LIESSE
Fourier representation of random signals

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Outline

The Fourier transforms of sequences and functions

- Let $u = (u_t)_{t \in \mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$. Its Fourier series is defined by

$$u^*(\lambda) = \sum_t u_t \mathrm{e}^{-\mathrm{i}\lambda t} , \quad \lambda \in \mathbb{R} .$$

  It is well defined if $u \in \ell^1(\mathbb{Z})$. But it can be extended to $u \in \ell^2(\mathbb{Z})$ using a convergence in $L^2(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.
- Let $f : \mathbb{R} \to \mathbb{R}$. Its Fourier transform is defined by

$$f^*(\lambda) = \int f(t)\, \mathrm{e}^{-\mathrm{i}\lambda t} , \quad \lambda \in \mathbb{R} .$$

  It is well defined if $f \in L^1(\mathbb{R})$ but can be extended to $f \in L^2(\mathbb{R})$ or even to the case where f is a tempered distribution.

Well known properties

- Fourier transforms can be inverted using a similar transform.
- They have nice properties with respect to standard transformations: shifting, scaling, deriving...
- The Fourier transform of a convolution product is a simple product. For instance, for $u, v \in \ell^1(\mathbb{Z})$,

$$(u \star v)^*(\lambda) = u^*(\lambda)\, v^*(\lambda) , \quad \lambda \in \mathbb{R} .$$

  where $u \star v \in \ell^1(\mathbb{Z})$ is defined by

$$(u \star v)_k = \sum_{j \in \mathbb{Z}} u_j v_{k-j} , \quad k \in \mathbb{Z} .$$

Examples of applications

Time series analysis based on stochastic modeling is applied in various fields :

- Health : physiological signal analysis (image analysis).
- Engineering : monitoring, anomaly detection, localizing/tracking.
- Audio data : analysis, synthesis, coding.
- Ecology : climatic data, hydrology.
- Econometrics : economic/financial data.
- Insurance : risk analysis.

Heartbeats

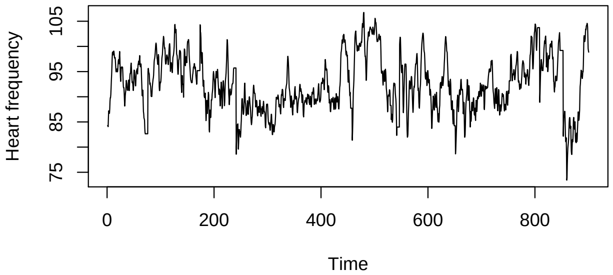

Figure: Heart rate of a resting person over a period of 900 seconds. This rate is defined as the number of heartbeats per unit of time. Here the unit is the minute and is evaluated every 0.5 seconds.

Internet traffic

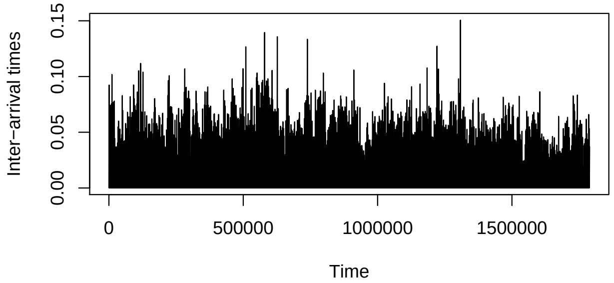

Figure: Inter-arrival times of TCP packets, expressed in seconds, obtained from a 2 hours record of the traffic going through an Internet link. `http://ita.ee.lbl.gov/`.

Speech audio data

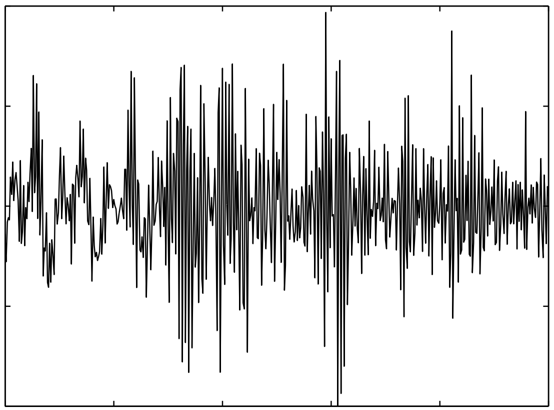

Figure: A speech audio signal with a sampling frequency equal to 8000 Hz. Record of the unvoiced fricative phoneme *sh* (as in *sh*arp).

Climatic data: wind speed

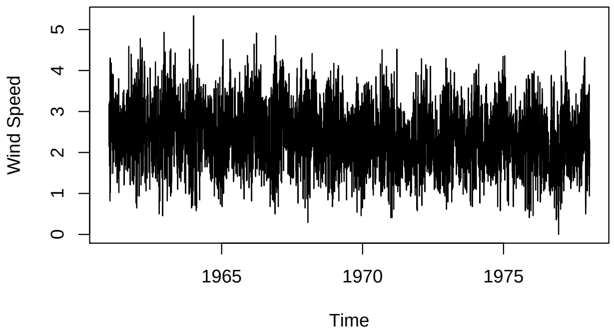

Figure: Daily record of the wind speed at Kilkenny (Ireland) in knots (1 knot = 0.5148 metres/second).

Climatic data: temperature changes

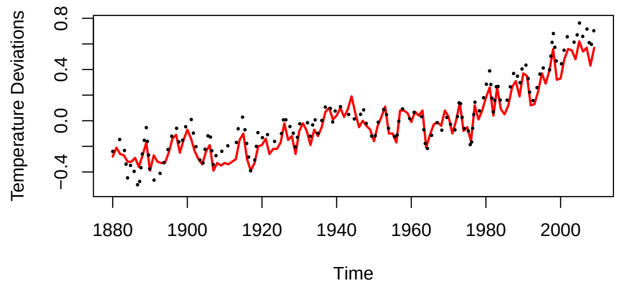

Figure: Global mean land-ocean temperature index (solid red line) and surface-air temperature index (dotted black line).
http://data.giss.nasa.gov/gistemp/graphs/.

Gross National Product of the USA

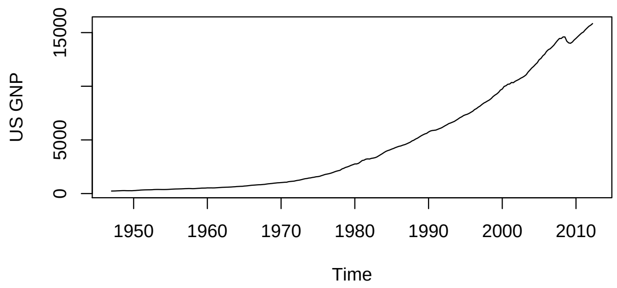

Figure: Growth national product (GNP) of the USA in Billions of $s.
http://research.stlouisfed.org/fred2/series/GNP.

GNP quarterly rate

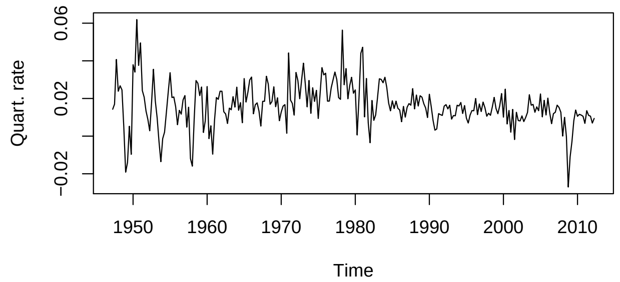

Figure: Quarterly rate of the US GNP.

Financial index

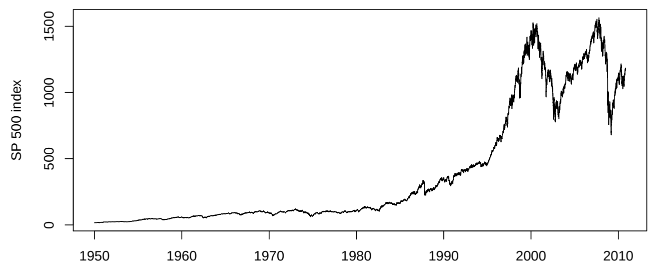

Figure: Daily open value of the Standard and Poor 500 index. This index is computed as a weighted average of the stock prices of 500 companies traded at the New York Stock Exchange (NYSE) or NASDAQ.

Financial index: log returns

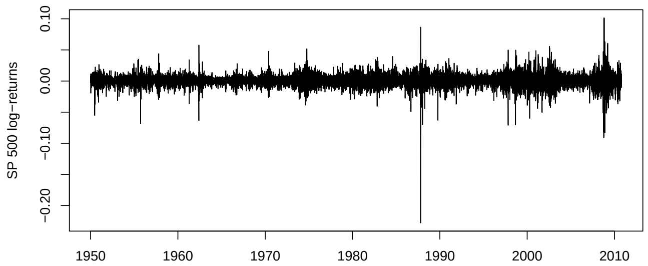

Figure: SP500 log-returns.

Stochastic modelling

Definition : time series

A time series valued in $(E,\mathcal{E})$ and indexed on $T=\mathbb{Z}$ is a collection of random variables $(X_t)_{t\in T}$ defined on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Definition : path

Let $(X_t)_{t\in T}$ be a random process defined on $(\Omega,\mathcal{F},\mathbb{P})$. The path of the random experiment $\omega\in\Omega$ is defined as $(X_t(\omega))_{t\in T}$ viewed as an element of E^T.

Definition : law

Let $X=(X_t)_{t\in T}$ be a random process. The law of X is defined as the image probability measure $\mathbb{P}^X=\mathbb{P}\circ X^{-1}$ on $(E^T,\mathcal{E}^{\otimes T})$.

Finite dimensional (fidi) distributions

For all $I\in\mathcal{I}(T)$ (a finite subset of T),

(i) denote by Π_I is the canonical projection $(x_t)_{t\in T}\mapsto(x_t)_{t\in I}$,

(ii) denote by X_I the random vector $(X_t)_{t\in I}=\Pi_I\circ X$,

(iii) denote by $\mathbb{P}^{X_I}$ the distribution of X_I, which is defined by

$$\mathbb{P}^{X_I}\left(\prod_{t\in I}A_t\right)=\mathbb{P}\left(X_t\in A_t,\,t\in I\right),\quad \text{where } A_t\in\mathcal{E}\text{ for all } t\in I\,.$$

Remark: $\mathbb{P}^X$ is characterized by the collection of fidi distributions $\left(\mathbb{P}^{X_I}\right)_{I\in\mathcal{I}(T)}$.

Backshift operator, stationarity

Definition : backshift operators

Let the backshift operator $\mathrm{B} : E^{\mathbb{Z}} \to E^{\mathbb{Z}}$ be defined by

$$\mathrm{B}(x) = (x_{t-1})_{t\in\mathbb{Z}} \quad \text{for all} \quad x = (x_t)_{t\in\mathbb{Z}} \in E^{\mathbb{Z}} .$$

For all $\tau \in \mathbb{Z}$, we define B^τ by

$$\mathrm{B}^\tau(x) = (x_{t-\tau})_{t\in\mathbb{Z}} \quad \text{for all} \quad x = (x_t)_{t\in\mathbb{Z}} \in E^{\mathbb{Z}} .$$

A process $X = (X_t)_{t\in T}$ is said to be stationary if X and $\mathrm{B} \circ X$ have the same distributions.

Examples: constant process, i.i.d. processes, Gaussian processes, ...

L^2 space

We set $E = \mathbb{C}$. We denote

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \left\{X \ \mathbb{C}\text{-valued r.v. such that } \mathbb{E}\left[|X|^2\right] < \infty\right\} .$$

$(L^2, \langle , \rangle)$ is a Hilbert space with

$$\langle X, Y \rangle = \mathbb{E}\left[X\overline{Y}\right] .$$

Definition : L^2 Processes

The process $X = (X_t)_{t \in T}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{C}$ is an L^2 process if $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \in T$.

Mean and covariance functions

Let $X = (X_t)_{t \in T}$ be an L^2 process.

- Its mean function is defined by $\mu(t) = \mathbb{E}[X_t]$,
- Its covariance function is defined by

$$\gamma(s,t) = \mathrm{cov}(X_s, X_t) = \mathbb{E}\left[X_s\overline{X_t}\right] - \mathbb{E}[X_s]\,\mathbb{E}\left[\overline{X_t}\right] .$$

Hermitian symmetry, non-negative definiteness

For all $I \in \mathcal{I}(T)$, $\Gamma_I = \mathrm{Cov}([X(t)]_{t \in I}) = [\gamma(s,t)]_{s,t \in I}$ is a hermitian non-negative definite matrix.

Examples

- L^2 independent random variables $(X_t)_{t \in \mathbb{Z}}$ have mean function $\mu(t) = \mathbb{E}(X_t)$ and covariance

$$\gamma(s,t) = \begin{cases} \mathrm{var}(X_t) & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases}$$

- A Gaussian process is an L^2 process whose law is entirely determined by its mean and covariance functions.

Weakly stationary processes

Let $T = \mathbb{Z}$. Let X be an L^2 strictly stationary process with mean function μ and covariance function γ.

Then $\mu(t) = \mu(0)$ and $\gamma(s,t) = \gamma(s-t,0)$ for all $s,t \in T$.

Definition : Weak stationarity

We say that a random process X is weakly stationary with mean μ and autocovariance function $\gamma : \mathbb{Z} \to \mathbb{C}$ if it is L^2 with mean function $t \mapsto \mu$ and covariance function $(s,t) \mapsto \gamma(s-t)$.

The autocorrelation function is defined (when $\gamma(0) > 0$) by

$$\rho(t) = \frac{\gamma(t)}{\gamma(0)} .$$

Examples

An L^2 strictly stationary process is weakly stationary.

- The constant L^2 process has constant autocovariance function.

Strong and weak white noise

- A sequence of L^2 i.i.d. random variables is called a strong white noise, denoted by $X \sim \mathrm{IID}(\mu, \sigma^2)$.
- An L^2 process X with constant mean μ and constant diagonal covariance function equal to σ^2 is called a weak white noise. It is denoted by $X \sim \mathrm{WN}(\mu, \sigma^2)$. (It does not have to be i.i.d.)

Examples based on stationarity preserving linear filters

Let X be weakly stationary with mean μ and autocovariance γ.

In the following examples, $Y = g(X)$ is weakly stationary with mean μ' and autocovariance γ'.

- Let g be the time reversing operator $(x_t)_{t\in\mathbb{Z}} \mapsto (x_{-t})_{t\in\mathbb{Z}}$. Then

$$\mu' = \mu \qquad \text{and} \qquad \gamma' = \overline{\gamma} .$$

- Let $g = \sum_k \psi_k \mathrm{B}^k : x \mapsto \psi \star x$ for a finitely supported sequence ψ.

  Then

$$\mu' = \mu \sum_k \psi_k$$

$$\gamma'(\tau) = \sum_{\ell,k} \psi_k \overline{\psi_\ell} \gamma(\tau + \ell - k) \tag{1}$$

Heartbeats : autoregression

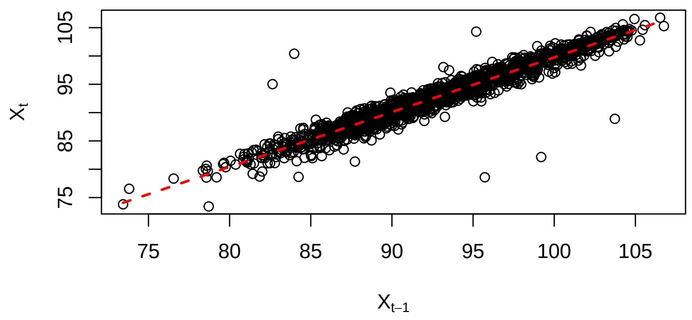

Figure: X_t VS X_{t-1} for the heartbeats data (see Figure 9). The red dashed line is the best linear fit.

Empirical estimates

Suppose you want to estimate the mean and the autocovariance from a sample $X_1, \dots, X_n$. Define the empirical mean as

$$\widehat{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \;,$$

and the empirical autocovariance and autocorrelation functions as

$$\widehat{\gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n-|h|} (X_k - \widehat{\mu}_n)(X_{k+|h|} - \widehat{\mu}_n) \quad \text{and}$$

$$\widehat{\rho}_n(h) = \frac{\widehat{\gamma}_n(h)}{\widehat{\gamma}_n(0)} \;.$$

Heartbeats : autocorrelation (empirical)

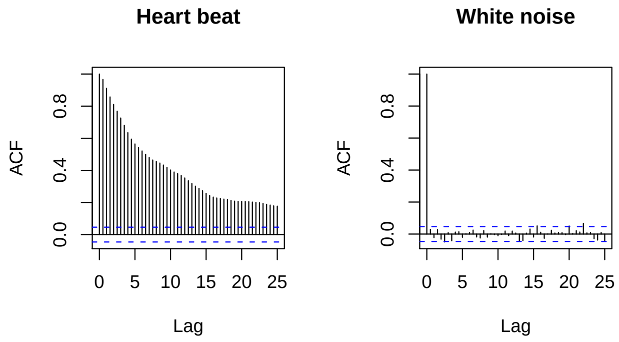

Figure: Left : empirical autocorrelation $\widehat{\rho}_n(h)$ of heartbeat data for $h = 0, \dots, 100$. Right : the same from a simulated white noise sample with same length.

Spectral measure

Given a function $\gamma : \mathbb{Z} \to \mathbb{C}$, does there exist a weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ with autocovariance γ?

Herglotz Theorem

Let $\gamma : \mathbb{Z} \to \mathbb{C}$. Then the two following assertions are equivalent:

(i) γ is hermitian symmetric and non-negative definite.

(ii) There exists a finite non-negative measure ν on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that,

$$\text{for all } t \in \mathbb{Z}, \quad \gamma(t) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t} \, \nu(\mathrm{d}\lambda) \;. \tag{2}$$

When these two assertions hold, ν is uniquely defined by (2).

Spectral density

If moreover $\gamma \in \ell^1(\mathbb{Z})$, these assertions are equivalent to

$$f(\lambda) := \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\lambda t} \gamma(t) \geq 0 \text{ for all } \lambda \in \mathbb{R} ,$$

and ν has density f (that is, $\nu(\mathrm{d}\lambda) = f(\lambda)\mathrm{d}\lambda$).

Definition : spectral measure and spectral density

If γ is the autocovariance of a weakly stationary process X, the corresponding measure ν is called the spectral measure of X. Whenever the spectral measure ν admits a density f, it is called the spectral density function.

Examples

- Let $X \sim \mathrm{WN}(\mu, \sigma^2)$. Then $f(\lambda) = \frac{\sigma^2}{2\pi}$.
- Let X be a weakly stationary process with covariance function γ/spectral measure ν. Define

$$Y = \sum_k \psi_k \, \mathrm{B}^k \circ X$$

for a finitely supported sequence ψ. Recall that Y is a weakly stationary process with covariance function

$$\gamma'(\tau) = \sum_{\ell,k} \psi_k \overline{\psi_\ell} \gamma(\tau + \ell - k) .$$

Then Y is a weakly stationary process with spectral measure ν' having density $\lambda \mapsto \left|\sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}\right|^2$ with respect to ν,

$$\nu'(\mathrm{d}\lambda) = \left| \sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k} \right|^2 \nu(\mathrm{d}\lambda) .$$

A special one : the harmonic process

Let $(A_k)_{1 \leq k \leq N}$ be N real valued L^2 random variables. Denote $\sigma_k^2 = \mathbb{E}\left[A_k^2\right]$. Let $(\Phi_k)_{1 \leq k \leq N}$ be N i.i.d. random variables with a uniform distribution on $[0, 2\pi]$, and independent of $(A_k)_{1 \leq k \leq N}$. Define

$$X_t = \sum_{k=1}^{N} A_k \cos(\lambda_k t + \Phi_k) , \qquad (3)$$

where $(\lambda_k)_{1 \leq k \leq N} \in [-\pi, \pi]$ are N frequencies. The process (X_t) is called a harmonic process. It satisfies $\mathbb{E}[X_t] = 0$ and, for all $s, t \in \mathbb{Z}$,

$$\mathbb{E}[X_s X_t] = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k (s - t)) .$$

Hence X is weakly stationary with autocovariance

$$\gamma(t) = \frac{1}{2} \sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k t) .$$

Example: Spectral representation of the harmonic process

We deduce that X has spectral measure

$$\mu = \frac{1}{4}\sum_{k=1}^{N} \sigma_k^2 \left(\delta_{\lambda_k} + \delta_{-\lambda_k}\right) ,$$

where we denote by δ_λ the Dirac mass at point λ.

Similarly, we can write

$$\begin{aligned} X_t &= \frac{1}{2}\sum_{k=1}^{N} \left(A_k \mathrm{e}^{\mathrm{i}\Phi_k}\, \mathrm{e}^{\mathrm{i}\lambda_k t} + A_k \mathrm{e}^{-\mathrm{i}\Phi_k}\, \mathrm{e}^{-\mathrm{i}\lambda_k t}\right) \\ &= \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}W(\lambda) , \qquad (4) \end{aligned}$$

where W is the random (complex valued) measure

$$W = \frac{1}{2}\sum_{k=1}^{N} \left(A_k \mathrm{e}^{\mathrm{i}\Phi_k}\, \delta_{\lambda_k} + A_k \mathrm{e}^{-\mathrm{i}\Phi_k}\, \delta_{-\lambda_k}\right) .$$

Spectral representation

One can interpret the relation between X and W in (4) as saying that W is the Fourier transform of X, so we denote it by $\widehat{X}$:

$$X_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}\widehat{X}(\lambda), \qquad t \in \mathbb{Z} .$$

This spectral representation of X can be extended to any weakly stationary processes with some remarkable properties on $\widehat{X}$.

But some work is necessary.

- ▷ The paths of X are random sequences, usually unbounded (no decay at infinity can be used!) so $\mathrm{d}\widehat{X}$ cannot be in the "nice" form $\widehat{X}(\lambda)\mathrm{d}\lambda$.
- ▷ On the contrary, $\widehat{X}$ is always a random measure defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.
- ▷ For the same reason, there is no simple formula for defining $\widehat{X}$ from X : we rely on Hilbert geometry.

Construction of the spectral random field

Let $(X_t)_{t\in\mathbb{Z}}$ be a centered weakly stationary with spectral measure η. The relation

$$X_t = \int_{\mathbb{T}} e^{i\lambda t}\, d\widehat{X}(\lambda), \qquad t \in \mathbb{Z}\ , \tag{5}$$

in fact means that one can define an isometric operator from $L^2(\eta)$ to $L^2(\mathbb{P})$ that maps $\lambda \mapsto e^{i\lambda t}$ to X_t for all t.

Step 1 Define

$$\mathcal{H}_\infty^X = \overline{\mathrm{Span}}\,(X_t,\, t\in\mathbb{Z})\ .$$

Step 2 As previously, we can extend $X_t \mapsto e^{it\cdot}$ linearly and continuously as a unitary linear operator from $\mathcal{H}_\infty^X$ to $L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\eta)$.

Step 3 Since $\overline{\mathrm{Span}}\left(e^{it\cdot},\, t\in\mathbb{Z}\right) = L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\eta)$, this operator is bijective.

Step 4 Let $\widehat{X}$ be its inverse operator.

Then $\widehat{X}$ is a random field with orthogonal increments with intensity measure η on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$, and (5) holds true.

Spectral representation

Moreover, by construction, every $Y_0 \in \mathcal{H}_\infty^X$ can be represented as

$$Y_0 = \int g(\lambda)\, d\widehat{X}(\lambda)\ .$$

for a (unique) well chosen $g \in L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\eta)$.

In particular, for all $t\in\mathbb{Z}$,

$$Y_t = B^t Y_0 = \int e^{it\lambda}\, g(\lambda)\, d\widehat{X}(\lambda)\ .$$

Observe that, for all $s,t\in\mathbb{Z}$,

$$\mathrm{Cov}\,(Y_s, Y_t) = \int_{\mathbb{T}} e^{i\lambda(s-t)}\, |g(\lambda)|^2\, d\nu(\lambda)\ .$$

Hence $Y = (Y_t)_{t\in\mathbb{Z}}$ is a centered weakly stationary process and its spectral measure has density $|g|^2$ with respect to ν, the spectral measure of X.

Example: complex-valued Harmonic processes

The previous definition of harmonic processes can be extended as follows.

Definition : Harmonic processes

The process $(X_t)_{t\in\mathbb{Z}}$ is an harmonic process if its spectral representation $\widehat{X}$ is of the form

$$\widehat{X} = \sum_{k=1}^{n} Z_k \delta_{\lambda_k}\ ,$$

where $\lambda_1,\dots,\lambda_n$ are deterministic frequencies in $\mathbb{T}$ and $Z_1,\dots,Z_n$ are uncorrelated centered $\mathbb{C}$-valued random variables.

Example: real-valued Harmonic processes

To obtained a real valued process $\widehat{X}$ must satisfy an hermitian symmetry $\widehat{X}(-A) = \overline{\widehat{X}(A)}$.

Hence, for a real valued harmonic process, we obtain for $0 < \lambda_0 < \cdots < \lambda_n \leq \pi$,

$$\widehat{X} = Z_0 \delta_0 + \sum_{k=1}^{N} (Z_k \delta_{\lambda_k} + \overline{Z_k} \delta_{-\lambda_k}) ,$$

where $Z_0, Z_1, \ldots, Z_N, \overline{Z_1}, \ldots, \overline{Z_N}$ are uncorrelated centered $\mathbb{C}$-valued random variables and Z_0 is real valued.

(Recall our previous example where $Z_k = \frac{1}{2} A_k \mathrm{e}^{\mathrm{i}\Phi_k}$.)

Examples

Centered white noise

If $(X_t)_{t \in \mathbb{Z}} \sim \mathrm{WN}(0, \sigma^2)$ then $\widehat{X}$ satisfies

$$\mathrm{Var}\left(\widehat{X}((\lambda', \lambda])\right) = \frac{\sigma^2}{2\pi} (\lambda - \lambda') , \quad \lambda' < \lambda < \lambda' + 2\pi .$$

Linear filtering

Let $(X_t)_{t \in \mathbb{Z}}$ be centered, weakly stationary with spectral measure ν and spectral representation $\widehat{X}$. Then for any $\widehat{g} \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$, one can define a centered, weakly stationary process $(Y_t)_{t \in \mathbb{Z}}$ by its spectral representation $\widehat{Y}(\mathrm{d}\lambda) = \widehat{g}(\lambda) \widehat{X}(\mathrm{d}\lambda)$,

$$Y_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i} t \lambda} \, \widehat{Y}(\mathrm{d}\lambda) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i} t \lambda} \, \widehat{g}(\lambda) \widehat{X}(\mathrm{d}\lambda) ,$$

and $(Y_t)_{t \in \mathbb{Z}}$ is centered, weakly stationary with spectral measure $\nu'(\mathrm{d}\lambda) = |\widehat{g}(\lambda)|^2 \, \nu(\mathrm{d}\lambda)$.