Towards a Minimal Axiomatization of Quantum Information: Schrödinger *Ex Nihilo*

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Abstract

Starting from only two simple quantum postulates, one can derive Schrödinger's equation governing the evolution of any finite-dimensional quantum system. This fundamental equation of physics, which may appear rather abstruse to the signal processing community, is in fact merely a consequence of orthogonality properties in Hilbert spaces together with a probabilistic assumption on quantum measurement. The mathematical proof relies on simplified versions of Wigner's and Stone's theorems.

1. Introduction

The literature on quantum information—including the outstanding reference textbook by Nielsen and Chuang [4]—typically begins by stating four, five, or six postulates of quantum mechanics as its foundation. However, previous exposure to quantum mechanics is not necessarily of great help: wave functions, wave-particle duality, wave-function collapse, the uncertainty principle, Schrödinger's equation postulating the evolution of a system, not to mention the Hilbert-space formalism and infinite-dimensional operators, all have relatively little direct relevance to quantum computation and quantum information, where the systems under consideration are finite-dimensional.

The purpose of this paper is therefore to propose a **minimal axiomatization** (restricted to finite-dimensional systems), based essentially on two postulates:

1. a postulate describing the state of a system;
2. a postulate describing quantum measurement together with its probabilistic nature.

These postulates are, in fact, closer to definitions establishing the physical vocabulary than to genuine mathematical axioms. Everything else relevant to quantum computation and quantum information follows naturally from them. As an illustration, we shall derive Schrödinger's equation, governing the evolution of an isolated quantum system, solely from these basic assumptions (together with a stationarity assumption introduced in Section 3).

2. Two Axioms

Axiom 1

The state of a physical system is represented by a vector in a Hermitian vector space (called the state space).

A quantum system is therefore characterized by a Hermitian space (equivalently, a finite-dimensional Hilbert space) H, namely a complex vector space of dimension d, where d is called the **dimension of the system**. Every possible state of the system belongs to this space.

The celebrated **superposition principle** of quantum mechanics is simply a consequence of the linear structure of this vector space.

Using Dirac notation, an orthonormal basis—called the **computational basis**—is denoted simply by

$$|1\rangle,\ |2\rangle,\ \ldots,\ |d\rangle.$$

Any state is therefore expressed as the linear superposition

$$|\phi\rangle = c1|1\rangle + c2|2\rangle + \cdots + cd|d\rangle,$$

where the complex coefficients satisfy

$$ck = \langle k|\phi\rangle.$$

Axiom 2

Information about the state $|\phi\rangle$ of a system can only be obtained through a measurement. A given measurement randomly transforms $|\phi\rangle$ into one of several possible post-measurement states $|\psi\rangle$, with transition probability

$$P(|\phi\rangle \to |\psi\rangle) = \left(\frac{|\langle\phi|\psi\rangle|}{\|\phi\|\,\|\psi\|}\right)^2.$$

Thus, every quantum measurement is inherently probabilistic and instantaneously changes the state of the system.

Expression (1) may be interpreted as the **coefficient of determination** (the square of a correlation coefficient) between the two states. By the Cauchy–Schwarz inequality, it naturally takes values between 0 and 1.

Beyond its symmetry,

$$P(|\phi\rangle \to |\psi\rangle) = P(|\psi\rangle \to |\phi\rangle),$$

this expression possesses several important consequences.

Non-vanishing

The zero vector cannot represent a physical state, since it would imply a vanishing transition probability toward every possible state.

Normalization and phase freedom

If two vectors differ only by a nonzero scalar factor,

$$|\phi\rangle \quad \text{and} \quad \alpha|\phi\rangle,$$

they produce exactly the same probabilities, because the scalar cancels in Equation (1). Consequently, they are physically indistinguishable and must be regarded as representing the same quantum state.

Accordingly, one may always normalize a state,

$$|\phi\rangle \longrightarrow \frac{|\phi\rangle}{\|\phi\|},$$

although normalization is not always required during intermediate calculations.

Furthermore, since the scalar α may be complex, a state is defined only up to an arbitrary global phase:

$$|\phi\rangle \sim e^{i\theta}|\phi\rangle.$$

Orthogonality of measurement outcomes

Once a measurement has produced the state $|\psi\rangle$, repeating the same measurement necessarily yields the same state with probability one. Consequently, every alternative outcome must occur with zero probability, implying that distinct outcomes of a given measurement are mutually orthogonal.

Since a d-dimensional Hilbert space can contain at most d mutually orthogonal vectors, the number of distinct outcomes of a projective measurement cannot exceed the dimension of the system.

2. Two Axioms (continued)

Projection and Born's Rule

Let

$$\Pi\psi|\phi\rangle = \frac{|\psi\rangle}{\|\psi\|}\left\langle \frac{\psi}{\|\psi\|}\middle|\phi\right\rangle = \frac{|\psi\rangle\langle\psi|}{\|\psi\|^2}|\phi\rangle$$

denote the orthogonal projection of the state $|\phi\rangle$ onto the one-dimensional subspace spanned by $|\psi\rangle$. Equation (1) then becomes the familiar **Born rule**

$$P(|\phi\rangle \to |\psi\rangle) = \frac{\|\Pi\psi\phi\|^2}{\|\phi\|^2},$$

which states that the transition probability is proportional to the squared norm of the orthogonal projection of the initial state onto the final state. When $|\phi\rangle$ is normalized, this reduces simply to

$$P(|\phi\rangle \to |\psi\rangle) = \|\Pi\psi\phi\|^2.$$

More generally, for two orthogonal states $|\psi\rangle$ and $|\psi'\rangle$, the probability of obtaining either outcome is the sum of the individual probabilities,

$$P(|\phi\rangle \to |\psi\rangle) + P(|\phi\rangle \to |\psi'\rangle),$$

which equals

$$\frac{\|\Pi\psi\phi\|^2}{\|\phi\|^2} + \frac{\|\Pi\psi'\phi\|^2}{\|\phi\|^2} = \frac{\|\Pi\psi,\psi'\phi\|^2}{\|\phi\|^2},$$

by the Pythagorean theorem. Here $\Pi\psi,\psi'$ denotes the orthogonal projector onto the two-dimensional subspace generated by $|\psi\rangle$ and $|\psi'\rangle$.

For dimensions greater than two, the same argument extends immediately to any collection of mutually orthogonal states. Thus, the square appearing in Equation (1) is precisely what ensures that probabilities of mutually exclusive events add according to the Pythagorean theorem.

Finally, suppose that the possible outcomes of a measurement form an orthonormal basis

$$|\psi1\rangle,\ |\psi2\rangle,\ \ldots,\ |\psi d\rangle.$$

Expanding an arbitrary state as

$$|\phi\rangle = c1|\psi1\rangle + c2|\psi2\rangle + \cdots + cd|\psi d\rangle,$$

Born's rule becomes

$$Pk = \frac{|ck|^2}{|c1|^2 + \cdots + |cd|^2} = \frac{\|\Pi k\phi\|^2}{\|\phi\|^2},$$

where $\Pi k = |\psi k\rangle\langle\psi k|$ is the orthogonal projector onto the k-th basis vector.

As expected,

$$\sum_k Pk = 1.$$

This naturally leads to the general notion of **von Neumann projective measurement**, together with the associated concept of a measurement outcome, defined through an observable.

3. Evolution of an Isolated System

3.1 Wigner's Theorem

Let us now consider the evolution of the possible states of an isolated quantum system, i.e., one that does not interact with its environment.

After an arbitrarily short time interval, every state undergoes a transformation

$$E : H \to H, \qquad |\phi\rangle \mapsto |\phi\rangle',$$

where, for convenience, we write either $|\phi\rangle'$ or $|\phi'\rangle$.

Without loss of generality, the transformed states may always be normalized so that

$$\|\phi'\| = \|\phi\|,$$

and this normalization will henceforth be assumed.

We further assume that the statistical properties of a given measurement apparatus are **stationary**, so that transition probabilities remain unchanged during the evolution:

$$P(|\phi\rangle \to |\psi\rangle) = P(|\phi'\rangle \to |\psi'\rangle).$$

Since the norms are preserved, this condition is equivalent to

$$|\langle\phi|\psi\rangle| = |\langle\phi'|\psi'\rangle|$$

for every pair of states.

Each vector of the computational basis,

$$|k\rangle,$$

is transformed into another vector

$$|k\rangle',$$

and because

$$|\langle k'|\ell'\rangle| = |\langle k|\ell\rangle| = \delta k\ell,$$

the transformed vectors also constitute an orthonormal basis.

Writing

$$|\phi\rangle = \sum_k ck|k\rangle, \qquad |\phi'\rangle = \sum_k c'k|k\rangle',$$

one immediately obtains

$$|ck| = |c'k|$$

for every coefficient.

The remaining question concerns the phases of these complex coefficients, which determine the exact nature of the evolution.

Since the global phase of any quantum state is physically irrelevant, one may always choose the phase of $|\phi'\rangle$ arbitrarily.

Theorem 1 (Wigner)

By suitable choices of phase, exactly one of the following two possibilities holds for every state:

$$c\overset{\prime}{k} = ck$$

for every k, or

$$c\overset{\prime}{k} = c\overset{*}{k}$$

for every k.

Wigner originally provided only a sketch of the proof of this theorem. Since then, numerous demonstrations have appeared, either concise but incomplete, technically involved, or relying on additional assumptions such as bijectivity or regularity.

The proof presented here appears to be new. It relies only on elementary geometric properties of the complex coefficients ck, viewed as points in the Euclidean plane identified with the complex plane.

In what follows, $|k\rangle$ and $|\ell\rangle$ denote vectors of the computational basis.

The proof begins with several elementary examples in which all coefficients are real, implying

$$c\overset{\prime}{k} = ck = c\overset{*}{k}.$$

3.1 Wigner's Theorem (continued)

Example 1

By an appropriate choice of phase for the transformed basis vectors $|k\rangle'$, one can ensure that, for every $k \neq 1$,

$$(|k\rangle - |1\rangle)' = |k\rangle' - |1\rangle'.$$

Indeed, one knows that

$$(|k\rangle - |1\rangle)' = c|k\rangle' - d|1\rangle',$$

with $|c| = |d| = 1$. By suitably choosing the phase of $|1\rangle'$, one may set $d = 1$, and similarly choose the phase of $|k\rangle'$ so that $c = 1$.

Example 2

Assume that $d > 2$, and let k, ℓ, and 1 be distinct indices. By a suitable choice of phase,

$$(|k\rangle + |\ell\rangle - |1\rangle)' = |k\rangle' + |\ell\rangle' - |1\rangle'.$$

Indeed, one may write

$$(|k\rangle + |\ell\rangle - |1\rangle)' = c|k\rangle' + d|\ell\rangle' - |1\rangle',$$

where again $|c| = |d| = 1$.

Applying Equation (6) to

$$|\phi\rangle = |k\rangle + |\ell\rangle - |1\rangle$$

and

$$|\psi\rangle = |k\rangle - |1\rangle,$$

one finds

$$\langle\psi|\phi\rangle = 2,$$

while

$$\langle\psi'|\phi'\rangle = c + 1.$$

Hence,

$$|c + 1| = 2.$$

Geometrically, the only point in the complex plane satisfying both

$$|c| = 1 \quad \text{and} \quad |c + 1| = 2$$

is

$$c = 1.$$

The same reasoning shows that $d = 1$.

Example 3 (Generalization of Example 1)

By an appropriate phase choice, for every pair $k > \ell$,

$$(|k\rangle - |\ell\rangle)' = |k\rangle' - |\ell\rangle'.$$

Indeed, assuming $d > 2$, one may write

$$(|k\rangle - |\ell\rangle)' = |k\rangle' - c|\ell\rangle',$$

where $|c| = 1$.

Applying Equation (6) to

$$|\phi\rangle = |k\rangle - |\ell\rangle$$

and

$$|\psi\rangle = |k\rangle + |\ell\rangle - |1\rangle,$$

gives

$$\langle\psi|\phi\rangle = 0,$$

whereas

$$\langle\psi'|\phi'\rangle = 1 - c.$$

Therefore,

$$|1 - c| = 0,$$

which immediately implies

$$c = 1.$$

Corollary 1

For every evolution

$$|\phi\rangle \mapsto |\phi\rangle',$$

the coefficients satisfy

$$|ck - c\ell| = |c'k - c'\ell|$$

for all indices k, ℓ.

Proof
Assume $k > \ell$. Applying Equation (6) to the state

$$|\psi\rangle = |k\rangle - |\ell\rangle,$$

one has

$$\langle\psi|\phi\rangle = ck - c\ell,$$

and

$$\langle\psi'|\phi'\rangle = c'k - c'\ell.$$

Taking absolute values immediately yields the desired identity.

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Lemma 1

A point c in the complex plane is uniquely determined by its distances to at least three given non-collinear points.

If all reference points are collinear, distances to two distinct points determine c uniquely up to reflection across the line joining those points.

Proof
The locus of points equidistant from two distinct points is the perpendicular bisector of the segment joining them.

Hence ambiguity arises only when all reference points are collinear, in which case the two possible solutions are mirror images with respect to that common line.

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Corollary 2

For every evolution

$$|\phi\rangle \mapsto |\phi\rangle',$$

one may choose the global phase of the transformed state so that either

$$c\acute{k} = ck$$

for every k, or

$$c\acute{k} = c\overset{*}{k}$$

for every k.

Proof
Suppose that all pairwise distances among the $d + 1$ points

$$0,\ c1,\ c2,\ \ldots,\ cd$$

are known.

If all these points are collinear, the conclusion is immediate.

Otherwise, once one non-collinear point has been fixed, every remaining point is uniquely determined by its distances to three non-collinear reference points, according to Lemma 1.

Consequently, the entire configuration is determined uniquely, except possibly for a reflection.

If no reflection occurs, one chooses the global phase so that

$$c\acute{1} = c1,$$

and Equations (8) and (12) imply

$$c\acute{k} = ck$$

for every k.

If a reflection occurs instead, the global phase is chosen so that

$$c\acute{1} = c\overset{*}{1},$$

yielding

$$c\acute{k} = c\overset{*}{k}$$

for every coefficient.

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3.1 Wigner's Theorem (conclusion)

Corollary 3

Assume that there exist indices $k > \ell$ such that the three points 0, ck, and $c\ell$ are not collinear.

If

$$(|k\rangle - i|\ell\rangle)' = |k\rangle' - i|\ell\rangle',$$

then

$$\overset{\prime}{cj} = cj$$

for every index j.

Conversely, if

$$(|k\rangle - i|\ell\rangle)' = |k\rangle' + i|\ell\rangle',$$

then

$$\overset{\prime}{cj} = c\overset{*}{j}$$

for every j.

Proof

Since 0, ck, and $c\ell$ are not collinear, both coefficients are nonzero and the ratio

$$\alpha = \frac{ck}{c\ell}$$

is not real.

Applying Equation (6) to the states

$$|\phi\rangle \quad \text{and} \quad |\psi\rangle = |k\rangle - i|\ell\rangle,$$

gives

$$\langle\psi|\phi\rangle = ck + ic\ell.$$

Under the first hypothesis, two possibilities remain:

$$\langle\psi'|\phi'\rangle = ck + ic\ell$$

or

$$c\overset{*}{k} + ic\overset{*}{\ell}.$$

The second possibility would imply

$$|ck + ic\ell| = |ck - ic\ell|,$$

which forces

$$\operatorname{Im}\alpha = 0,$$

contradicting the assumption that the three points are not collinear.

Hence only the first possibility is admissible, yielding

$$\overset{\prime}{cj} = cj$$

for every coefficient.

The second hypothesis is treated identically, leading instead to

$$\overset{\prime}{cj} = c\overset{*}{j}.$$

Since the particular choice of k and ℓ is irrelevant, one concludes that exactly one of these two possibilities holds for **every** quantum state.

This completes the proof of Wigner's theorem.

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3.2 Principle of Unitary Evolution

Theorem 2

Every continuous, stationary evolution of an isolated quantum system is represented by a unitary operator U, such that

$$|\phi\rangle' = U|\phi\rangle$$

for every state.

Introducing time explicitly, let

$$|\phi 0\rangle = |\phi\rangle$$

be the initial state and

$$|\phi t\rangle = |\phi\rangle'$$

its state after time t.

The unitary evolution principle then reads

$$|\phi t\rangle = Ut|\phi 0\rangle,$$

where Ut depends on time.

Proof

According to Wigner's theorem, two possibilities exist.

In the first, the evolution merely changes the orthonormal basis while leaving the expansion coefficients unchanged. Such a transformation is linear and preserves inner products,

$$\langle\phi'|\psi'\rangle = \langle\phi|\psi\rangle,$$

and therefore corresponds to a **unitary operator**.

The second possibility additionally complex-conjugates the coefficients. The corresponding transformation is **anti-linear** and is called **anti-unitary**.

Suppose every continuous time evolution were anti-unitary.

Considering three successive instants 0, t, and $t+\tau$, stationarity gives

$$\bar{U}t{+}\tau = \bar{U}t\bar{U}\tau.$$

However, the product of two anti-unitary operators is unitary, contradicting the above identity.

Therefore continuous physical evolution cannot be anti-unitary.

It necessarily follows that

$$|\phi\rangle' = U|\phi\rangle,$$

with U unitary.

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3.3 Stone's Theorem

Theorem 3 (Stone)

There exists a Hermitian operator Ω such that

$$Ut = e^{-i\Omega t}.$$

Although Stone and von Neumann established this theorem in the general setting of infinite-dimensional Hilbert spaces, the finite-dimensional proof is remarkably simple.

Proof

Stationarity implies the group property

$$Ut{+}\tau = UtU\tau,$$

for every $t, \tau \geq 0$.

Since all operators Ut are unitary and commute,

$$UtU\tau = U\tau Ut,$$

they can be simultaneously diagonalized in an orthonormal basis

$$|u1\rangle, \ldots, |ud\rangle.$$

Thus,

$$Ut = \sum_k \chi k(t)|uk\rangle\langle uk|,$$

where each eigenvalue satisfies

$$|\chi k(t)| = 1.$$

The group property becomes

$$\chi k(t+\tau) = \chi k(t)\chi k(\tau).$$

Integrating this functional equation shows that every eigenvalue is continuous and differentiable.

Differentiation yields

$$\frac{d}{dt}\chi k(t) = -i\omega k \chi k(t),$$

whose solution is

$$\chi k(t) = e^{-i\omega k t}.$$

Since every eigenvalue has unit modulus,

$$\omega k \in \mathbb{R}.$$

Therefore,

$$Ut = \sum_k e^{-i\omega k t}|uk\rangle\langle uk| = e^{-i\Omega t},$$

where

$$\Omega = \sum_k \omega k|uk\rangle\langle uk|$$

is Hermitian.

∎

The quantities ωk are angular frequencies (rad·s^{-1}). Introducing the ordinary frequencies

$$\nu k = \frac{\omega k}{2\pi},$$

their associated energies are

$$Ek = h\nu k = \hbar\omega k,$$

which are precisely the eigenvalues of the Hamiltonian

$$H = \hbar\Omega.$$

Differentiating

$$|\phi t\rangle = Ut|\phi 0\rangle$$

and using the exponential representation above immediately gives

$$i\hbar\frac{d}{dt}|\phi t\rangle = H|\phi t\rangle.$$

Corollary 4 (Schrödinger Equation)

$$\boxed{i\hbar\frac{d}{dt}|\phi t\rangle = H|\phi t\rangle}$$

where $H = \hbar\Omega$ is the Hamiltonian of the system.

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4. Conclusion

This work has shown that Schrödinger's differential equation governing the evolution of an isolated quantum system is ultimately nothing more than the logical consequence of a remarkably small set of elementary quantum postulates.