(μ) Tensor Products of Operators

Definition. *The* ***tensor product*** *of two* ***linear maps*** $M : \mathcal{H}_A \to \mathcal{H}_{A'}$ *and* $N : \mathcal{H}_B \to \mathcal{H}_{B'}$ *is the unique linear map*

$$M \otimes N : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$$

such that, for all vectors $|\psi_A\rangle \in \mathcal{H}_A$ *and* $|\psi_B\rangle \in \mathcal{H}_B$*,*

$$(M \otimes N)(|\psi_A\rangle \otimes |\psi_B\rangle) = M|\psi_A\rangle \otimes N|\psi_B\rangle.$$

This linear map is uniquely determined by its image on any orthonormal basis $|ab\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$, since

$$(M \otimes N)|ab\rangle = (M \otimes N)(|a\rangle \otimes |b\rangle) = M|a\rangle \otimes N|b\rangle.$$

Remark. The image of $M \otimes N$ is therefore spanned by the vectors $M|a\rangle \otimes N|b\rangle$; it is the tensor product of the images of M and N, whose dimension is the product of the dimensions of the images, that is, the product of the ranks $r_M r_N$ of M and N. The rank of the tensor product $M \otimes N$ is therefore $r_{M\otimes N} = r_M r_N$.

By its definition, the tensor product $M \otimes N$ of linear maps is evidently **bilinear** in its arguments (linear in M and linear in N).

Let $|a\rangle$ $(a \in A)$, $|a'\rangle$ $(a' \in A')$, $|b\rangle$ $(b \in B)$, and $|b'\rangle$ $(b' \in B')$ denote orthonormal bases of $\mathcal{H}_A$, $\mathcal{H}_{A'}$, $\mathcal{H}_B$, and $\mathcal{H}_{B'}$, of dimensions d_A, $d_{A'}$, d_B, and $d_{B'}$, respectively. For these bases, we may therefore identify $M \otimes N$ with a **matrix** of size $d_{A'}d_{B'} \times d_A d_B$ whose coefficients are

$$\langle a'b'|M \otimes N|ab\rangle = (\langle a'| \otimes \langle b'|)(M|a\rangle \otimes N|b\rangle) = \langle a'|M|a\rangle\, \langle b'|N|b\rangle\,.$$

In other words, if we write $M = \sum_{a'\in A',\, a\in A} M_{a'a}|a'\rangle\langle a|$ and $N = \sum_{b'\in B',\, b\in B} N_{b'b}|b'\rangle\langle b|$, then

$$M \otimes N = \sum_{a'\in A',\, a\in A}\ \sum_{b'\in B',\, b\in B} M_{a'a}N_{b'b}|a'b'\rangle\langle ab|.$$

This matrix $M \otimes N$ is called the **Kronecker product** of the matrices M and N; for given bases, it is identified with the **tensor product** of the two linear maps M and N. When a particular order is specified for the indices a and b, for example the ordered index sets $A = \{1, 2, \ldots, d_A\}$ and $B = \{1, 2, \ldots, d_B\}$, the indices $ab = (a, b) \in A \times B$ are ordered lexicographically. Setting $\alpha_{i,j} = \langle i|M|j\rangle$ and $\beta_{k,\ell} = \langle k|N|\ell\rangle$ for the coefficients of the matrices M and N, respectively, the matrix $M \otimes N$ is written in the form of a block matrix:

$$\begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,d_A} \\ \vdots & \ddots & \vdots \\ \alpha_{d_{A'},1} & \cdots & \alpha_{d_{A'},d_A} \end{pmatrix} \otimes \begin{pmatrix} \beta_{1,1} & \cdots & \beta_{1,d_B} \\ \vdots & \ddots & \vdots \\ \beta_{d_{B'},1} & \cdots & \beta_{d_{B'},d_B} \end{pmatrix} = \begin{pmatrix} \alpha_{1,1}N & \cdots & \alpha_{1,d_A}N \\ \vdots & \ddots & \vdots \\ \alpha_{d_{A'},1}N & \cdots & \alpha_{d_{A'},d_A}N \end{pmatrix}.$$

that is, expanding:

$$\begin{pmatrix}
\alpha_{1,1}\beta_{1,1} & \cdots & \alpha_{1,1}\beta_{1,d_B} & & \alpha_{1,d_A}\beta_{1,1} & \cdots & \alpha_{1,d_A}\beta_{1,d_B} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
\alpha_{1,1}\beta_{d_{B'},1} & \cdots & \alpha_{1,1}\beta_{d_{B'},d_B} & & \alpha_{1,d_A}\beta_{d_{B'},1} & \cdots & \alpha_{1,d_A}\beta_{d_{B'},d_B} \\
 & \vdots & & \ddots & & \vdots & \\
\alpha_{d_{A'},1}\beta_{1,1} & \cdots & \alpha_{d_{A'},1}\beta_{1,d_B} & & \alpha_{d_{A'},d_A}\beta_{1,1} & \cdots & \alpha_{d_{A'},d_A}\beta_{1,d_B} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
\alpha_{d_{A'},1}\beta_{d_{B'},1} & \cdots & \alpha_{d_{A'},1}\beta_{d_{B'},d_B} & & \alpha_{d_{A'},d_A}\beta_{d_{B'},1} & \cdots & \alpha_{d_{A'},d_A}\beta_{d_{B'},d_B}
\end{pmatrix}$$

Remark. In the special case $d_A = d_B = 1$, by identifying $|\psi_A\rangle$ and $|\psi_B\rangle$ with linear maps $\mathbb{C} \to \mathcal{H}_A$ and $\mathbb{C} \to \mathcal{H}_B$, respectively, we recover the definition of the tensor product $|\psi_A \otimes \psi_B\rangle$, which may be viewed as a special Kronecker product.

As in this special case, the Kronecker product is not commutative in general: $M \otimes N \neq N \otimes M$; the two matrices contain the same entries but in different arrangements.

Example. The map $\langle a| \otimes \mathbb{1}_B$, applied to a vector $|\Psi\rangle = \sum_{a',b} \alpha_{a',b}|a'\rangle|b\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$, gives

$$|\phi_a\rangle = (\langle a| \otimes \mathbb{1}_B)|\Psi\rangle = \sum_{a'}\sum_{b} \alpha_{a',b} \langle a|a'\rangle \; \mathbb{1}_B|b\rangle = \sum_{b} \alpha_{a,b}|b\rangle \in \mathcal{H}_B.$$

We can therefore write $|\Psi\rangle = \sum_a |a\rangle \otimes |\phi_a\rangle$. The operator $\langle a| \otimes \mathbb{1}_B$ takes the partial coordinate only along the orthonormal basis vector $|a\rangle$ of $\mathcal{H}_A$, producing a vector $|\phi_a\rangle \in \mathcal{H}_B$. By a similar calculation,

$$(|a\rangle\langle a| \otimes \mathbb{1}_B)|\Psi\rangle = |a\rangle \otimes |\phi_a\rangle.$$

The operator $|a\rangle\langle a| \otimes \mathbb{1}_B$ partially projects $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ onto the basis vector $|a\rangle$.

Given two other linear maps $M' : \mathcal{H}_{A'} \to \mathcal{H}_{A''}$ and $N' : \mathcal{H}_{B'} \to \mathcal{H}_{B''}$, one can define the products (by composition) $M'M : \mathcal{H}_A \to \mathcal{H}_{A''}$ and $N'N : \mathcal{H}_B \to \mathcal{H}_{B''}$. The Kronecker product $M'M \otimes N'N : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{A''} \otimes \mathcal{H}_{B''}$ has coefficients (in the orthonormal bases of the spaces under consideration):

$$\begin{aligned}
\langle a''b''|M'M \otimes N'N|ab\rangle &= \langle a''|M'M|a\rangle \; \langle b''|N'N|b\rangle \\
&= \sum_{a'} \langle a''|M'|a'\rangle\langle a'|M|a\rangle \sum_{b'} \langle b''|N'|b'\rangle\langle b'|N|b\rangle \\
&= \sum_{a',b'} \langle a''b''|M' \otimes N'|a'b'\rangle\langle a'b'|M \otimes N|ab\rangle \,.
\end{aligned}$$

These are precisely the coefficients of the matrix product of $M' \otimes N'$ by $M \otimes N$. We have proved the following fundamental theorem.

Theorem (Product Formula). *The tensor product is multiplicative: with the notation above, products of tensor products are the tensor products of the products:*

$$(M' \otimes N')(M \otimes N) = M'M \otimes N'N.$$

More generally, by applying this formula successively,

$$(M_1 \otimes N_1)(M_2 \otimes N_2) \cdots (M_n \otimes N_n) = (M_1 M_2 \cdots M_n) \otimes (N_1 N_2 \cdots N_n).$$

Example. By interpreting the vectors $|\psi_A\rangle$ and $|\psi_B\rangle$ as linear maps $\mathbb{C} \to \mathcal{H}_A$ and $\mathbb{C} \to \mathcal{H}_B$, we recover the formula

$$(M \otimes N)(|\psi_A\rangle \otimes |\psi_B\rangle) = M|\psi_A\rangle \otimes N|\psi_B\rangle,$$

which served to define $M \otimes N$, as a special case of the product formula.

Remark (Block extraction). Consider an arbitrary matrix $\mathbf{M}$ of size $d_{A'}d_{B'} \times d_A d_B$, representing a linear map $\mathbf{M} : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{A'} \otimes \mathcal{H}_{B'}$, written in (rectangular) blocks of size $d_{B'} \times d_B$:

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_{1,1} & \cdots & \mathbf{M}_{1,d_A} \\ \vdots & \ddots & \vdots \\ \mathbf{M}_{d_{A'},1} & \cdots & \mathbf{M}_{d_{A'},d_A} \end{pmatrix},$$

where the $\mathbf{M}_{a',a}$ denote the $d_{A'}d_A$ blocks extracted from $\mathbf{M}$.

In the special case $\mathbf{M} = M \otimes N$, by looking at the Kronecker product one simply has

$$\mathbf{M}_{a',a} = \langle a'|M|a\rangle \; N,$$

where $\langle a'|M|a\rangle = \alpha_{a',a}$ is a scalar. This can be written directly in terms of the matrix $M \otimes N$ as

$$\mathbf{M}_{a',a} = (\langle a'| \otimes \mathbb{1}_{B'})(M \otimes N)(|a\rangle \otimes \mathbb{1}_B)$$

by the product formula. In the general case, $\mathbf{M}$ is always a linear combination of tensor products of the form $M \otimes N$ (more specifically, of $|a'b'\rangle\langle ab| = |a'\rangle\langle a| \otimes |b'\rangle\langle b|$). By linearity, we obtain the **block-extraction formula**:

$$\mathbf{M}_{a',a} = (\langle a'| \otimes \mathbb{1}_{B'})\mathbf{M}(|a\rangle \otimes \mathbb{1}_B).$$

This is a generalization of the formula $M_{a',a} = \langle a'|M|a\rangle$ giving the entries of a matrix M, which corresponds to the case $d_B = d_{B'} = 1$. In the special case where the domain is one-dimensional, $\mathbf{M}$ may be identified with a column vector $|\Psi\rangle = \sum_{a,b} \alpha_{a,b}|a\rangle|b\rangle$; its a-th block is the vector $|\phi_a\rangle = \sum_b \alpha_{a,b}|b\rangle$. We recover the formula

$$|\phi_a\rangle = (\langle a| \otimes \mathbb{1}_B)|\Psi\rangle.$$

Another way to see the construction of a matrix by blocks $\mathbf{M}_{a',a}$ is to note that the matrix with the single block $\mathbf{M}_{a',a}$, all other blocks being zero, corresponds to the tensor product $|a'\rangle\langle a| \otimes \mathbf{M}_{a',a}$, because $|a'\rangle\langle a|$ has just one nonzero coefficient, equal to 1, at the intersection of row a' and column a. Thus, by linearity, we have the **block-composition formula**:

$$\mathbf{M} = \sum_{a',a} |a'\rangle\langle a| \otimes \mathbf{M}_{a',a},$$

from which we recover the block-extraction formula

$$(\langle a'| \otimes \mathbb{1}_{B'})\mathbf{M}(|a\rangle \otimes \mathbb{1}_B) = \mathbf{M}_{a',a}.$$

Remark. One can easily generalize what has just been explained to the tensor product of several maps $M_1 \otimes M_2 \otimes \cdots \otimes M_n$, where $M_k : \mathcal{H}_k \to \mathcal{H}'_k$, which defines a linear map

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n \to \mathcal{H}'_1 \otimes \mathcal{H}'_2 \otimes \cdots \otimes \mathcal{H}'_n.$$

This tensor product is associative and satisfies the product formula

$$(M_1 \otimes M_2 \otimes \cdots \otimes M_n)(N_1 \otimes N_2 \otimes \cdots \otimes N_n) = M_1N_1 \otimes M_2N_2 \otimes \cdots \otimes M_nN_n.$$

In what follows, we will always reduce to the simple case of the tensor product of two maps $M \otimes N$ by using associativity. For example, the tensor product $M_1 \otimes M_2 \otimes M_3$ of three maps can be viewed as $(M_1 \otimes M_2) \otimes M_3$, or equivalently as $M_1 \otimes (M_2 \otimes M_3)$.

In the rest of this chapter we consider linear **operators** (endomorphisms) $M : \mathcal{H}_A \to \mathcal{H}_A$ and $N : \mathcal{H}_B \to \mathcal{H}_B$, so that $M \otimes N$ is also a linear operator $\mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A \otimes \mathcal{H}_B$ represented by a square matrix of size $d_A d_B \times d_A d_B$. When interpreting $M \otimes N$ on the compound system AB, one can view $M \otimes N$ as the simultaneous application of the operator M to system A and of the operator N to system B. Indeed, specifying $(M \otimes N)(|\psi_A\rangle \otimes |\psi_B\rangle) = M|\psi_A\rangle \otimes N|\psi_B\rangle$ amounts to specifying the states $M|\psi_A\rangle$ and $N|\psi_B\rangle$ separately.

Example. In particular, $M \otimes \mathbb{1}_B$ applies the operator M to system A and does nothing (applies the identity operator) to system B. Likewise, $\mathbb{1}_A \otimes N$ applies N to B and does nothing to A.

Example. In the special case in which one of the dimensions d_A or d_B is equal to 1, say $d_B = 1$, the operators on $\mathcal{H}_B$ are scalars α, and the tensor product $M \otimes \alpha$ simply amounts to multiplication by the scalar α: $M \otimes \alpha = \alpha M$.

By conjugating the relation

$$\langle a'b'|M \otimes N|ab\rangle = \langle a'|M|a\rangle \, \langle b'|N|b\rangle \,,$$

one obtains the relation

$$\langle ab|(M \otimes N)^*|a'b'\rangle = \langle a|M^*|a'\rangle \, \langle b|N^*|b'\rangle$$

for the dual (conjugate-transpose) operators, hence the following result.

Theorem (Dual of a Tensor Product). *The **dual** of a tensor product is the tensor product of the duals:*

$$(M \otimes N)^* = M^* \otimes N^*.$$

The determinant of the matrix $M \otimes N$ is denoted by $|M \otimes N|$. Since it does not depend on the chosen basis, taking the orthonormal vectors $|ab\rangle$ in reverse lexicographic order gives $|M \otimes N| = |N \otimes M|$: the order of the factors does not matter. By the product formula,

$$M \otimes N = (\mathbb{1}_A M) \otimes (N \mathbb{1}_B) = (\mathbb{1}_A \otimes N)(M \otimes \mathbb{1}_B),$$

and hence

$$|M \otimes N| = |\mathbb{1}_A \otimes N| \cdot |M \otimes \mathbb{1}_B| = |\mathbb{1}_A \otimes N| \cdot |\mathbb{1}_B \otimes M|.$$

Now, from the definition of the Kronecker product as a block matrix, it follows immediately that $|\mathbb{1}_A \otimes N| = |N|^{d_A}$ and likewise $|\mathbb{1}_B \otimes M| = |M|^{d_B}$, yielding the following result.

Theorem (Determinant of a Tensor Product).

$$|M \otimes N| = |M|^{d_B}|N|^{d_A},$$

where the exponent of $|M|$ is the size of N, and the exponent of $|N|$ is the size of M.

It follows from the determinant formula that $M \otimes N$ is invertible (has nonzero determinant) if and only if M and N are invertible. In this case, by the product formula,

$$\mathbb{1}_{AB} = \mathbb{1}_A \otimes \mathbb{1}_B = (MM^{-1}) \otimes (NN^{-1}) = (M \otimes N)(M^{-1} \otimes N^{-1}),$$

which gives the inversion formula

$$(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}.$$

If M and N are **normal** operators, then $MM^* = M^*M$ and $NN^* = N^*N$. Hence, by the Product Formula and the duality theorem,

$$\begin{aligned}(M \otimes N)(M \otimes N)^* &= (M \otimes N)(M^* \otimes N^*)\\ &= MM^* \otimes NN^*\\ &= M^*M \otimes N^*N\\ &= (M^* \otimes N^*)(M \otimes N)\\ &= (M \otimes N)^*(M \otimes N),\end{aligned}$$

so the tensor product $M \otimes N$ is itself normal. We can make its spectral decomposition explicit: if, in orthonormal bases of eigenvectors, $M = \sum_a \lambda_a |a\rangle\langle a|$ and $N = \sum_b \mu_b |b\rangle\langle b|$, then by bilinearity

$$M \otimes N = \sum_a \sum_b \lambda_a \mu_b \, |a\rangle\langle a| \otimes |b\rangle\langle b| = \sum_{a,b} \lambda_a \mu_b \, |ab\rangle\langle ab|,$$

where we have used the Product Formula $|a\rangle\langle a| \otimes |b\rangle\langle b| = |ab\rangle\langle ab|$. This spectral decomposition of $M \otimes N$ shows that the eigenvalues of $M \otimes N$ are all the pairwise products $\lambda_a \mu_b$ of the eigenvalues of M and N.

Remark. We recover the fact that the determinant of $M \otimes N$ is the product $\prod_{a,b} \lambda_a \mu_b = (\prod_a \lambda_a)^{d_B}(\prod_b \mu_b)^{d_A} = |M|^{d_B}|N|^{d_A}$.

Observing that $|\lambda_a| = |\mu_b| = 1 \Rightarrow |\lambda_a \mu_b| = 1$, $\lambda_a, \mu_b \in \mathbb{R} \Rightarrow \lambda_a \mu_b \in \mathbb{R}$, $\lambda_a, \mu_b \geqslant 0 \Rightarrow \lambda_a \mu_b \geqslant 0$, $\lambda_a, \mu_b > 0 \Rightarrow \lambda_a \mu_b > 0$, and $\lambda_a, \mu_b \in \{0, 1\} \Rightarrow \lambda_a \mu_b \in \{0, 1\}$, we obtain, by the characterizations of the different types of operators, the following results.

Theorem.

$$\begin{aligned}
M \text{ and } N \text{ are normal} &\implies M \otimes N \text{ is normal}\\
M \text{ and } N \text{ are unitary} &\implies M \otimes N \text{ is unitary}\\
M \text{ and } N \text{ are Hermitian} &\implies M \otimes N \text{ is Hermitian}\\
M \text{ and } N \text{ are positive} &\implies M \otimes N \text{ is positive}\\
M \text{ and } N \text{ are positive definite} &\implies M \otimes N \text{ is positive definite}\\
M \text{ and } N \text{ are orthogonal projectors} &\implies M \otimes N \text{ is an orthogonal projector.}
\end{aligned}$$

Remark. For every real- or complex-valued function f whose domain contains the relevant eigenvalues and their products, and which is **multiplicative**, meaning $f(\lambda\mu) = f(\lambda)f(\mu)$, the spectral decomposition $M \otimes N = \sum_{a,b} \lambda_a \mu_b \, |ab\rangle\langle ab|$ gives

$$f(M \otimes N) = \sum_{a,b} f(\lambda_a) f(\mu_b) \, |ab\rangle\langle ab| = f(M) \otimes f(N).$$

In particular, this recovers the formulas $(M \otimes N)^* = M^* \otimes N^*$ for $f(\lambda) = \lambda^*$ and $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$ for $f(\lambda) = \lambda^{-1}$. We also have, for $f(\lambda) = \sqrt{\lambda}$,

$$\sqrt{M \otimes N} = \sqrt{M} \otimes \sqrt{N}$$

if M and N are positive.

Remark (Unitary Evolution of a Compound System). A compound system AB undergoes a unitary evolution U just as in the case of a single system, with a square unitary matrix U of size $d_A d_B$. If each system undergoes a separate evolution (A undergoes a unitary evolution U_A and B a unitary evolution U_B), the compound system as a whole undergoes a unitary evolution of the form

$$U = U_A \otimes U_B,$$

which is indeed unitary, as one sees directly from the Product Formula:

$$U^* U = (U_A^* \otimes U_B^*)(U_A \otimes U_B) = U_A^* U_A \otimes U_B^* U_B = \mathbb{1}_A \otimes \mathbb{1}_B = \mathbb{1}.$$

If one of the systems undergoes a unitary evolution while nothing changes for the other system, say a unitary operation U_A is performed on A and B remains unchanged, then the compound system AB undergoes a unitary evolution of the form

$$U = U_A \otimes \mathbb{1}_B.$$

These considerations generalize easily to several systems $ABC\cdots$.