(μ) Multiple Systems and the Tensor Product

Up to now, we have considered only the case of a quantum system **isolated** from the rest of the world (or: closed). In general, a system is not isolated, because it interacts with other systems, which form its environment. To model this type of open (non-isolated) system mathematically, we first describe the simplest case of a **multiple system** composed of two systems A and B, also called the compound system AB (also denoted (A, B)).

The state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of these two systems have respective dimensions $d_A = \dim A$ and $d_B = \dim B$ (not necessarily equal), and respective standard orthonormal bases $|a\rangle$ $(a \in A)$ and $|b\rangle$ $(b \in B)$. The d_A possible values of $a \in A$ are the classical states of the system A, obtained by its standard basis measurement, and correspond to the quantum states $|a\rangle$ of the standard basis of $\mathcal{H}_A$. Similarly, the d_B possible values of $b \in B$ are the classical states of the system B, obtained by its standard basis measurement, and correspond to the quantum states $|b\rangle$ of the standard basis of $\mathcal{H}_B$.

If we now consider the systems A and B together, all possible choices of the classical states of the two systems correspond to all $d_A d_B$ pairs $(a, b) \in A \times B$ (the Cartesian product of the sets A and B). The compound system AB must therefore admit all these pairs (a, b) as classical states, corresponding to quantum states $|a, b\rangle$ in the standard orthonormal basis of the state space $\mathcal{H}_{AB}$ of the compound system AB:

Definition (State Space of a Compound System). *Given two systems A and B with respective state spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, admitting respectively $|a\rangle$ $(a \in A)$ and $|b\rangle$ $(b \in B)$ as standard orthonormal bases, the multiple system AB is described by the state space $\mathcal{H}_{AB}$ admitting the orthonormal basis $|a, b\rangle$ $(a \in A,\ b \in B)$ as its standard orthonormal basis.*

The space $\mathcal{H}_{AB}$ is sometimes denoted $\mathcal{H}_A \otimes \mathcal{H}_B$, the ***tensor product*** *space of the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.*

For simplicity of notation, we will write simply $|ab\rangle$ instead of $|a, b\rangle$. A state of the compound system, $|\Psi\rangle \in \mathcal{H}_{AB}$, is therefore written in the standard basis in the form

$$|\Psi\rangle = \sum_{a \in A} \sum_{b \in B} \alpha_{a,b} |ab\rangle = \sum_{(a,b) \in A \times B} |ab\rangle\langle ab|\Psi\rangle\,,$$

where the $\alpha_{a,b} = \langle ab|\Psi\rangle$ are complex coefficients. The system AB has dimension equal to the product of the dimensions of the two systems of which it is composed:

$$\dim AB = \dim A \dim B.$$

The above definition of $\mathcal{H}_{AB}$ depends on the choices of the standard bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, but we will see later how to remove this restriction. Once the standard bases are fixed, $\mathcal{H}_A$ is isomorphic to $\mathbb{C}^{d_A}$, $\mathcal{H}_B$ is isomorphic to $\mathbb{C}^{d_B}$, and hence the compound state space $\mathcal{H}_{AB}$ is isomorphic to $\mathbb{C}^{d_A d_B}$. We can therefore identify

$$\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} = \mathbb{C}^{d_A d_B}.$$

Example. If the two systems A and B are qubits, $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$, with a standard orthonormal basis $\{|0\rangle, |1\rangle\}$ for each of the two systems, then the compound system $\mathcal{H}_{AB} = \mathbb{C}^4$ is the system consisting of two qubits, which admits the orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

If $|\psi_A\rangle = \sum_a \alpha_a |a\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle = \sum_b \beta_b |b\rangle \in \mathcal{H}_B$, we can define the **tensor product** of these **vectors** $\left(\sum_a \alpha_a |a\rangle\right) \otimes \left(\sum_b \beta_b |b\rangle\right) \in \mathcal{H}_{AB}$ by the usual distributive law, identifying $|a\rangle \otimes |b\rangle = |ab\rangle$. This gives the following definition:

Definition (Tensor Product). *The **tensor product** of $|\psi_A\rangle = \sum_a \alpha_a |a\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle = \sum_b \beta_b |b\rangle \in \mathcal{H}_B$ is the vector $|\psi_A\rangle \otimes |\psi_B\rangle \in \mathcal{H}_{AB}$ defined by*

$$|\psi_A\rangle \otimes |\psi_B\rangle = \sum_{a,b} \alpha_a \beta_b |ab\rangle.$$

We will write either $|\psi_A\rangle \otimes |\psi_B\rangle$ or $|\psi_A \otimes \psi_B\rangle$.

Remark. From this definition, in particular, we indeed have $|a\rangle \otimes |b\rangle = |a \otimes b\rangle = |a, b\rangle = |ab\rangle$ for every $(a, b) \in A \times B$. For example, for two qubits, we have $|00\rangle = |0\rangle \otimes |0\rangle$, $|01\rangle = |0\rangle \otimes |1\rangle$, $|10\rangle = |1\rangle \otimes |0\rangle$, and $|11\rangle = |1\rangle \otimes |1\rangle$.

By definition, the tensor product is **bilinear**, that is, the expression $|\psi_A\rangle \otimes |\psi_B\rangle$ is linear in $|\psi_A\rangle$ and in $|\psi_B\rangle$:

$$\begin{aligned}
\left(\alpha|\psi_A\rangle + \beta|\psi'_A\rangle\right) \otimes |\psi_B\rangle &= \alpha|\psi_A \otimes \psi_B\rangle + \beta|\psi'_A \otimes \psi_B\rangle, \\
|\psi_A\rangle \otimes \left(\alpha|\psi_B\rangle + \beta|\psi'_B\rangle\right) &= \alpha|\psi_A \otimes \psi_B\rangle + \beta|\psi_A \otimes \psi'_B\rangle.
\end{aligned}$$

We can therefore define the tensor product as the unique bilinear map from $\mathcal{H}_A \times \mathcal{H}_B$ to $\mathcal{H}_{AB}$ satisfying $|a\rangle \otimes |b\rangle = |ab\rangle$ for every $(a, b) \in A \times B$.

Note that by bilinearity, multiplication by a scalar $\alpha \in \mathbb{C}$ can be placed on either factor of the tensor product:

$$\alpha|\psi_A \otimes \psi_B\rangle = \alpha|\psi_A\rangle \otimes |\psi_B\rangle = |\psi_A\rangle \otimes \alpha|\psi_B\rangle.$$

Example. In the particular case where one of the dimensions d_A or d_B is equal to 1, say $d_B = 1$, the vectors of $\mathcal{H}_B$ are scalars α, $\mathcal{H}_B$ is identified with $\mathbb{C}$, and $\mathcal{H}_A \otimes \mathcal{H}_B$ is identified with $\mathbb{C}^{d_A d_B} = \mathbb{C}^{d_A}$, hence with $\mathcal{H}_A$. In this case, the tensor product $|\psi_A\rangle \otimes \alpha$ simply amounts to multiplication by the scalar α: $|\psi_A\rangle \otimes \alpha = \alpha|\psi_A\rangle$.

The above definition $|\psi_A\rangle \otimes |\psi_B\rangle = \sum_{a,b} \alpha_a \beta_b |ab\rangle$ of the tensor product amounts to writing, in coordinates on the standard bases,

$$\langle ab|\psi_A \otimes \psi_B\rangle = \langle a|\psi_A\rangle \langle b|\psi_B\rangle .$$

It follows that for all vectors $|\phi_A\rangle = \sum_a \alpha'_a |a\rangle \in \mathcal{H}_A$ and $|\phi_B\rangle = \sum_b \beta'_b |b\rangle \in \mathcal{H}_B$,

$$\sum_{a,b} {\alpha'_a}^* {\beta'_b}^* \langle ab|\psi_A \otimes \psi_B\rangle = \sum_a {\alpha'_a}^* \langle a|\psi_A\rangle \sum_b {\beta'_b}^* \langle b|\psi_B\rangle \,,$$

whence the following result by conjugate linearity in the first argument of the inner products:

Theorem. *For all $|\psi_A\rangle, |\phi_A\rangle \in \mathcal{H}_A$ and $|\psi_B\rangle, |\phi_B\rangle \in \mathcal{H}_B$, the inner product of tensor products is the product of the inner products:*

$$\langle \phi_A \otimes \phi_B|\psi_A \otimes \psi_B\rangle = \langle \phi_A|\psi_A\rangle \, \langle \phi_B|\psi_B\rangle \,.$$

In particular, for $|\psi_A\rangle = |\phi_A\rangle$ and $|\psi_B\rangle = |\phi_B\rangle$, the norm of the tensor product is the product of the norms:

$$\|\psi_A \otimes \psi_B\| = \|\psi_A\| \cdot \|\psi_B\|.$$

This property of the inner product does not depend on the standard bases chosen for $\mathcal{H}_A$ and $\mathcal{H}_B$. It implies that orthogonality in $\mathcal{H}_A$ and $\mathcal{H}_B$ is preserved under tensor products: $\phi_A \perp \psi_A$ and $\phi_B \perp \psi_B$ imply $\phi_A \otimes \phi_B \perp \psi_A \otimes \psi_B$, and it also preserves normalization: $\|\psi_A\| = \|\psi_B\| = 1$ implies $\|\psi_A \otimes \psi_B\| = 1$. Consequently, for any choice of orthonormal bases $|a'\rangle$ of $\mathcal{H}_A$ and $|b'\rangle$ of $\mathcal{H}_B$, the $d_A d_B$ tensor products $|a'\rangle \otimes |b'\rangle$ also form an orthonormal basis of $\mathcal{H}_{AB}$.

It follows that the definition of the compound state space for another choice of orthonormal bases $|a'\rangle$ of $\mathcal{H}_A$ and $|b'\rangle$ of $\mathcal{H}_B$ gives, by identifying $|a'b'\rangle = |a'\rangle \otimes |b'\rangle \in \mathcal{H}_{AB}$, the same Hermitian space $\mathcal{H}_{AB}$ (up to an isomorphism of Hermitian spaces). Moreover, the definition of the tensor product for this other choice of bases gives the same tensor product, since for all vectors $|\psi_A\rangle = \sum_a \alpha'_a |a'\rangle$ and $|\psi_B\rangle = \sum_b \beta'_b |b'\rangle$ written in these bases, one has

$$\sum_{a,b} \alpha'_a \beta'_b |a'b'\rangle = \left(\sum_a \alpha'_a |a'\rangle\right) \otimes \left(\sum_b \beta'_b |b'\rangle\right) = |\psi_A\rangle \otimes |\psi_B\rangle.$$

Theorem. *The tensor product of spaces $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, as well as the tensor product of vectors $|\psi_A\rangle \otimes |\psi_B\rangle$, does not depend (up to an isomorphism of Hermitian spaces) on the choices of orthonormal bases in the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$.*

These are therefore **intrinsic** definitions, independent of choices of bases.

Remark. The notation $\mathcal{H}_A \otimes \mathcal{H}_B$ does not mean that all vectors of $\mathcal{H}_A \otimes \mathcal{H}_B$ are of the form $|\psi_A\rangle \otimes |\psi_B\rangle$. Indeed, the space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ contains all vectors of this type, but also linear combinations of such vectors, which, as we shall see, are not necessarily themselves of this form.

When a particular order is specified for the indices a and b, for example the ordered index sets $A = \{1, 2, \dots, d_A\}$ and $B = \{1, 2, \dots, d_B\}$, we shall always agree that the compound indices $ab = (a, b) \in A \times B$ are ordered in the **lexicographic order**

$$A \times B = \{11, 12, \dots, 1d_B, 21, 22, \dots, d_A1, d_A2, \dots, d_Ad_B\}.$$

Consequently, the tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$ of the column vectors

$$|\psi_A\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d_A} \end{pmatrix} \quad \text{and} \quad |\psi_B\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{pmatrix}$$

is represented by the column vector obtained by stacking d_A copies of $|\psi_B\rangle$ multiplied by the d_A coefficients of $|\psi_A\rangle$:

$$\begin{aligned} |\psi_A\rangle \otimes |\psi_B\rangle &= \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{d_A} \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{d_B} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \\ \vdots \\ \alpha_{d_A} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_{d_B} \end{bmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_{d_B} \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_2\beta_{d_B} \\ \vdots \\ \alpha_{d_A}\beta_1 \\ \vdots \\ \alpha_{d_A}\beta_{d_B} \end{pmatrix}. \end{aligned}$$

Remark. The tensor product depends on the order in which the systems A and B are considered. Although $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{H}_B \otimes \mathcal{H}_A$ can be identified, and are both isomorphic to $\mathbb{C}^{d_A d_B} = \mathbb{C}^{d_B d_A}$, the vector $|\psi_B\rangle \otimes |\psi_A\rangle$ has the same components as $|\psi_A\rangle \otimes |\psi_B\rangle$ but in a different order; consequently, the tensor product is not commutative: $|\psi_A\rangle \otimes |\psi_B\rangle \neq |\psi_B\rangle \otimes |\psi_A\rangle$. In calculations, it is important to keep the order in which the systems have been considered.

The tensor product of states $|\psi_A\rangle \otimes |\psi_B\rangle$ represents a very particular state of the space $\mathcal{H}_{AB}$ of compound states. Indeed, because a quantum state is always represented by a nonzero vector, $|\psi_B\rangle \neq 0$ has at least one coordinate $\beta_b \neq 0$. Consequently, knowing the tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$ makes it possible to recover all coefficients $\alpha_a\beta_b$ of the vector $\beta_b|\psi_A\rangle$. Since a state is defined only up to multiplication by a nonzero complex scalar, this completely determines the state $|\psi_A\rangle$. By a symmetric argument, knowing the tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$ also makes it possible to recover the state $|\psi_B\rangle$. Thus the two states $|\psi_A\rangle$ and $|\psi_B\rangle$ of the two systems A and B are uniquely determined, as states, by their tensor product:

Theorem. *Specifying the tensor product of states $|\psi_A\rangle\otimes|\psi_B\rangle$ is equivalent to specifying the two states $|\psi_A\rangle$ and $|\psi_B\rangle$ separately.*

Remark. Another type of product, the **outer product** $|\psi_A\rangle\langle\psi_B|$, has some similarities with the tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$. Indeed, by writing $|\psi_A\rangle = \sum_a \alpha_a|a\rangle$ and $|\psi_B\rangle = \sum_b \beta_b|b\rangle$, the outer product $|\psi_A\rangle\langle\psi_B|$ has components $\alpha_a\beta_b^*$, whereas the tensor product $|\psi_A\rangle \otimes |\psi_B\rangle$ has components $\alpha_a\beta_b$. In both cases, specifying the product of two states is equivalent to specifying the two states separately.

However, they are not the same kind of mathematical object: the tensor product is a vector, represented as a column vector, whereas the outer product is a linear map from $\mathcal{H}_B$ to $\mathcal{H}_A$, represented as a matrix.

Definition (Product State, Entangled State). *A **product state** is any state of $\mathcal{H}_{AB}$ of the form $|\psi_A\rangle \otimes |\psi_B\rangle$. A state of $\mathcal{H}_{AB}$ that is not of this form is said to be **entangled**.*

When a system A is independently prepared in a state $|\psi_A\rangle$ and a system B in a state $|\psi_B\rangle$ (for example in two different places), these two systems being isolated from one another, the compound state of the system AB is the product state $|\psi_A\rangle \otimes |\psi_B\rangle$, since it is equivalent to specifying the two states $|\psi_A\rangle$ and $|\psi_B\rangle$ separately. Owing to this equivalence, one often writes

$$|\psi_A\rangle|\psi_B\rangle \quad \text{or} \quad |\psi_A, \psi_B\rangle$$

instead of $|\psi_A\rangle \otimes |\psi_B\rangle$ or $|\psi_A \otimes \psi_B\rangle$, without specifying the product symbol, and one says that the systems A and B are **independent**. Two normalized states $|\psi_A\rangle$ and $|\psi_B\rangle$ have an automatically normalized product state $|\psi_A\rangle \otimes |\psi_B\rangle = |\psi_A\rangle|\psi_B\rangle$ since $\|\psi_A \otimes \psi_B\| = \|\psi_A\| \cdot \|\psi_B\| = 1 \cdot 1 = 1$.

The fact that a compound state is **entangled** (at a given instant) therefore reflects the fact that the two systems **interact**. Note that in the trivial case where one of the two subsystems has dimension 1, there can be no entangled state (all states are trivially product states). The notion of an entangled state of $\mathcal{H}_{AB}$ only makes sense if $d_A > 1$ and $d_B > 1$.

Example. The state of a system AB consisting of two qubits A and B is of the form

$$\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle,$$

where a state of the orthonormal basis, such as $|01\rangle$, can equivalently be written as $|01\rangle = |0\rangle|1\rangle = |0\rangle_A|1\rangle_B = |0\rangle \otimes |1\rangle = |0 \otimes 1\rangle$, depending on what is most convenient in the calculations.

The state $\frac{1}{2}(|00\rangle - |01\rangle + i|10\rangle - i|11\rangle)$ is a **product state**, since it factors as

$$\frac{1}{2}(|00\rangle - |01\rangle + i|10\rangle - i|11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

By contrast, a state of the form

$$\gamma_{00}|00\rangle + \gamma_{11}|11\rangle$$

with nonzero coefficients γ_{00} and γ_{11} is always **entangled**. Indeed, if it factored as $|\phi\rangle \otimes |\psi\rangle$, then on the coordinate 01, for example, we would have $\langle 01|\phi \otimes \psi\rangle = \langle 0|\phi\rangle \langle 1|\psi\rangle = 0$, hence $\langle 0|\phi\rangle = 0$ or $\langle 1|\psi\rangle = 0$, which contradicts

$$\gamma_{00} = \langle 00|\phi \otimes \psi\rangle = \langle 0|\phi\rangle \langle 0|\psi\rangle \neq 0 \quad \text{and} \quad \gamma_{11} = \langle 11|\phi \otimes \psi\rangle = \langle 1|\phi\rangle \langle 1|\psi\rangle \neq 0.$$

Similarly, a state of the form $\gamma_{01}|01\rangle + \gamma_{10}|10\rangle$ with nonzero coefficients γ_{01} and γ_{10} is always entangled.

Everything just set out in this chapter readily generalizes to a multiple system $A_1A_2\cdots A_N$ composed of any finite number of systems $A_1, A_2, \ldots, A_N$. Each system is described by its state space $\mathcal{H}_{A_k}$ of dimension d_k and standard basis $|a_k\rangle$, where $a_k \in A_k$. The space $\mathcal{H}_{A_1A_2\cdots A_N}$ of compound states of the system $A_1A_2\cdots A_N$ is the tensor product of spaces $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N}$, with orthonormal basis $|a_1a_2\cdots a_N\rangle$. The order of these systems $A_1A_2\cdots A_N$ is arbitrary but fixed by convention.

If each system A_k is in a state $|\psi_{A_k}\rangle = \sum_{a_k} \alpha_{a_k}|a_k\rangle$, we define the product state $|\psi_{A_1}\rangle|\psi_{A_2}\rangle\cdots|\psi_{A_N}\rangle$ as the tensor product of the vectors

$$|\psi_{A_1}\rangle \otimes |\psi_{A_2}\rangle \otimes \cdots \otimes |\psi_{A_N}\rangle = \sum_{a_1,a_2,\ldots,a_N} \alpha_{a_1}\alpha_{a_2}\cdots\alpha_{a_N}|a_1a_2\cdots a_N\rangle,$$

which is the unique multilinear map from $\mathcal{H}_{A_1}\times\mathcal{H}_{A_2}\times\cdots\times\mathcal{H}_{A_N}$ to $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\cdots\otimes\mathcal{H}_{A_N}$ satisfying

$$|a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_N\rangle = |a_1a_2\cdots a_N\rangle.$$

The tensor product does not depend on the chosen bases $|a_k\rangle$, and one may identify $\mathcal{H}_{A_1A_2\cdots A_N} = \mathbb{C}^{d_1d_2\cdots d_N}$. The inner product of tensor products is the product of the inner products:

$$\langle \phi_{A_1} \otimes \phi_{A_2} \otimes \cdots \otimes \phi_{A_N}|\psi_{A_1} \otimes \psi_{A_2} \otimes \cdots \otimes \psi_{A_N}\rangle = \langle \phi_{A_1}|\psi_{A_1}\rangle \langle \phi_{A_2}|\psi_{A_2}\rangle \cdots \langle \phi_{A_N}|\psi_{A_N}\rangle,$$

and the norm of the tensor product is the product of the norms:

$$\|\psi_{A_1} \otimes \psi_{A_2} \otimes \cdots \otimes \psi_{A_N}\| = \|\psi_{A_1}\|\,\|\psi_{A_2}\|\cdots\|\psi_{A_N}\|.$$

Specifying the product state $|\psi_{A_1}\rangle|\psi_{A_2}\rangle\cdots|\psi_{A_N}\rangle$ is equivalent to specifying the N states $|\psi_{A_k}\rangle$ separately, and it is normalized if all the $|\psi_{A_k}\rangle$ are normalized.

By identifying spaces through isomorphisms of Hermitian spaces, it is immediate that the tensor product of spaces $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_N}$, or of vectors $|\psi_{A_1}\rangle \otimes |\psi_{A_2}\rangle \otimes \cdots \otimes |\psi_{A_N}\rangle$, is **associative**: these tensor products do not depend on the association order chosen to define them from the tensor product of two spaces or of two vectors. For example, one can identify

$$\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{A_3} = (\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}) \otimes \mathcal{H}_{A_3} = \mathcal{H}_{A_1} \otimes (\mathcal{H}_{A_2} \otimes \mathcal{H}_{A_3}),$$

and similarly

$$|\psi_{A_1}\rangle \otimes |\psi_{A_2}\rangle \otimes |\psi_{A_3}\rangle = (|\psi_{A_1}\rangle \otimes |\psi_{A_2}\rangle) \otimes |\psi_{A_3}\rangle = |\psi_{A_1}\rangle \otimes (|\psi_{A_2}\rangle \otimes |\psi_{A_3}\rangle),$$

and so on for a tensor product of $N > 3$ factors. Thus, by associativity, one can always ultimately reduce to the case of a tensor product of two factors.

Remark. Since a compound system can be viewed as a single system, considering any number of systems can be reduced to the case of two systems AE by dividing the systems into two groups: those that are to be studied and those that are not. The systems to be studied constitute the system of interest, called the **principal system** A, and the rest of these systems is viewed as an **environment** E. By associativity, the order in which the systems present are grouped to form A or E is immaterial. The compound system AE describes the possible interaction (or coupling) of the principal system A with its environment E. Equivalently, it is sometimes more convenient to consider the compound system EA.

The environment E may be unknown or only partially known and thus constitute a source of **noise** or interference, with which the principal system evolves in interaction. In other contexts, E may constitute a **workspace** or an **auxiliary system**, coupled to the system under study A, which gathers together a number of prepared states useful for a measurement or a quantum computation.

The point of defining a compound system such as AE in this way is that it can itself be considered as a system isolated from other systems, and can therefore satisfy all the postulates already stated in the case of a single system – in particular those concerning its measurement and its evolution. This description will then make it possible to obtain results about the system A regarded as an open system, in interaction with E.

Before describing the quantum properties of open systems, it is important to have mathematical tools related to the notion of tensor product. In the mathematical digression that follows, we study tensor products of linear maps (in particular, of operators).