Fourier and Wavelet Spectrograms seen as smoothed Wigner-Ville Distributions

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Abstract: In this paper, we give an exhaustive bilinear generalization of the Continuous Wavelet Transform and emphasize on links between time-frequency and time-scale energy distributions. The well-known Cohen's Class gives a full description of the former, in which a smoothing function acts on the Wigner-Ville Distribution (WVD). We here provide a full description of the latter: the result is a new, versatile class of representations in which the smoothing of the WVD is scale-dependent (mathematically speaking, a correlation on the affine group). Through proper choices of the smoothing function, interesting properties may be further imposed on the representation. Also, specific choices allow to recover known definitions (including the Bertrands' and the scalogram = wavelet spectrogram). Another, very flexible, choice uses separable smoothing functions to provide a continuous transition between spectrograms and scalograms *via* Wigner-Ville. This "do it yourself" property makes of affine smoothing of the WVD a very flexible tool for nonstationary signal analysis.

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1 - The Short-Time Fourier Transform and the Wavelet Transform

Let $x(t)$ be a finite energy signal and $h(t)$ be a sliding window. A well-known linear *time-frequency* representation is the so-called *short-time Fourier transform* (STFT), computed as

$$F_x(t, \nu) \triangleq \int_{-\infty}^{+\infty} x(u)\, h^*(u - t)\, e^{-i2\pi\nu u}\, du \ .$$

In recent years, an alternative linear representation, called the continuous *wavelet transform* (CWT), has been extensively studied [1-2]. The fundamental idea underlying the CWT is to replace the *frequency shifting* operation that occurs in the STFT by a *time* (or frequency) *scaling* operation. This results in

$$T_x(t, a) \triangleq \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} x(u)\, h^*\!\left(\frac{u - t}{a}\right) du \ .$$

The so-called *analyzing wavelet* $h(t)$ is assumed to be localized in time in order to obtain a local analysis. The explicit dependence of this definition on the dilation/compression (or *scale*) parameter a makes the WT a *time-scale* representation rather than a time-frequency one [3].

As for the STFT, the WT may be inverted, but under different conditions, namely that the Fourier transform $H(\nu)$ of the analyzing wavelet $h(t)$ satisfies [1]

$$\int_{-\infty}^{+\infty} |H(\nu)|^2 \frac{d\nu}{|\nu|} < +\infty \, .$$

This basically means that $h(t)$ is the impulse response of some *band-pass* filter (hence it must be localized in frequency as well as in time). This is equivalent, in the time domain, to the condition that its mean value vanishes. $h(t)$ thus oscillates, hence the name *wavelet*.

It is important to note that both transforms analyze the signal by means of an *inner product* with analyzing waveforms depending on two parameters. The main difference between the STFT and the WT is actually related to the *generated structure* of the respective analyzing waveforms.

- In the STFT case, modulated versions of a *low-pass* filter are used to explore the spectral content of the analyzed signal (*uniform* filterbank). This amounts, in the time-domain, to using an analyzing waveform of constant envelope with an increasing number of oscillations as higher frequencies are analyzed.

- In the WT case, the waveforms are generated from $h(t)$ by time-shift and dilation operations and are referred to as the *wavelets*. Time evolutions of signals are thus analyzed by means of a waveform whose envelope is narrowed as higher frequencies are analyzed, whereas its number of oscillations, hence its *shape*, remains constant. They can therefore be seen as dilated or compressed versions of a *band-pass* filter, whose relative bandwidths are constant (*constant-Q* filterbank).

2 - Spectrograms and Scalograms

A. Definitions and comparison

STFTs and WTs are defined as complex-valued functions and thus convey both modulus and phase informations. For some applications [4], phase information can be of interest, but a description based only on the squared modulus - providing an energy density distribution, is often preferred. Indeed, the *spectrogram* $|F_x(t, \nu)|^2$, defined as the energy distribution associated to the STFT, has been widely used for many signal processing tasks. A similar quantity, $|T_x(t, a)|^2$, can be defined in the case of the WT: we propose to refer to this "wavelet spectrogram" as a *scalogram*.

There is a classical time and frequency resolution trade-off that underlies the structure of the spectrogram and should be mentioned here: the choice of an analyzing window of short duration ensures a good time localization, but at the expense of a poor frequency resolution (by Fourier duality), and *vice-versa*. Moreover, once an analyzing window has been chosen, the resolution capabilities of the spectrogram remain fixed for all time and frequency parameters.

The situation is different for scalograms: owing to the constant-Q structure described above, resolution capabilities are *frequency-dependent*.

B. Smoothing interpretation within the Cohen's class

Both spectrograms and scalograms are defined as bilinear functions of the analyzed signal. In the following, we provide a simple interpretation of them with the help of the general class of bilinear, shift-covariant, time-frequency energy distributions. This class, referred to as *Cohen's* , is given by [5]

$$C_x(t, \nu; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\, \Pi(u - t, n - \nu)\, du\, dn \quad , \qquad (1)$$

where $\Pi(t, \nu)$ is some arbitrary function and where

$$W_x(t, \nu) = \int_{-\infty}^{+\infty} x\left(t + \frac{\tau}{2}\right)\, x^*\left(t - \frac{\tau}{2}\right)\, e^{-i2\pi\nu\tau}\, d\tau$$

is the so-called *Wigner-Ville Distribution* (WVD) [5]. Note that whenever $\Pi(t, \nu)$ happens to be a low-pass function in the time-frequency plane, the general class (1) may be considered as composed of *smoothed* versions of the WVD.

For description purposes, it is convenient to introduce two-dimensional Fourier transformations in (1). Changing variables accordingly yields a dual characterization:

$$C_x(t, v; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(n, \tau) A_x(n, \tau)\, e^{-i2\pi(nt + \tau v)}\, dn\, d\tau\ ,$$

where the *weighting function* $f(n, \tau)$ and the (narrowband) *ambiguity function* [5] $A_x(n, \tau)$ are the (direct and inverse, respectively) 2D Fourier transforms of $\Pi(t, v)$ and $W_x(t, v)$.

We now use members of the Cohen's class to give a simple interpretation of both spectrograms and scalograms. What happens in the spectrogram case [6-7] can be summarized by the following :

Proposition 1. *Provided that the weighting function has modulus unity, a classical smoothing of the signal distribution by the window distribution results in a spectrogram* :

$$|f(n, \tau)| = 1 \;\Leftrightarrow\; |F_x(t, v)|^2 =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_x(u, n; \Pi)\, C_h^*(u - t, n - v; \Pi)\, du\, dn\ . \qquad (2)$$

This condition is met by numerous distributions, including the class of *generalized Wigner distributions* [7] among which the *Rihaczek's* distribution [5] and the WVD itself are special cases.

We state a similar specification of scalograms from Cohen's distributions [6]:

Proposition 2. *Provided that the weighting function has modulus unity and depends on its variables only through their product, an* _*affine smoothing*_ *of the signal distribution by the wavelet distribution results in a scalogram* :

$$\{\exists\ \phi(.)\ /\ f(n, \tau) \triangleq \phi(n\tau) \text{ and } |\phi(n\tau)| = 1\} \Rightarrow$$

$$|T_x(t, a)|^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_x(u, n; \Pi)\, C_h^*\!\left(\frac{u - t}{a}, an; \Pi\right) du\, dn. \qquad (3)$$

Although the condition on the weighting function is more restrictive, it is still fulfilled by any generalized Wigner distributions [8]. Hence C_x in (3) can be chosen as any generalized Wigner-Ville distribution, including the WVD itself.

3 - Time-Scale Energy Distributions

A. A general class and its interpretation

In order to derive the general formulation of time-scale energy distributions, it is appropriate, at this point, to interpret Proposition 2 in the restrictive case where WVDs are used. Proposition 2 then reads:

A scalogram results from the affine smoothing of the WVD of the analyzed signal by the WVD of the analyzing wavelet.

It is this affine smoothing concept that enables us to generalize scalograms to general time-scale energy distributions, in a similar way as spectrograms are generalized to the Cohen's class. More precisely, consider the following affine transformation (a square integrable representation of the affine group):

$$[L_A(t, a)h](u) = \frac{1}{\sqrt{|a|}}\, h\left(\frac{u - t}{a}\right) \quad ,$$

where the factor $1/\sqrt{|a|}$ is introduced for normalization purposes. The main result of [6] presented in this paper is given by the

Proposition 3. *If a bilinear time-scale distribution $\Omega_x(t, a)$ is covariant to affine transformations, i.e.*

$$\Omega_{L_A(\theta,\alpha)x}(t, a) = \Omega_x\left(\frac{t - \theta}{\alpha}, \frac{a}{\alpha}\right) ,$$

then, it is necessarily parameterized as :

$$\Omega_x(t, a; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\, \Pi\left(\frac{u - t}{a}, an\right) du\, dn \ , \quad (4)$$

where Π is some arbitrary time-frequency function. Eq.(4) characterizes the general class of time-scale energy distributions.

The operation described in (4) is nothing else but a two-dimensional correlation on the affine group [8]. More precisely, {t, a} are elements of the affine group for the product $\{t, a\}\{u, \alpha\} = \{au+t, a\alpha\}$. This yields $\{t, a\}^{-1}\{u, \alpha\} = \{(u-t)/a, \alpha/a\}$, which, with the identification $n = 1/\alpha$, indeed interprets (4) as a correlation.

A similar approach has been investigated by the Bertrands [9]. Precise links between our formulation and theirs will be given later in subsection III-C.

Alternative characterizations of the class (4) may be given. An interesting one makes use of the weighting function f and reads :

$$\Omega_x(t, a; \Pi) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(an, \frac{\tau}{a})\, A_x(n, \tau)\, e^{i2\pi nt}\, dn\, d\tau \quad .$$

Just as the WT uses band-pass filters, the smoothing function Π is preferrably chosen to be band-pass as a function of frequency. We thus define $\Pi(t, \nu) = \Pi_0(t, \nu - \nu_0)$, where ν_0 is some non-zero frequency. Using this notation, an interesting identification between time-scale and time-frequency distributions may be found:

$$\Omega_x(t, a; \Pi) = C_x(t, \frac{\nu_0}{a}; \Pi_0) \quad ,$$

provided that the associated weighting function $f_0(n, \tau)$ depends only on the product $n\tau$. This condition is met by numerous distributions. In addition to the class of generalized Wigner distributions, we can mention the *Choï-Williams'* distribution [5], which has recently received special attention.

B. Properties

The general formulation (4) enables us to find distributions satisfying various specific requirements. This approach, which closely parallels the one used for the Cohen's class, is illustrated on some examples in the following.

1) Energy. The terminology «energy distribution» is justified by the following:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega_x(t, a; \Pi)\, \frac{dt\, da}{a^2} = [\int_{-\infty}^{+\infty} \pi(0, m)\, \frac{dm}{|m|}]\, E_x \ ,$$

where $\pi(n, m)$ is the partial Fourier transform of Π over time. This means that energy is properly spread over the time-scale plane if the quantity into brackets is unity.

2) Marginal in frequency. The spectral energy density of x is recovered from the marginal in frequency as long as:

$$\int_{-\infty}^{+\infty} \Omega_x(t,\, a;\, \Pi)\, dt = |\, X(\frac{\nu_0}{a})\,|^2 \quad \Leftrightarrow \quad f(0,\, \tau) = e^{-i2\pi\nu_0\tau} \;.$$

3) Marginal in time. Similarly, the instantaneous power of x is obtained as time marginal if:

$$\int_{-\infty}^{+\infty} \Omega_x(t,\, a;\, \Pi) \frac{da}{a^2} = |\, x(t)\,|^2 \Leftrightarrow \int_{-\infty}^{+\infty} f(an,\, \frac{\tau}{a})\, \frac{da}{a^2} = \delta(\tau),\ \forall n \,.$$

4) Moyal-type formula. Finally, a Moyal-type formula relating inner products of signals and distributions may be obtained as:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega_x(t,\, a;\, \Pi)\, \Omega_y^*(t,\, a;\, \Pi)\, \frac{dt\, da}{a^2} = |\int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt\,|^2$$

$$\Leftrightarrow \quad \int_{-\infty}^{+\infty} f(an,\, \frac{\tau}{a})\, f^*(an,\, \frac{\tau'}{a})\, \frac{da}{a^2} = \delta(\tau - \tau')\,,\ \forall\, n \;.$$

C. Special cases

As for the Cohen's class, specific choices for smoothing (or weighting) functions allow to obtain special cases of time-scale distributions: some of them will now be reviewed.

1) Scalograms. The simplest example is the scalogram which, according to (3), can be seen as the affine smoothing of the WVD of the analyzed signal by the WVD of the analyzing wavelet [8]:

$$|\, T_x(t,\, a)\,|^2 = \Omega_x(t,\, a;\, W_h) \;.$$

2) Bertrands' class. Another choice yields the *Bertrands'* class [9]:

$\Omega_x(t,\, a;\, \Pi_B) =$

$$= \frac{1}{|a|} \int_{-\infty}^{+\infty} \mu(u)\, X(\frac{1}{a}\lambda(u))\, X^*(\frac{1}{a}\lambda(-u))\, e^{-i2\pi(t/a)(\lambda(u)-\lambda(-u))}\, du, \qquad (5)$$

where $\lambda(u)$ and $\mu(u)$ are two arbitrary functions.

Although (5) is quite general, it can be noted that (4) better reveals the affine smoothing concept underlying time-scale distributions and certainly is more suited for combining time-scale and time-frequency into a unified perspective. Moreover, the formulation (5) is obtained as a special case of (4) corresponding to the following choice for the weighting function:

$$f_B(n, \tau) = \int_{-\infty}^{+\infty} \mu(u)\delta(n+[\lambda(u)-\lambda(-u)])e^{-i2\pi(\tau/2)(\lambda(u)+\lambda(-u))}\,du\,. \qquad (6)$$

In the following paragraph we show how to simply recover a particular distribution used by the Bertrands [9] :

$$B_x(t, a) =$$

$$\frac{1}{|a|}\int_{-\infty}^{+\infty}\frac{(n/2)}{\sinh(n/2)}\,X\Big(\frac{(n/2)e^{-(n/2)}}{a\,\sinh(n/2)}\Big)\,X^*\Big(\frac{(n/2)e^{+(n/2)}}{a\,\sinh(n/2)}\Big)\,e^{-i2\pi(t/a)n}dn, \quad (7)$$

for which the formulation (6) appears to be unnecessarily complicated.

<u>3) Localized bi-frequency kernels</u>. A useful subclass of (4) consists in characterization functions which are perfectly localized on some curve $m = F(n)$ in their bi-frequency representation:

$$\pi_\delta(n, m) \triangleq G(n)\delta(m - F(n)) \Leftrightarrow f_\delta(n, \tau) \triangleq G(n)\,e^{-i2\pi F(n)\tau}\,,$$

where $G(n)$ is an arbitrary function. Those distributions can be written

$$\Omega_x(t, a; \Pi_\delta) =$$

$$= \frac{1}{|a|}\int_{-\infty}^{+\infty} G(n)\,X\Big(\frac{1}{a}\Big[F(n)-\frac{n}{2}\Big]\Big)\,X^*\Big(\frac{1}{a}\Big[F(n)+\frac{n}{2}\Big]\Big)\,e^{-i2\pi(t/a)n}\,dn\,.$$

Specifying

$$G(n) = \frac{(n/2)}{\sinh(n/2)} \quad ; \quad F(n) = (n/2)\,\coth(n/2)$$

allows to recover the particular Bertrands' distribution (7). More important is the fact that this specific definition may be constructed starting from a localized bi-frequency kernel by imposing *a priori* requirements (namely *time-localization* and a *Moyal-type* formula) with the help of the results of subsection III-B. This is detailed in [6]. Of course, the same requirements led the Bertrands to their definition (5). Our construction, however, takes place within the more general framework of (4).

<u>4) Separable kernels and affine smoothed Wigner-Ville</u>. It is known from the theory of time-frequency distributions that the trade-off underlying time and frequency behaviors of the spectrogram can be overcome by replacing the associated WVD smoothing by a smoothing function which is *separable* in time and frequency [10]. The resulting distribution (called the *smoothed pseudo-WVD*) offers a great versatility for balancing *e.g.* time-frequency resolution and cross-terms reduction,

although this is necessarily at the expense of the loss of other properties such as marginals.

We propose a similar approach for time-scale distributions and define the *affine smoothed WVD* by

$$\Pi_S(t, \nu) = g(t)\, H_0(\nu - \nu_0) \quad \Rightarrow$$

$$\Omega_x(t, a; \Pi_S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_x(u, n)\, g\left(\frac{u-t}{a}\right) H_0(an - \nu_0)\, du\, dn.$$

This is a versatile representation which allows a flexible choice of time and scale resolutions in an independent manner through the choice of g and H_0. An illustration of this, with additional interesting properties, is given next.

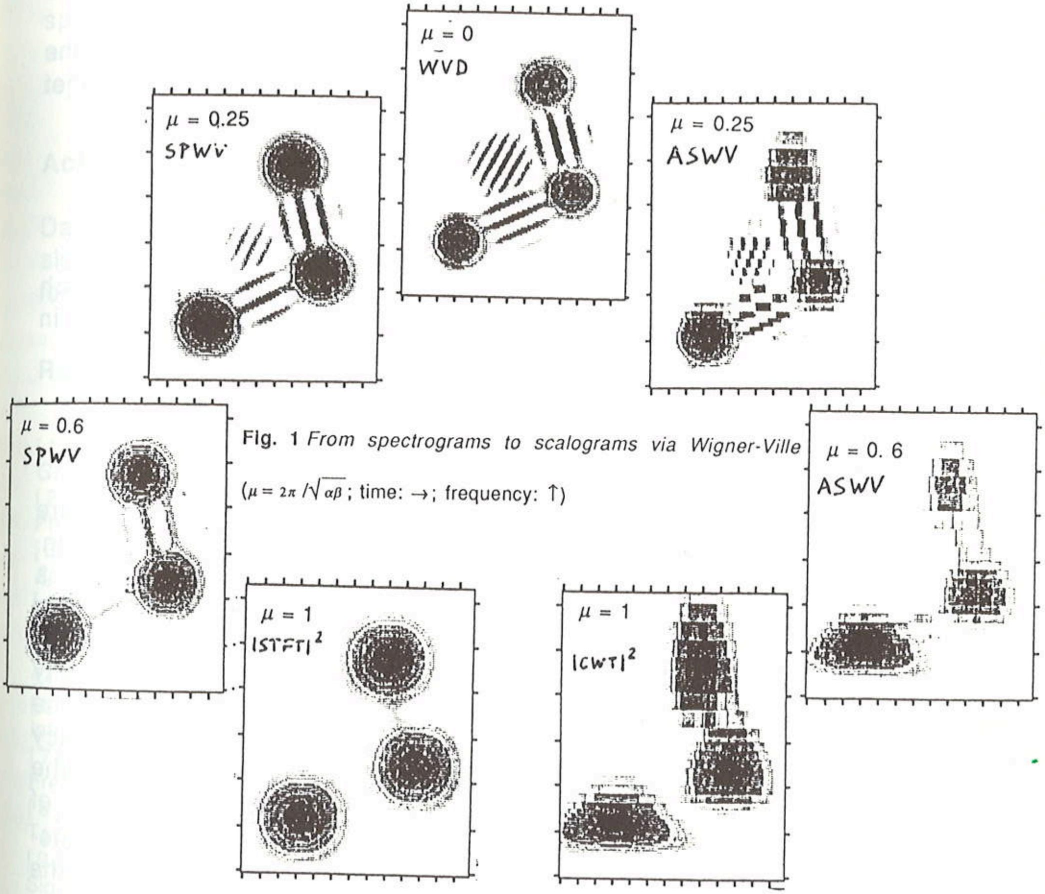

Fig. 1 *From spectrograms to scalograms via Wigner-Ville*

($\mu = 2\pi / \sqrt{\alpha\beta}$; time: →; frequency: ↑)

$|STFT|^2$: Spectrogram - S.P.W.V.: Smoothed Pseudo-Wigner-Ville

WVD: Wigner-Ville - A.S.W.V.: Affine Smoothed Wigner-Ville - $|CWT|^2$: Scalogram

D. From spectrograms to scalograms via Wigner-Ville

The smoothing functions acting on the WVD to obtain spectrograms on one hand and scalograms on the other hand, are found, by Propositions 1 and 2, to be of the form of a WVD. This suggests a continuous transition from spectrograms to scalograms *via* the WVD by suitably controlling the evolution of the smoothing function between a WVD and a delta function.The following proposition claims that this can be achieved using separable kernels, which allow an independent control of the time and frequency (or scale) behaviors of the associated distributions (see [6] for details).

Proposition 4. *A continuous transition from spectrograms to scalograms via Wigner-Ville is possible by means of separable kernels if and only if these latter are Gaussian.*

A suitable choice of separable smoothing function which allows the continuous transition from WVD to spectrograms or spectrograms is of the form

$$\Pi_S(t, \nu) = \frac{\sqrt{\alpha\beta}}{\pi} e^{-\alpha t^2} e^{-\beta(\nu-\nu_0)^2}$$

The transition is controlled by the parameter $\mu = 2\pi/\sqrt{\alpha\beta}$ which runs from 0 (WVD) to 1 (spectrogram/scalogram). This is illustrated in Fig. 1 which shows several analyses of three Gaussian wave packets.

Conclusion

Owing to their constant-Q structure, time-scale distributions are expected to play an important role for transient analysis and detection. The above development has demonstrated that it is possible to build a general class of time-scale energy distributions in a systematic manner, in which affine smoothing of the WVD plays a central role.

An additional benefit of our presentation is that it closely parallels the one used for the Cohen's class, thereby unifying the derivation of time-scale distributions and their time-frequency counterparts. Again, the WVD is shown to be a central part of the analysis in which the simple identification $a = \nu_0/\nu$ (scale = inverse of frequency) holds: the WVD thus belongs to both classes of time-frequency and time-scale distributions. This is well illustrated by the last result presented in this paper, which shows a continuous transition from spectrograms to scalograms with the WVD as a middle step.

In light of this, we recommend that various properties of time-frequency and time-scale methods be compared keeping in mind that both result from a smoothing operation acting on a common kernel (the

WVD), the difference being related to the nature of the smoothing operation (time-frequency or time-scale smoothing). Moreover, this continuous transition permits to balance time-frequency resolution and cross-terms reduction in the time-scale representation, in a similar (but different) way as for the smoothed pseudo-WVD [10]. Other specific requirements (such as energy normalization, time marginal, *etc.*) and associated parameterizations of the representation were also studied in this paper. This results in a great versatility for the choice of representations depending on precise requirements.

Since a large class of time-scale and time-frequency representations is now available, with many possible (and sometimes, exclusive) properties, some analysis should be done on the analysis tool itself in order to express particular needs: starting from the most general formulation, one can, for instance, build a subset of time-scale energy representations, suitable for a given application, by imposing specific requirements. Controlling a few parameters on this set of analyses should help in many ways, *e.g.* for determining which representation best reveals a given time-scale signature.

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