sum_n c[n]\, g^*[2k-n] \quad (2)$$

where g is the impule response of a low-pass filter (called *scaling filter*). Several types of description, in terms of z-transforms, operators or vector decomposition may be used. This makes this presentation of *scale* a very powerful one.

For exemple, (1) means filtering + decimation whereas (2) performs up-sampling followed by filtering. In terms of basis decomposition, we have: (1) performs an inner product of the signal with shifted versions of $\{g[n]\}$ by $2k$ (called *scaling functions*). When applied to the result of (1), (2) accordingly sums the *projections* on these scaling functions to give an *approximation* of $\{x[n]\}$ at twice its *resolution*: $\{A(x)[n]\}$, after a "roundtrip" to half the scale has been made.

A fundamental requirement is that $A(x)$ is left unchanged when re-approximated by A. The basis functions $g[.-2k]$ are then mutually *orthogonal* and A is an orthogonal projection (more details may be found in [1], especially for biorthonormal bases [2].)

Due to self-similarity at all scales, multiresolution schemes have only to be described at one level of decomposition: the whole process then results from simple iteration.

2. Orthonormal DWT and Filters Banks with Perfect Reconstruction

One level of wavelet decomposition can be written as $x = A(x) + B(x)$, where A is the orthogonal projection on the scaling functions and where B is *complementary* orthogonal projection defined from a *high-pass* filter h in a similar way as A is defined from g. $B(x)$ is supported by the so-called *wavelets* $h[\,.\,-2k]$.

In z-transform notations, we considerer the two-band *self-transposed* filter bank depicted in Fig. 1. It performs one level of wavelet decomposition/reconstruction and orthonormality of wavelets is here equivalent to perfect reconstruction. The complete DWT structure is then obtained by simple iteration on the low-pass components which provides a tree-structure algorithm.

3. Reconstruction after Quantization.

Assume the wavelet coefficients are quantized for each resolution 2^{-j} ("octave") on B_j bits, and

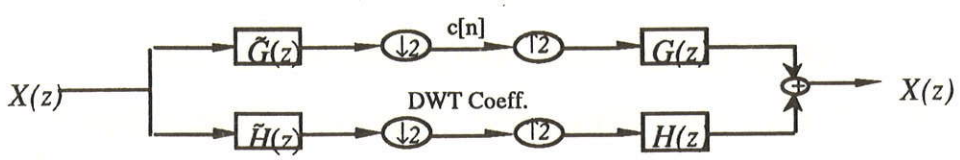

Fig. 1. The elementary cell of the DWT/IDWT composed of a self-transposed two-band filter bank structure.

that the quantization step is small so that the standard deviation σ_j is proportional to 2^{-Bj}, even for a non-uniform quantizer[1]. A simple model for the quantization noise (ergodic processes, expectation zero), along with the study of the complete DWT structure, permits to compute the reconstruction error's variance as $\Sigma\, 2^{-j}\sigma_j$. Minimization with constant bit rate $\Sigma 2^{-j}B_j$ yields the result that the quantizers should be chosen so as their error variance σ_j are mutually *equal* [1]. The minimized error's variance is then the common value of the σ_j's. Also, orthogonality relations imply this error to be second order stationnary white.

This result is based on the sole SNR criterium and is independent of the way the wavelets are designed. Other criteria, e.g. perceptual criteria on the reconstructed image for 2D decomposition, may involve some interesting properties of the wavelets such as "filter regularity" [2], [3].

4. FFT-Based Computation of the DWT.

When the DWT is computed with direct implementation of the QMF structure with filters $G(z)$ and $H(z)$ of length N, the subsampling operation at each stage already allows for a reasonable small complexity of $2N$ multiplications per point for the whole DWT, even for a large number of octaves. Further reduction can be achieved by convolving using FFTs (overlap-add or overlap-save techniques). The elementary cell of the DWT depicted in Fig. 2 is used instead that of Fig. 1, in order to make "true" filters and demultiplexing operations (i.e. even and odd sequences' extraction) appear in the structure.

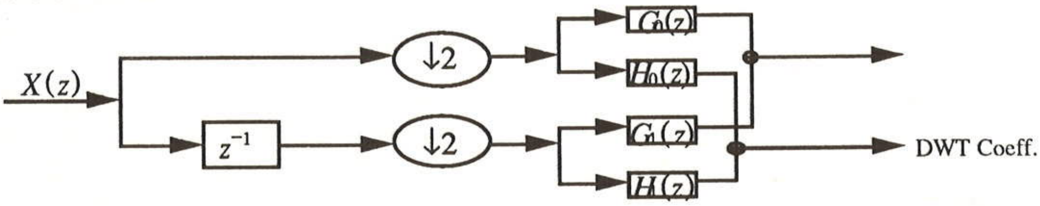

Fig. 2. The elementary cell of the DWT composed of a demultiplexing operation and four decimated filters

The idea is to perform all calculations in the Fourier domain. Vetterli's method [2] halves the FFT lengths at each stage due to down-sampling. The length of the initial FFT has therefore to be taken exponentially large (depending on the number of octaves on which the DWT is computed) so that wrap-around effects are avoided down to the last stage. The resulting algorithm is efficient only for a small number of octaves and requires storage of many filter DFTs.

These drawbacks can be overcome by gathering two subsequent length-L FFT blocks at each stage before performing the convolutions. Hence all FFTs used are of same length and are applied to filters of same order; their length can therefore be fully adapted to the order of the filter. Assuming that a split-radix FFT algorithm is used, we found that the overall DWT requires less than $2\, log_2 N$ multiplications per point (and a comparable numner of additions) independently of the number of octaves, instead of $2N$ mults per point in the direct implementation method.

References

[1] O. Rioul, "A Unifying Multiresolution Theory for the Discrete Wavelet Transform, Regular Filter Banks and Pyramid Transforms," submitted to *IEEE Trans. Acoust. Speech, Signal Proc.*

[2] M. Vetterli and C. Herley, "Wavelet and Filter Banks: Relationships and New Results," in *Proc. 1990 IEEE Int. Conf. Acoust. Speech, Signal Proc.*, Albuquerque, NM, Apr. 3-6, 1990.

[3] I. Daubechies, "Orthonormal Bases of Compactly Supported Wavelets," *Comm. in Pure and Applied Math.*, Vol.41, No.7, pp.909-996, 1988.