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AFFINE SMOOTHING OF THE WIGNER-VILLE DISTRIBUTION

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ABSTRACT

A new formalism of signal energy representations depending on time and scale is presented as an alternative method to time-frequency representations. In addition, precise links between time-frequency and time-scale energy distributions are provided. It is known that a full description of the former is given by the Cohen's class, which can be described as a generalization of the spectrogram appropriately parameterized by a smoothing function acting on the Wigner-Ville distribution. In the present paper, we provide a full description of the latter, resulting in a new class of representations in which the smoothing of the Wigner-Ville distribution is scale-dependent. Through proper choices of the smoothing function, interesting properties may be imposed on the representation, which makes it a versatile tool for the analysis of nonstationary signals. Also, specific choices allow to recover known definitions (including the Bertrands' and the energetic version of the wavelet transform, referred to as the «scalogram»). Another, very flexible, choice uses separable smoothing functions: it is shown, in particular, that Gaussian kernels provide a continuous transition between spectrograms and scalograms *via* Wigner-Ville.

I - THE SHORT-TIME FOURIER TRANSFORM AND THE WAVELET TRANSFORM

Given a finite energy signal $x(t)$ and a sliding window $h(t)$, a classical linear *time-frequency* representation can be obtained by computing the *short-time Fourier transform* (STFT):

$$F_x(t, \nu) \triangleq \int_{-\infty}^{+\infty} x(u)\, h^*(u - t)\, e^{-i2\pi\nu u}\, du \ .$$

In recent years, an alternative representation, called the *wavelet transform* (WT), has been widely addressed in the literature [1-2]. The fundamental idea here is to replace the frequency *shifting* operation which occurs in the STFT by a time (or frequency) *scaling* operation. The resulting definition is

$$T_x(t, a) \triangleq \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(u)\, h^*\!\left(\frac{u - t}{a}\right) du \ .$$

The function $h(t)$ (called the *analyzing wavelet*) is supposed to have some localization properties in time. The explicit dependence of this definition on the dilation/compression (or *scale*) parameter a makes the WT a *time-scale* representation rather than a time-frequency one [3].

As for the STFT, the WT may be inverted provided that the Fourier transform $H(\nu)$ of the analyzing wavelet $h(t)$ satisfies [1]

$$\int_{-\infty}^{+\infty} |H(\nu)|^2 \frac{d\nu}{|\nu|} < +\infty .$$

This means that $h(t)$ is the impulse response of some band-pass filter. In the time domain, its mean value must be zero, which implies that $h(t)$ will oscillate, hence the name *wavelet*. Both transforms analyze the signal by means of an inner product with analyzing waveforms depending on two parameters. In the WT case, the waveforms onto which the signal is decomposed is generated from the analyzing wavelet $h(t)$ by time-shift and dilation operations and are referred to as the *wavelets*.

The main difference between the STFT and the WT is related to the structure of their respective analyzing waveforms. The former uses modulated versions of a *low-pass* filter to explore the spectral content of the analyzed signal (*uniform* filterbank). This amounts, in the time-domain, to using an analyzing waveform of constant envelope with an increasing number of oscillations as higher frequencies are analyzed. The latter uses dilated or compressed versions of a *band-pass* filter, whose relative bandwidths are constant (*constant-Q* filterbank). Therefore, time evolutions of signals are analyzed by means of a waveform whose envelope is narrowed as higher frequencies are analyzed, whereas its number of oscillations, hence its shape, remains constant.

II - SPECTROGRAMS AND SCALOGRAMS

A. Definitions and comparison

Owing to their definition, STFTs and WTs are complex-valued functions and they convey both modulus and phase informations. For some applications [4], these latter can be of interest but a description based only on the squared modulus, providing an energy density distribution, is often preferred. Indeed, the *spectrogram* $|F_x(t, \nu)|^2$, defined as the energy distribution associated to the STFT, has been widely used for many signal processing tasks. A similar quantity, $|T_x(t, a)|^2$, can be defined in the case of the WT: we propose to refer to it as a *scalogram*.

A classical time and frequency resolution trade-off underlies the structure of the spectrogram: the choice of an analyzing window of short duration ensures a good time localization, but at the expense of a poor frequency resolution (by Fourier duality), and *vice-versa*. Moreover, once an analyzing window has been chosen, the resolution capabilities of the spectrogram remain fixed all over the time-frequency plane. The situation is different for scalograms: owing to the constant-Q structure described above, resolution capabilities are *frequency-dependent.*

B. Smoothing interpretation within the Cohen's class

Both spectrograms and scalograms have a bilinear dependence on the analyzed signal. As shown below, a simple interpretation may be given considering the general class of bilinear (shift-covariant) time-frequency energy distributions. Recall that this class, referred to as *Cohen's* [5], is given by

$$C_x(t, \nu; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\, \Pi(u - t, n - \nu)\, du\, dn \quad , \ (1)$$

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where $\Pi(t, \nu)$ is some arbitrary time-frequency function and where

$$W_x(t, \nu) = \int_{-\infty}^{+\infty} x(t + \frac{\tau}{2})\, x^*(t - \frac{\tau}{2})\, e^{-i2\pi\nu\tau}\, d\tau$$

is the so-called *Wigner-Ville distribution* (WVD) [5]. If $\Pi(t, \nu)$ behaves like a low-pass function in the time-frequency plane, the general class (1) may be considered as composed of *smoothed* versions of the WVD. It is convenient to introduce 2D Fourier transformations in (1). Changing variables accordingly yields a dual characterization:

$$C_x(t, \nu; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(n, \tau) A_x(n, \tau)\, e^{-i2\pi(nt + \tau\nu)}\, dn\, d\tau\ ,$$

where the *weighting function* $f(n, \tau)$ and the (narrowband) *ambiguity function* [5] $A_x(n, \tau)$ are the 2D Fourier transforms of $\Pi(t, \nu)$ and $W_x(t, \nu)$, respectively .

Structure constraints of spectrograms and scalograms can be made explicit using members of the Cohen's class. The following proposition gives a well-known example of this for the spectrogram [6-7] :

Proposition 1. *For time-frequency energy distributions characterized by a weighting function of modulus unity, a spectrogram results from the smoothing of the signal distribution by the window distribution* :

$$|f(n, \tau)| = 1 \iff |F_x(t, \nu)|^2 =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_x(u, n; \Pi)\, C_h^*(u - t, n - \nu; \Pi)\, du\, dn\ . \quad (2)$$

This condition is met by many distributions, including the class of *generalized Wigner distributions* [7] and, in particular, the *Rihaczek's* distribution [5] and the WVD itself.

We state a similar specification of scalograms from Cohen's distributions [6]:

Proposition 2. *For time-frequency energy distributions characterized by a weighting function of modulus unity which depends on its variables only through their product, a scalogram results from the affine smoothing of the signal distribution by the wavelet distribution* :

$$\{\exists\ \phi(.)\ /\ f(n, \tau) \triangleq \phi(n\tau) \text{ and } |\phi(n\tau)| = 1\} \Rightarrow$$

$$|T_x(t, a)|^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_x(u, n; \Pi) C_h^*(\frac{u - t}{a}, an; \Pi) du\, dn. \quad (3)$$

Although this condition seems to be more restrictive, it is still fulfilled by the whole family of generalized Wigner distributions, including the WVD itself [8].

III - TIME-SCALE ENERGY DISTRIBUTIONS

A. A general class and its interpretation

In order to derive the general formulation of time-scale energy distributions, it is appropriate, at this point, to interpret Proposition 2 in the restrictive case where WVDs are used: a scalogram results from the affine smoothing of the WVD of the analyzed signal by the WVD of the analyzing wavelet. However, scalograms are only a special case of time-scale energy distributions. It is this affine smoothing concept that enables us to generalize scalograms to general time-scale energy distributions, in a similar way as spectrograms are generalized to the Cohen's class. More precisely, consider the affine transformation:

$$[\mathbf{L}_A(t, a)h](u) = \frac{1}{\sqrt{|a|}} h(\frac{u - t}{a})\quad ,$$

where the factor $1/\sqrt{|a|}$ is introduced for normalization purposes. The main result of [6] presented in this paper states:

Proposition 3. *If a bilinear time-scale distribution $\Omega_x(t, a)$ is covariant to affine transformations, i.e.*

$$\Omega_{\mathbf{L}_A(\theta,\alpha)x}(t, a) = \Omega_x(\frac{t - \theta}{\alpha}, \frac{a}{\alpha})\ ,$$

then, it is necessarily parameterized as :

$$\Omega_x(t, a; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\ \Pi(\frac{u - t}{a}, an)\ du\ dn\ , \quad (4)$$

where Π is some arbitrary time-frequency function. Eq.(4) characterizes the general class of time-scale energy distributions.

A similar approach has been investigated by the Bertrands [9]. Precise links between our formulation and theirs are given in subsection III-C. It can be noted for the moment that (4) better reveals the affine smoothing concept underlying time-scale distributions and certainly is more suited for combining time-scale and time-frequency into a unified perspective.

Alternative characterizations of the class (4) may be given. An interesting one makes use of the weighting function f and reads :

$$\Omega_x(t, a; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(an, \frac{\tau}{a})\, A_x(n, \tau)\, e^{i2\pi nt}\, dn\, d\tau$$

Just as the WT uses band-pass filters, the smoothing function Π is preferrably chosen to be band-pass as a function of frequency. We thus define $\Pi(t, \nu) = \Pi_0(t, \nu - \nu_0)$, where ν_0 is some non-zero frequency. Using this notation, an interesting identification between time-scale and time-frequency distributions may be found:

$$\Omega_x(t, a; \Pi) = C_x(t, \frac{\nu_0}{a}; \Pi_0)\quad ,$$

provided that the associated weighting function $f_0(n, \tau)$ depends only on the product $n\tau$. This condition is met by numerous distributions. In addition to the class of generalized Wigner distributions, we can mention the *Choï-Williams'* distribution [5], which has recently received a special attention.

B. Properties

The general formulation (4) enables us to find distributions satisfying various specific requirements. This approach, which closely parallels the one used for the Cohen's class, is illustrated on some examples in the following.

1) Energy. The terminology «energy distribution» is justified by the following:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega_x(t, a; \Pi)\, \frac{dt\, da}{a^2} = [\int_{-\infty}^{+\infty} \pi(0, m)\, \frac{dm}{|m|}]\, E_x ,$$

where $\pi(n, m)$ is the partial Fourier transform of Π over time. This means that energy is properly spread over the time-scale plane if the quantity into brackets is unity.

2) Marginal in frequency. The spectral energy density of x is recovered from the marginal in frequency as long as:

$$\int_{-\infty}^{+\infty} \Omega_x(t, a; \Pi)\, dt = |X(\frac{\nu_0}{a})|^2 \Leftrightarrow f(0, \tau) = e^{-i2\pi\nu_0\tau} .$$

3) Marginal in time. Similarly, the instantaneous power of x is obtained as time marginal if:

$$\int_{-\infty}^{+\infty} \Omega_x(t, a; \Pi)\frac{da}{a^2} = |x(t)|^2 \Leftrightarrow \int_{-\infty}^{+\infty} f(an, \frac{\tau}{a})\, \frac{da}{a^2} = \delta(\tau), \forall n .$$

4) Moyal-type formula. Finally, a Moyal-type formula relating inner products of signals and distributions may be obtained as:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega_x(t, a; \Pi)\, \Omega_y^*(t, a; \Pi)\, \frac{dt\, da}{a^2} = |\int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt\, |^2$$

$$\Leftrightarrow \int_{-\infty}^{+\infty} f(an, \frac{\tau}{a})\, f^*(an, \frac{\tau'}{a})\, \frac{da}{a^2} = \delta(\tau - \tau') , \forall n .$$

C. Special cases

As for the Cohen's class, specific choices for smoothing (or weighting) functions allow to obtain special cases of time-scale distributions: some of them will now be reviewed.

1) Scalograms. The simplest example is the scalogram which, according to (3), can be seen as the affine smoothing of the WVD of the analyzed signal by the WVD of the analyzing wavelet [8]:

$$|T_x(t, a)|^2 = \Omega_x(t, a; W_h) .$$

2) Bertrands' class. Another choice yields the *Bertrands'* class [9]:

$$\Omega_x(t, a; \Pi_B) = \frac{1}{|a|} \int_{-\infty}^{+\infty} \mu(u)\, X(\frac{1}{a}\lambda(u))\, X^*(\frac{1}{a}\lambda(-u))\, e^{-i2\pi(t/a)(\lambda(u)-\lambda(-u))}\, du, \quad (5)$$

where $\lambda(u)$ and $\mu(u)$ are two arbitrary functions. Several points should be noted here. 1) Eq.(5) is explicitly written in terms of time and *scale*, whereas the Bertrands' formulation uses frequency as a formal parameter playing the role of the inverse of scale. 2) Bertrands' approach puts emphasis on *analytic* signals and the integration in (5) is therefore limited to positive values of u. However, the formulation (5) is obtained as a special case of (4) corresponding to the following choice for the weighting function:

$$f_B(n, \tau) = \int_{-\infty}^{+\infty} \mu(u)\delta(n+[\lambda(u)-\lambda(-u)])e^{-i2\pi(\tau/2)(\lambda(u)+\lambda(-u))}\, du . \quad (6)$$

The following example will allow to simply recover a particular distribution used by the Bertrands [9]:

$$B_x(t, a) = \frac{1}{|a|} \int_{-\infty}^{+\infty} \frac{(n/2)}{\sinh(n/2)}\, X(\frac{(n/2)e^{-(n/2)}}{a \sinh(n/2)})\, X^*(\frac{(n/2)e^{+(n/2)}}{a \sinh(n/2)})\, e^{-i2\pi(t/a)n} dn, (7)$$

for which the formulation (6) appears to be unnecessarily complicated.

3) Localized bi-frequency kernels. A useful subclass of (4) consists in characterization functions which are perfectly localized on some curve $m = F(n)$ in their bi-frequency representation:

$$\pi_\delta(n, m) \triangleq G(n)\delta(m - F(n)) \Leftrightarrow f_\delta(n, \tau) \triangleq G(n)\, e^{-i2\pi F(n)\tau} ,$$

where $G(n)$ is an arbitrary function. Those distributions can be written

$$\Omega_x(t, a; \Pi_\delta) = \frac{1}{|a|} \int_{-\infty}^{+\infty} G(n)\, X(\frac{1}{a}[F(n)-\frac{n}{2}])\, X^*(\frac{1}{a}[F(n)+\frac{n}{2}])\, e^{-i2\pi(t/a)n}\, dn .$$

Specifying

$$G(n) = \frac{(n/2)}{\sinh(n/2)} \quad ; \quad F(n) = (n/2)\coth(n/2)$$

allows to recover the particular Bertrands' distribution (7). More important is the fact that this specific definition may be constructed starting from a localized bi-frequency kernel by imposing *a priori* requirements (namely *time-localization* and a *Moyal-type* formula) with the help of the results of subsection III-B. This is detailed in [6]. It is, of course, these same requirements that led the Bertrands to their definition (7). Our construction, however, takes place within the more general framework of (4).

4) Separable kernels and affine smoothed Wigner-Ville. It is known from the theory of time-frequency distributions that the trade-off underlying time and frequency behaviors of the spectrogram can be overcome by replacing the associated WVD smoothing by a smoothing function which is *separable* in time and frequency [10]. The resulting distribution (called the *smoothed pseudo-WVD*) offers a great versatility for balancing *e.g.* time-frequency resolution and cross-terms reduction, although this is necessarily at the expense of the loss of other properties such as marginals.

We propose a similar approach for time-scale distributions and define the *affine smoothed WVD* by

$$\Pi_S(t, \nu) = g(t)\, H_0(\nu - \nu_0) \quad \Rightarrow$$

$$\Omega_x(t, a; \Pi_S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\, g(\frac{u-t}{a})\, H_0(an - \nu_0)\, du\, dn.$$

This is a versatile representation which allows a flexible choice of time and scale resolutions in an independent manner through the choice of g and H_0. An illustration of this, with additional interesting properties, is given next.

D. From spectrograms to scalograms via Wigner-Ville

The smoothing functions acting on the WVD to obtain spectrograms on one hand and scalograms on the other hand, are found, by Propositions 1 and 2, to be of the form of a WVD. This suggests a continuous transition from spectrograms to scalograms *via* the WVD by suitably controlling the evolution of the smoothing function between a WVD and a delta function. The following proposition shows that this can be achieved using separable kernels, which allow an independent control of the time and frequency (or scale) behaviors of the associated distributions.

Proposition 4. *A continuous passage from spectrograms to scalograms via Wigner-Ville is possible by means of separable kernels if and only if these latter are Gaussian.*

Outline of the proof. If either spectrograms or scalograms are supposed to be attainable through separable kernels, then their associated smoothing function, which is a WVD, must necessarily be itself a separable function of time and frequency. Since separable WVDs are necessarily everywhere non-negative, we deduce from Hudson's theorem and the separability condition that this WVD must be of the form of a normalized product of Gaussians. Therefore, a suitable choice of separable smoothing function which allows a continuous passage from WVD to spectrograms or spectrograms is of the form

$$\Pi_S(t, \nu) = \frac{\sqrt{\alpha\beta}}{\pi} e^{-\alpha t^2} e^{-\beta(\nu-\nu_0)^2} \qquad \blacksquare$$

The transition is controlled by the parameter $\mu = 2\pi/\sqrt{\alpha\beta}$ which runs from 0 (WVD) to 1 (spectrogram/scalogram). This is illustrated in Fig. 1 which shows several analyses of three Gaussian wave packets.

CONCLUSION

We may envision that, owing to their constant-Q structure, time-scale distributions are to play an important role for transient analysis and detection. The above development has demonstrated that it is possible to build a general class of time-scale energy distributions in a systematic manner, in which affine smoothing of the WVD plays a central role. An additional benefit of our presentation is that it closely parallels the one used for the Cohen's class, thereby unifying the derivation of time-scale distributions and their time-frequency counterparts. Again, the WVD is shown to be a central part of the analysis in which the simple identification $a = \nu_0/\nu$ (scale = inverse of frequency) holds: the WVD thus belongs to both classes of time-frequency and time-scale distributions. This is well illustrated by the last result presented in this paper, which shows a continuous transition from spectrograms to scalograms with the WVD as a middle step. In light of this, we recommend that various properties of time-frequency and time-scale methods be compared keeping in mind that both result from a smoothing operation acting on a common kernel (the WVD), the difference being related to the nature of the smoothing operation used (time-frequency or affine (time-scale) smoothing). Moreover, this continuous transition permits to balance time-frequency resolution and cross-terms reduction in the time-scale representation, in a similar (but different) way as for the smoothed pseudo-WVD [10]. Other specific requirements (such as energy normalization, time marginal, *etc.*) and associated parameterizations of the representation were also studied in this paper. This results in a great versatility for the choice of representations appropriate for various applications.

Since a large class of time-scale and time-frequency representations is now available, with many possible (and sometimes, exclusive) properties, some analysis should be done on the analysis tool itself in order to express particular needs: starting from the most general formulation, one can, for instance, build a subset of time-scale energy representations, suitable for a given application, by imposing specific requirements. Controlling a few parameters on this set of analyses should help in many ways, *e.g.* for determining which representation best reveals a given time-scale signature.

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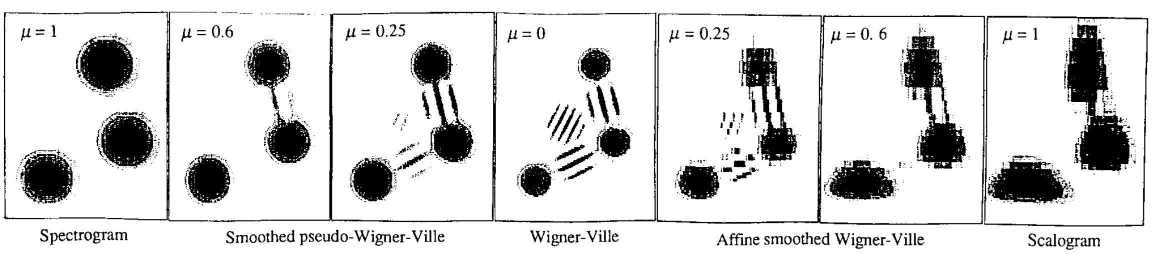

Fig. 1 From spectrograms to scalograms *via* Wigner-Ville (time: →; frequency: ↑; $\mu = 2\pi/\sqrt{\alpha\beta}$)