AT&T Bell Laboratories

subject: **Wigner-Ville representations of signals adapted to shifts and dilatations**

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from: **Olivier Rioul***
MH 11227
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ABSTRACT

General results concerning Wigner-Ville representations with respect to the *Weyl-Heisenberg* group (roughly, the group of time and frequency shifts) are transposed into another formalism, using *ax+b group representations* (*ax+b* denotes the group of shifts and dilatations). This provides analytic, painless representations of signals, in which the Wigner-Ville distribution is smoothed with windows adapted to a logarithmic scale in frequency.

A particular choice for the smoothing kernel, given by the Wigner-Ville distribution itself, leads to the squared modulus of the *Wavelet Transform*, introduced several years ago by Jean Morlet and Alex Grossmann[3]).

* IGE - Ecole Nationale Supérieure des Télécommunications - 46 rue Barrault - 75013 PARIS, France.

1. NOTATIONS AND DEFINITIONS.

Let $s(t)$ be a time-varying real or complex-valued signal with finite energy, i.e.: $s(t) \in L^2(\mathbf{R}, dt)$. Its Fourier Transform is denoted by

$$S(\nu) = \int s(t)\, e^{-2i\pi\nu t} dt,$$

its Wigner-Ville distribution is given by

$$W_s(t,\nu) = \int s(t+\frac{\tau}{2})\, s^*(t+\frac{\tau}{2})\, e^{-2i\pi\nu\tau} d\tau. \tag{1}$$

In the following we use an extended form of (1),

$$W_s^\alpha(t,\nu) = \int s(t+(1-\alpha)\frac{\tau}{2})\, s^*(t-(1+\alpha)\frac{\tau}{2})\, e^{-2i\pi\nu\tau} d\tau, \tag{2}$$

where $-1 \le \alpha \le 1$. It is known that the Wigner Ville distribution and the Rihaczek distribution, defined by

$$R_s(t,\nu) = s(t)\, S^*(\nu)\, e^{-2i\pi\nu t},$$

can be recovered from (2) by letting $\alpha=0$ and $\alpha=\pm 1$ respectively.

2. THE COHEN CLASS REPRESENTATIONS.

A common choice for so-called *smoothed Wigner-Ville distributions* is given by the Cohen class, which can be obtained with general assumptions on a certain function $C(s)(t,\nu)$. For instance, if we assume that $C(s)(t,\nu)$ is bilinear in s, and that shifting $s(t)$ in time and frequency amounts to shifting $C(s)$ in the time-frequency plane, we obtain the following formula:

$$C_\Pi(s)(t,\nu) = \int\int \Pi(u-t, x-\nu)\, W_s(u,x)\, du\, dx. \tag{3}$$

Without loss of generality, we have taken $\alpha=0$ in (3). Any $W_s^\alpha(t,\nu)$ can in fact be recovered from (3), with a suitable kernel Π_α[1]. A choice commonly adopted for practical computations is to take a separable kernel $\Pi(t,\nu) = \Pi_1(t)\, \Pi_2(\nu)$. Note that the smoothing procedure uses windows whose dimension and shape remain constant for every value of τ and x.

The following lemma shows that another particular choice for Π leads to the squared modulus of the *Short Time Fourier Transform.*

2.1 Lemma 1:

Let $s(t)$, $s'(t) \in L^2(\mathbf{R}, dt)$. One has:

$$|<s\,|\,s'>|^2 = \int\int W_s^\alpha(t,\nu)\; W_{s'}^{\alpha\,*}(t,\nu)\; dt\; d\nu, \tag{4}$$

where $<s|s'>$ *denotes the dot product of s and s',* $<s\,|\,s'> = \int s(t)s'^*(t)dt$.

Proof: $W_s^\alpha(t,\nu)$ is given by a Fourier Transform of a specific function that can be recovered via the Fourier inversion formula:

$$s(t+(1-\alpha)\frac{\tau}{2})\; s^*(t-(1+\alpha)\frac{\tau}{2}) = \int W_s^\alpha(t,\nu)\; e^{2i\pi\nu\tau} d\nu.$$

Using Plancherel's isometry formula for $W_s^\alpha(t,\nu)$ and $W_{s'}^\alpha(t,\nu)$, one has:

$$\int s(t+(1-\alpha)\frac{\tau}{2})\; s^*(t-(1+\alpha)\frac{\tau}{2})\; s'^*(t+(1-\alpha)\frac{\tau}{2})\; s'(t-(1+\alpha)\frac{\tau}{2})\; d\tau = \int W_s^\alpha(t,\nu) W_{s'}^{\alpha\,*}(t,\nu)\; d\nu.$$

Integrating over the variable t, and using the transformation:

$$\begin{cases} u = t-(\alpha-1)\dfrac{\tau}{2} \\ v = t-(\alpha+1)\dfrac{\tau}{2} \end{cases}$$

(whose Jacobian is 1) in the first integral leads to (4) ■

The squared modulus of the Short Time Fourier Transform can now be obtained easily if we consider a representation of the Weyl-Heisenberg group, defined by

$$U_x^\tau(s(t)) = e^{2i\pi xt} s(t-\tau). \tag{5}$$

U_x^τ simply consists in shifting the signal in time and frequency. An easy computation leads to:

$$W_{U_x^\tau(s)}^\alpha(t,\nu) = W_s^\alpha(t-\tau,\nu-x) \tag{6}$$

If we define the Short Time Fourier Transform to be:

$$S_s(\tau,x) = \int s(t)\; g^*(t-\tau)\; e^{-2i\pi xt} dt, \tag{7}$$

where $g(t)$ is a given "analysing window," we have the following well known result:

2.2 Theorem 1:

Let $s(t) \in L^2(\mathbf{R},dt)$ *and* $S_s(\tau,x)$ *be the Short Time Fourier Transform of* $s(t)$. *One has:*

$$|S_s(\tau,x)|^2 = \int\int W_s^\alpha(t,\nu)\; W_g^{\alpha\,*}(t-\tau,\nu-x)\; dt\; d\nu. \tag{8}$$

Proof: Using lemma 1 and (6), we have:

$$| S_s(\tau,x) |^2 = | \langle s | U_x^\tau(g) \rangle |^2 = \int\int W^\alpha_{U_x^\tau(g)}(t,\nu)\, W_s^{\alpha\,*}(t,\nu)\; dt\; d\nu \blacksquare$$

3. A WIGNER-VILLE REPRESENTATION OF THE WAVELET TRANSFORM.

We use the same scheme as in § 2 to show that the squared modulus of the wavelet transformation with respect to a analysing wavelet[3] $g \in L^2(\mathbf{R},dt)$ can be recovered by correlating the two distributions $W_s(t,\nu)$ and $W_g(t,\nu)$. This is formally equivalent to § 2, except that this correlation must now be computed in the $ax{+}b$-group sense, as shown in § 3.2.

Let U_a^b be the left-invariant square integrable representation of the $ax{+}b$ group:

$$U_a^b(s(t)) = \frac{1}{\sqrt{|a|}}\, s\left(\frac{t-b}{a}\right), \tag{8}$$

where a and $b \in \mathbf{R}$, $a>0$. Using the decomposition $U_a^b = U_1^b\; U_a^0$ one can easily prove the following lemma, which is similar to (6):

3.1 Lemma 2:

Let $s(t) \in L^2(\mathbf{R},dt)$ and $U_a^b(s(t))$ be the left-invariant $ax{+}b$-group representation. One has:

$$W^\alpha_{U_a^b(s)}(t,\nu) = W_s^\alpha\left(\frac{t-b}{a}, a\nu\right). \tag{9}$$

Proof: We have to show that

$$\begin{cases} W^\alpha_{U_1^b(s)}(t,\nu) = W_s^\alpha(t-b,\nu) \\ W^\alpha_{U_a^0(s)}(t,\nu) = W_s^\alpha\left(\frac{t}{a}, a\nu\right) \end{cases}$$

The first assertion is obvious. A simple transformation of variables leads to the second ■

If we define the Wavelet Transform to be:

$$T_s(b,a) = \frac{1}{\sqrt{|a|}} \int s(t)\; g^*\left(\frac{t-b}{a}\right)\, dt, \tag{10}$$

where g(t) is a given "analysing wavelet[3]," we have the following:

3.2 Theorem 2:

Let $s(t) \in L^2(\mathbf{R},dt)$ and $T_s(b,a)$ be the Wavelet Transform of $s(t)$. One has:

$$| T_s(b,a) |^2 = \int\int W_s^\alpha(t,\nu)\; W_g^{\alpha\,*}\left(\frac{t-b}{a}, a\nu\right)\, dt\; d\nu. \tag{11}$$

Proof: Using lemma 1 and 2, we have:

$$
\begin{aligned}
|T_s(b,a)|^2 &= |\langle s \mid U_a^b(g)\rangle|^2 \\
&= \int\int W^{\alpha}_{U_a^b(g)}(t,\nu)\, W_s^{\alpha\,*}(t,\nu)\, dt\, d\nu \\
&= \int\int W_s^{\alpha}(t,\nu)\, W_g^{\alpha\,*}(\frac{t-b}{a}, a\nu)\, dt\, d\nu \blacksquare
\end{aligned}
$$

4. WIGNER-VILLE REPRESENTATIONS ADAPTED TO SHIFTS AND DILATATIONS.

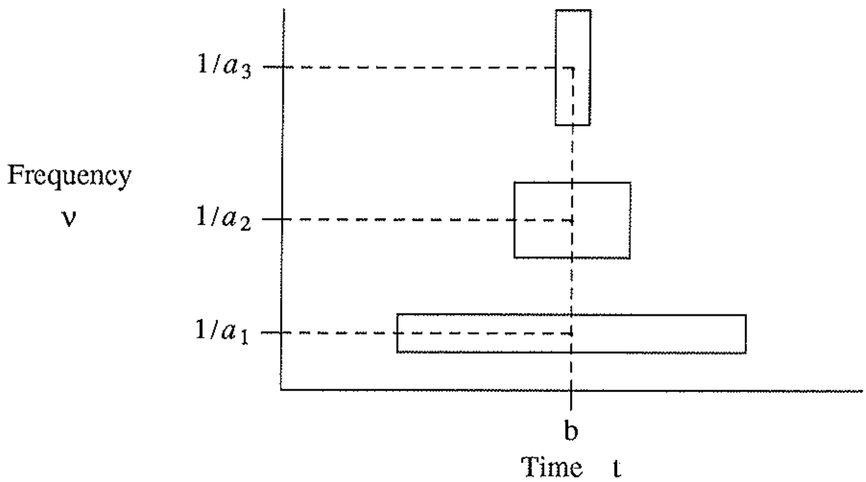
Following § 3, we can define a general "scale-smoothed" version of Wigner-Ville representations by taking

$$\Omega_{\Pi}(s)(b,a) = \int\int \Pi(\frac{t-b}{a}, a\nu)\, W_s^{\alpha\,*}(t,\nu) dt d\nu. \qquad (12)$$

This new class does not contain the Cohen class, but gives another smoothing procedure. Note that a enters in both the time and the frequency integration. For numerical computations, it may be useful to choose a separable kernel $\Pi(t,\nu) = \Pi_1(t)\,\Pi_2(\nu)$, which leads to the formula (for α=0):

$$\Omega_{\Pi}(s)(b,a) = \frac{1}{a}\int\int s(t-\frac{\tau}{2})\, s^*(t+\frac{\tau}{2})\, \Pi_1(\frac{t-b}{a})\, \Psi_2(\frac{\tau}{a})\, dt\, d\tau, \qquad (12')$$

where Ψ_2 is the Fourier Transform of Π_2. The smoothing procedure of (12) appears to be quite different from (3). This is due to the behaviour of the smoothing windows used, whose shapes are concentrated in time for high frequencies ($\nu \to \infty$), and dilated in time for lower frequencies (*see figure*). These windows have also the remarkable property to be automatically distributed in a logarithmic way in the frequency domain:



4.1 Remarks on special notations:

Wigner-Ville representations adapted to shifts and dilatations as defined in (12), can be expressed in a general formalism where the time-frequency notations (t,ν) are replaced by time-scale notations (b,a).

In order to define the Wigner-Ville distribution $W_s(t,\nu)$ depending on (b,a) we note that b is a time translation parameter, whereas $1/a$ can be considered as a frequency parameter. Moreover the sine wave function:

$$e^{-2i\pi\frac{\tau}{a}} = e^{-2i\pi\tau\nu}$$

is dilated (a) while translated (ν) in frequency. This leads us to define Wigner-Ville distribution adapted to time-shifts and time-dilatations as follows:

$$W_s\{b,a\} = \int s(b+\frac{\tau}{2})\, s^*(b-\frac{\tau}{2})\, e^{-2i\pi\frac{\tau}{a}}\, d\tau. \qquad (13)$$

The correspondance between $W_s(t,\nu)$ is:

$$W_s(t,\nu) = W_s\{b,\frac{1}{a}\}$$

if we assume t=b and ν=$1/a$. $W_s^\alpha\{b,a\}$ is defined in a similar manner. Note that we could have chosen t=b and ν=k/a, where k is any scale constant. This has no implication for the following definitions.

$\{b,a\}$ are elements of the ax+b-group for the product defined by $\{b,a\}\{\beta,\gamma\} = \{a\beta+b,a\gamma\}$. The inverse of $\{b,a\}$ is given by $\{b,a\}^{-1} = \{-b/a,1/a\}$, and the identity (unit element in the group) is $\{0,1\}$.

With these notations we can rewrite (11) as:

$$|\,T_s(b,a)\,|^2 = \int\int W_s^\alpha\{\beta,\gamma\}\, W_g^{\alpha\,*}(\{b,a\}^{-1}\{\beta,\gamma\})\, \frac{d\beta\, d\gamma}{\gamma^2}. \qquad (14)$$

Note that the total energy of $s(t)$ is given by integrating (14) with respect to the measure $db\ da/a^2$, if and only if g is said to be "admissible[3]," i.e.:

$$\int \frac{|\,G(\nu)\,|^2}{|\,\nu\,|}\, d\nu < \infty.$$

The general class of ax+b-smoothed Wigner-Ville representations is now defined by (compare (12)):

$$\Omega_\Pi(s)(b,a) = \int\int \Pi(\{b,a\}^{-1}\{\beta,\gamma\})\ W_s\{\beta,\gamma\}\ \frac{d\beta\ d\gamma}{\gamma^2}. \tag{15}$$

Note that $d(\beta,\gamma) = d\beta d\gamma/\gamma^2$ is the left-invariant measure of the $ax+b$-group, and that integrating (15) with respect to $d(b,a)$ leads to the total energy of $s(t)$ if Π is suitably normalized. The right term of (15) appears to be the general expression of a correlation with respect to the $ax+b$-group. We can taken α=0 since any $W_s^\alpha\{b,a\}$, with $\alpha\neq 0$, can be recovered from (15), with

$$\Pi_\alpha\{b,a\} = \frac{1}{\alpha} e^{-\frac{4i\pi}{\alpha} b(\frac{1}{a}-1)}.$$

We had a similar property for $W_s^\alpha(t,\nu)$ which can be recovered from (3) with

$$\Pi_\alpha(t,\nu) = \frac{1}{\alpha} e^{-\frac{4i\pi}{\alpha}\nu t}.$$

Note that we have $\Pi_\alpha\{b,a\} = \Pi_\alpha(b, 1/a\ -1)$, so that Π_α does not verify the same property (13') as W_s. This comes from the fact that the two smoothing procedures (3) and (15) act in a different way, so that no simple mapping $(t,\nu)\rightarrow\{b,a\}$ can make the connexion between (3) and (15) for every function of two variables involved. Besides, W_s^α can be recovered from W_s for α=0 with the limit of $\Pi_\alpha\{b,a\}$ as $\alpha\rightarrow 0$, i.e. $\delta(\{b,a\}=\{0,1\})$ as expected.

4.2 General assumptions providing $\Omega_\Pi(s)(b,a)$:

It is interesting to note that, in a similar manner to the Cohen's class, $\Omega_\Pi(s)(b,a)$ can be obtained with general assumptions as shown in Theorem 3:

4.2.1 Theorem 3: Let $\Omega(s)(b,a)$ be a bilinear form in $s(t)\in L^2(\mathbf{R},dt)$, i.e.:

$$\Omega(s)(b,a) = \int\int K(t,\tau,a,b)\ s(t)\ s^*(\tau)\ dt\ d\tau \tag{16}$$

where $K\in L^1(\mathbf{R}^4, dt\ d\tau\ db\ da/a)$. Assume that $\Omega(s)(b,a)$ is invariant under left $ax+b$-shifts, i.e.: for every element $\{b_o,a_o\}$ of the $ax+b$ group,

$$\Omega(U_{a_o}^{b_o}(s))(b,a) = \Omega(s)(\{b_o,a_o\}^{-1}\{b,a\}). \tag{17}$$

Then there exists a kernel $\Pi(b,a)$ such as:

$$\Omega_\Pi(s)(b,a) = \int\int W_s\{\beta,\alpha\}\ \Pi(\{b,a\}^{-1}\{\beta,\alpha\})\ \frac{d\alpha d\beta}{\alpha^2}. \tag{18}$$

Proof: $\Omega_\Pi(b,a)$ defined by (18) can obviously be written in the form (16) via (12'). The property (17) is also easy to verify, since (9) can be rewritten in the form:

$$W_{U_{a_o}^{b_o}(s)}\{b,a\} = W_s(\{b_o,a_o\}^{-1}\{b,a\})$$

and that the left-invariant measure $d\beta\ d\alpha/\alpha^2$ is by definition invariant under left $ax+b$ shifts $\{b,a\}\rightarrow\{b_o,a_o\}\{b,a\}$.

To prove the assertion, one just has to note that (17) implies:

$$K(a_o t+b_o,\ a_o\tau,\ b,\ a) = K(t,\ \tau, \frac{b-b_o}{a_o}, \frac{a_o}{a})$$

for every b_o,a_o. Thus K is only depending on $(t-b)/a$ and τ/a. Taking the partial Fourier Transform of $K((t-b)/a,\tau/a)$ in τ and making the transformation of variables: $\beta = t$ and $\alpha = 1/\nu$ (where ν is the dual Fourier variable of τ) leads to the announced formula ■

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