(μ) Dirac Notation

The notation $\langle\phi|\psi\rangle$ for the inner product, with angle brackets ("brackets" in English), inspired the physicist Paul Dirac to establish a formalism useful in quantum mechanics, based on the notations "bra" $\langle\phi|$ and "ket" $|\psi\rangle$, which, put together, form the "bra-ket" $\langle\phi|\psi\rangle$. This formalism can in fact be applied to any mathematical description of spaces equipped with an inner product. We describe this formalism here for Hermitian spaces (the state spaces of our quantum systems).

Definition (ket notation). *A vector $\psi \in \mathcal{H}$ is called a **ket** and denoted $|\psi\rangle$.*

This notation simplifies certain expressions. For example, if an orthonormal basis $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_d\rangle$ of $\mathcal{H}$ is given, one can omit the Greek letter ϕ and denote the basis vectors only by the indices themselves:

$$|1\rangle, |2\rangle, \ldots, |d\rangle$$

Here the indexing starts at 1, but nothing prevents us from indexing from 0 and writing an orthonormal basis in dimension d in the form

$$|0\rangle, |1\rangle, \ldots, |d-1\rangle$$

One can even index by arbitrary finite sets; for example, one may write an orthonormal basis in dimension 4 in the form

$$|\heartsuit\rangle, |\diamondsuit\rangle, |\clubsuit\rangle, |\spadesuit\rangle$$

without specifying any particular ordering of the indices.

Remark. The unit vector $|0\rangle$ should not be confused with the zero vector of $\mathcal{H}$, which we will always denote by 0.

Example. A system of dimension $d = 2$ is called a **qubit** (or q-bit), equipped with the basis $|0\rangle$, $|1\rangle$. An arbitrary state of a qubit is therefore of the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$. More generally, a system of arbitrary finite dimension d is sometimes called a **qudit** (or q-dit), equipped with the standard basis $|0\rangle$, $|1\rangle$, $\ldots$, $|d-1\rangle$. An arbitrary state is then written in the form

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle + \cdots + \alpha_{d-1}|d-1\rangle$$

with complex coefficients α_i.

When $\mathcal{H} = \mathcal{H}_A$ denotes the state space of a system A, we will often identify A with the set of $d = \dim A$ indices and write $|a\rangle$ (for every $a \in A$) for the orthonormal basis vectors, with no ordering of the indices specified. In the preceding example in dimension 4, $A = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$. Similarly, for another system B, the basis vectors will be denoted $|b\rangle$ (for every $b \in B$), and so on. This notational simplification makes it easy to distinguish states of different systems.

With Dirac notation, the **expansion formula in the orthonormal basis** of the $|a\rangle$ is written

$$|\psi\rangle = \sum_a \alpha_a |a\rangle \quad \text{where } \alpha_a = \langle a|\psi\rangle$$

(When the summation index is not further specified, we agree that the sum ranges over all elements $a \in A$.)

Remark. When a symbol denoting a state carries a diacritic, for example ψ' with a prime, we will write either $|\psi\rangle'$ or $|\psi'\rangle$ indifferently. There is no need to distinguish between these two notations.

To avoid overburdening the notation, we will always write $\|\psi\|$ (rather than $\||\psi\rangle\|$) for the norm of the ket $|\psi\rangle$.

Definition (bra notation). *For every ket $|\psi\rangle \in \mathcal{H}$, we define the **bra** $\langle\psi|$ to be the linear functional that maps $|\phi\rangle \in \mathcal{H}$ to $\langle\psi|\phi\rangle \in \mathbb{C}$.*

This is indeed a linear functional, by linearity in the second argument of the inner product. Thus, to every ket $|\psi\rangle$ there corresponds a bra $\langle\psi|$, which is a linear functional acting on kets.

The expansion formula above shows that the **coordinates** α_a of $|\psi\rangle$ in the basis of the **kets** $|a\rangle$ are obtained by applying the corresponding **bras** $\langle a|$ to $|\psi\rangle$:

$$\alpha_a = \langle a|(|\psi\rangle) = \langle a|\psi\rangle$$

The linear map $\psi \in \mathcal{H} \mapsto (\alpha_a = \langle a|\psi\rangle)_{a\in A} \in \mathbb{C}^d$ (where $d = \dim \mathcal{H}$) is bijective, and makes it possible to identify every Hermitian space of dimension d with the space $\mathbb{C}^d$ of complex d-tuples; and therefore to identify every quantum system of dimension d with the data of d complex numbers.

The bras $\langle\phi|$ themselves form a vector space over $\mathbb{C}$, where, by conjugate linearity in the first argument of the inner product, $\alpha\langle\phi| + \beta\langle\phi'| = \langle\alpha^*\phi + \beta^*\phi'|$ for all scalars $\alpha, \beta \in \mathbb{C}$. This is the mathematical notion of **duality**: the space of bras is called the dual of $\mathcal{H}$ and is denoted $\mathcal{H}^*$, and $\langle\phi|$ is the dual vector of $|\phi\rangle$, sometimes denoted with a star:

$$\langle\phi| = |\phi\rangle^*$$

We will specify below the notation * in the most general case.

Example (of duality). The simplest example of a (complex) Hermitian space is $\mathbb{C}$ itself equipped with the inner product $\langle\phi|\psi\rangle = \phi^*\psi$, where $\phi, \psi \in \mathbb{C}$ and the star * denotes complex conjugation. The induced norm is $\|\psi\| = \sqrt{\psi^*\psi} = |\psi|$ (absolute value or modulus). The kets $|\psi\rangle$ are simply complex numbers, and the bras $\langle\phi|$ are the linear functionals of multiplication by the complex conjugate ϕ^* of ϕ, which are identified with the conjugates ϕ^* of complex numbers. With this identification, the duality star in the expression $\langle\phi| = |\phi\rangle^*$ is here nothing but the usual notation for complex conjugation.

Remark. The bras have been defined as linear maps from $\mathcal{H}$ to $\mathbb{C}$. We can symmetrize this point of view by also identifying kets with linear maps from $\mathbb{C}$ to $\mathcal{H}$. Indeed, $|\psi\rangle$ can be viewed as a linear map that sends $\alpha \in \mathbb{C}$ to $|\psi\rangle\alpha \in \mathcal{H}$. This is simply multiplication of the vector by the scalar α, placed on the right rather than on the left.

In expressions made up of kets and bras, it is sometimes more convenient to place scalars to the right of vectors rather than to the left. Thus the expansion formula in the orthonormal basis of the $|a\rangle$ can also be written $|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$.

Definition. *The **outer product** of ϕ and ψ is $|\phi\rangle\langle\psi|$.*

It is product notation for the composition of the two linear maps $\langle\psi|$, then $|\phi\rangle$. In other words, the outer product $|\phi\rangle\langle\psi|$ is the map that sends $|\chi\rangle$ to $|\phi\rangle\langle\psi|(|\chi\rangle) = |\phi\rangle\langle\psi|\chi\rangle$, that is, the ket $|\phi\rangle$ multiplied by the scalar $\langle\psi|\chi\rangle$.

Remark. Unlike the case of the inner product (scalar product) $\langle\phi|\psi\rangle$, the definition of the outer product $|\phi\rangle\langle\psi|$ does not require $|\phi\rangle$ and $|\psi\rangle$ to belong to the same Hermitian space. It is well defined for $|\psi\rangle \in \mathcal{H}$ and $|\phi\rangle \in \mathcal{H}'$, in which case $|\phi\rangle\langle\psi|$ is a linear map from $\mathcal{H}$ to $\mathcal{H}'$, two Hermitian spaces that need not have the same dimension.

Example. The orthogonal projection of ψ onto $\mathcal{V} = \mathrm{Vect}(\phi)$ (a subspace of dimension 1) is $\Pi\psi = \langle\phi|\psi\rangle\,|\phi\rangle$, which can be written in the form $\Pi\psi = |\phi\rangle\langle\phi|\psi\rangle$ with the inner product placed on the right. Consequently, the orthogonal projector onto $\mathcal{V} = \mathrm{Vect}(\phi)$ is nothing other than the outer product of ϕ with itself:

$$\Pi = |\phi\rangle\langle\phi|$$

The orthogonal projection of ψ onto an arbitrary subspace $\mathcal{V}$ of dimension n is given by the expansion formula in an orthonormal basis $|\phi_1\rangle, \ldots, |\phi_n\rangle$ of $\mathcal{V}$:

$$\begin{aligned}
\Pi\psi &= \langle\phi_1|\psi\rangle\,|\phi_1\rangle + \langle\phi_2|\psi\rangle\,|\phi_2\rangle + \cdots + \langle\phi_n|\psi\rangle\,|\phi_n\rangle \\
&= |\phi_1\rangle\langle\phi_1|\psi\rangle + |\phi_2\rangle\langle\phi_2|\psi\rangle + \cdots + |\phi_n\rangle\langle\phi_n|\psi\rangle \\
&= \big(|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_n\rangle\langle\phi_n|\big)|\psi\rangle
\end{aligned}$$

The general form of the **orthogonal projector** onto $\mathcal{V}$ is therefore

$$\Pi = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_n\rangle\langle\phi_n|$$

where $\phi_1, \phi_2, \ldots, \phi_n$ are the vectors of an arbitrary orthonormal basis of $\mathcal{V}$ (this expression does not depend on the choice of this basis). In the trivial case $n = 0$, this expression remains valid with an empty sum, and the orthogonal projector onto the zero subspace reduces to $\Pi = 0$.

In the special case where $\mathcal{V}$ is the entire Hermitian space $\mathcal{H}$, we have $\Pi|\psi\rangle = |\psi\rangle$ for every ψ, that is, $\Pi = \mathbb{1}$ (identity operator). This proves the

Theorem (Resolution of the identity). *For every orthonormal basis $|a\rangle$ ($a \in A$) of the space $\mathcal{H}_A$,*

$$\sum_a |a\rangle\langle a| = \mathbb{1}$$

This resolution of the identity (also called the **closure relation** or completeness relation) makes it easy to recover many useful expansion formulas. Thus, by expanding the identity $\mathbb{1}$ to the left of the ket $|\psi\rangle = \mathbb{1}|\psi\rangle$, we immediately recover the **expansion formula in the orthonormal basis**:

$$|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle = \sum_a \alpha_a |a\rangle$$

where the $\alpha_a = \langle a|\psi\rangle$ are the coordinates of the ket $|\psi\rangle$ in the orthonormal basis of the $|a\rangle$ $(a \in A)$.

By expanding the identity $\mathbb{1}$ to the right of the bra $\langle\psi| = \langle\psi|\mathbb{1}$, we obtain the **dual expansion formula**:

$$\langle\psi| = \sum_a \langle\psi|a\rangle\langle a| = \sum_a \alpha_a^* \langle a|$$

since $\langle\psi|a\rangle = \langle a|\psi\rangle^* = \alpha_a^*$ by Hermitian symmetry of the inner product.

Finally, by expanding the identity in the middle of the inner product $\langle\psi|\phi\rangle = \langle\psi|\mathbb{1}|\phi\rangle$, we obtain the **expansion formula for the inner product**:

$$\langle\psi|\phi\rangle = \sum_a \langle\psi|a\rangle\langle a|\phi\rangle = \sum_a \alpha_a^* \beta_a$$

where we have written $\beta_a = \langle a|\phi\rangle$ for the coordinates of $|\phi\rangle$. It is easy to check that this expression $\langle\alpha|\beta\rangle = \sum_a \alpha_a^* \beta_a$ itself defines an inner product on $\mathbb{C}^d$ between the complex d-tuples $\alpha = (\alpha_a)_{a\in A}$ and $\beta = (\beta_a)_{a\in A}$. Consequently, the bijective linear map $\psi \in \mathcal{H} \mapsto \alpha = (\alpha_a)_{a\in A} \in \mathbb{C}^d$ preserves the inner product: $\langle\psi|\phi\rangle = \langle\alpha|\beta\rangle$. In particular, for $|\phi\rangle = |\psi\rangle$, it also preserves the Euclidean norm:

$$\|\psi\|^2 = \sum_a |\alpha_a|^2$$

hence $\|\psi\| = \|\alpha\|$. It is therefore an isometry that constitutes an isomorphism of Hermitian spaces:

Theorem. *Every Hermitian space of dimension d is isomorphic to $\mathbb{C}^d$, where the isomorphism depends on the choice of the orthonormal basis $(|a\rangle)_{a\in A}$.*

Dirac notation also makes it possible to describe simply an arbitrary linear map between Hermitian spaces:

$$M : \mathcal{H}_B \to \mathcal{H}_A$$

which maps $|\psi\rangle \in \mathcal{H}_B$ to $M|\psi\rangle \in \mathcal{H}_A$, which may also be denoted $|M\psi\rangle$.

The two Hermitian spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ need not have the same dimension; we write $d_A = \dim \mathcal{H}_A$ and $d_B = \dim \mathcal{H}_B$. The resolution of the identity in $\mathcal{H}_A$ is written $\mathbb{1}_A = \sum_a |a\rangle\langle a|$, where the $|a\rangle$ form an orthonormal basis of $\mathcal{H}_A$ and where $\mathbb{1}_A$ is the identity map of $\mathcal{H}_A$. The corresponding resolution in $\mathcal{H}_B$ is written similarly as $\mathbb{1}_B = \sum_b |b\rangle\langle b|$. By expanding the two identities in the product (composition of maps) $M = \mathbb{1}_A M \mathbb{1}_B$, we obtain

$$M = \sum_{a,b} |a\rangle\langle a|M|b\rangle\langle b| = \sum_{a\in A, b\in B} M_{a,b} |a\rangle\langle b|$$

where

$$M_{a,b} = \langle a|M|b\rangle$$

are the **coefficients** (or entries) of M for the two orthonormal bases $(|a\rangle)_{a\in A}$ and $(|b\rangle)_{b\in B}$. Thus, in these given bases, we can identify the map M with the **matrix** of the $d_A \times d_B$ coefficients $(\langle a|M|b\rangle)_{a\in A,b\in B}$. These are the coefficients, in the orthonormal basis of the $|a\rangle$, of the images under M of the orthonormal vectors $|b\rangle$.

When a particular ordering is specified for the indices a and b, for instance the ordered index sets $A = \{1, 2, \ldots, d_A\}$ and $B = \{1, 2, \ldots, d_B\}$, we can arrange the coefficients M_{ij} ($i = 1, \ldots, d_A$, $j = 1, \ldots, d_B$) of the matrix M in an array, where the entry $M_{ij} = \langle i|M|j\rangle$ is placed at the intersection of the ith row and the jth column:

$$M = \begin{pmatrix} \langle 1|M|1\rangle & \cdots & \langle 1|M|j\rangle & \cdots & \langle 1|M|d_B\rangle \\ \vdots & & \vdots & & \vdots \\ \langle i|M|1\rangle & \cdots & \langle i|M|j\rangle & \cdots & \\ \vdots & & \vdots & & \vdots \\ \langle d_A|M|1\rangle & \ldots & \ldots & \ldots & \langle d_A|M|d_B\rangle \end{pmatrix}$$

We thus identify the linear map M with its (rectangular) matrix $M \in \mathbb{C}^{d_A \times d_B}$ of size $d_A \times d_B$ (with d_A rows and d_B columns), which depends on the choice of the bases $|a\rangle$ and $|b\rangle$ of the two spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Since

$$M = \sum_{i,j} M_{ij}\ |i\rangle\langle j|$$

the outer product $|i\rangle\langle j|$ is nothing other than the $d_A \times d_B$ matrix with a single nonzero entry, equal to 1, located at the intersection of the ith row and the jth column:

$$|i\rangle\langle j| = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & \vdots \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

The space of linear maps $\mathcal{H}_B \longrightarrow \mathcal{H}_A$ is thus identified with the space of complex matrices of size $d_A \times d_B$, for which the $|i\rangle\langle j|$ form a basis.

The **product** MN of two linear maps $N : \mathcal{H}_C \longrightarrow \mathcal{H}_B$ and $M : \mathcal{H}_B \longrightarrow \mathcal{H}_A$ is the composite map $MN : \mathcal{H}_C \longrightarrow \mathcal{H}_A$ (sometimes denoted $M \circ N$). Its coefficients are the $\langle a|MN|c\rangle$, which, by inserting the resolution of the identity $\mathbb{1}_B = \sum_b |b\rangle\langle b|$ into the expression $MN = M\mathbb{1}_B N$, are computed by the **matrix product formula** on the coefficients:

$$\langle a|MN|c\rangle = \sum_b \langle a|M|b\rangle\langle b|N|c\rangle$$

which is the familiar formula $(MN)_{ac} = \sum_b M_{ab}N_{bc}$. When a particular ordering is specified for the indices, for example $A = \{1, 2, \ldots, d_A\}$ and $B = \{1, 2, \ldots, d_B\}$, the matrix product is computed by arranging the two matrices M and N according to the usual matrix multiplication procedure, in which the entries of the rows of M are multiplied by those of the columns of N and summed.

Since a ket $|\psi\rangle$ (in a Hermitian space $\mathcal{H}$ of dimension d) can be identified with a linear map $\mathbb{C} \to \mathcal{H}$, where $\dim \mathbb{C} = 1$, it can in particular be written as a matrix of size $d \times 1$, that is, as a **column vector** that depends on the choice of the orthonormal basis $|a\rangle$ of $\mathcal{H}$ (the orthonormal basis of $\mathbb{C}$ consists only of the scalar 1). The coefficients of $|\psi\rangle = \sum_a |a\rangle\langle a|\psi\rangle$ are simply the coordinates $\langle a|\psi\rangle \cdot 1 = \langle a|\psi\rangle$ in the basis of the $|a\rangle$.

For a particular ordering of the indices such as $A = \{1, 2, \ldots, d\}$, the coefficients $\alpha_i = \langle i|\psi\rangle$ are placed in the order $i = 1, 2, \ldots, d$, and the decomposition $|\psi\rangle = \sum_i \alpha_i |i\rangle = \sum_i |i\rangle\langle i|\psi\rangle$ makes it possible to identify $|\psi\rangle$ with the column vector of coordinates

$$|\psi\rangle = \begin{pmatrix} \langle 1|\psi\rangle \\ \langle 2|\psi\rangle \\ \vdots \\ \langle d|\psi\rangle \end{pmatrix}$$

In particular, $|i\rangle$ is simply the column vector with a single nonzero entry, equal to 1, in position i:

$$|i\rangle = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

The $|i\rangle$ form the **standard basis** (also called the canonical basis, or the computational basis) of $\mathbb{C}^d$.

The image, under a linear map M (a $d_A \times d_B$ matrix), of a ket $|\psi\rangle \in \mathbb{C}^{d_B}$ is $M|\psi\rangle \in \mathbb{C}^{d_A}$, whose coefficients are

$$\langle a|M|\psi\rangle = \langle a|M\mathbb{1}_B|\psi\rangle = \sum_b \langle a|M|b\rangle\langle b|\psi\rangle$$

which are computed by the usual matrix–column-vector product procedure. In particular, $M|j\rangle$ ($j = 1, \ldots, d_B$) is simply the jth column of the matrix M.

Remark. Another (more concise) way to view the action of M on the basis vectors $|b\rangle$ of $\mathcal{H}_B$ is to write

$$M = M\mathbb{1}_B = \sum_b M|b\rangle\langle b| = \sum_b |Mb\rangle\langle b|$$

where one reads off in the columns the action $|Mb\rangle$ of M on each input basis vector $|b\rangle$.

Similarly, since a bra $\langle\psi|$ is a linear map $\mathcal{H} \to \mathbb{C}$, where $\dim \mathbb{C} = 1$, it can be written as a matrix of size $1 \times d$, that is, as a **row vector** that depends on the choice of the orthonormal basis $|a\rangle$ of $\mathcal{H}$ (the orthonormal basis of $\mathbb{C}$ consists only of the scalar 1). The coefficients of $\langle\psi| = \sum_a \langle\psi|a\rangle\langle a|$ are simply the coordinates $1 \cdot \langle\psi|a\rangle = \langle\psi|a\rangle = \langle a|\psi\rangle^*$ in the basis of the $\langle a|$.

For a particular ordering of the indices such as $A = \{1, 2, \ldots, d\}$, the coefficients $\alpha_i^* = \langle\psi|i\rangle$ are placed in the order $i = 1, 2, \ldots, d$, and the dual decomposition formula $\langle\psi| = \sum_i \alpha_i^* \langle i| = \sum_i \langle i|\psi\rangle^* \langle i|$ makes it possible to identify $\langle\psi|$ with the row vector of coordinates

$$\langle\psi| = \begin{pmatrix}\langle 1|\psi\rangle^* & \langle 2|\psi\rangle^* & \cdots & \langle d|\psi\rangle^*\end{pmatrix}$$

In particular, $\langle i|$ is simply the row vector with a single nonzero entry, equal to 1, in position i:

$$\langle i| = \begin{pmatrix}0 & \cdots & 1 & \cdots & 0\end{pmatrix}$$

Thus, if $|\psi\rangle = \sum_i \alpha_i |i\rangle$ and $|\phi\rangle = \sum_i \beta_i |i\rangle$, we can write, in the standard basis:

$$\langle\psi| = \begin{pmatrix}\alpha_1^* & \alpha_2^* & \cdots & \alpha_d^*\end{pmatrix} \quad \text{and} \quad |\phi\rangle = \begin{pmatrix}\beta_1 \\ \beta_2 \\ \vdots \\ \beta_d\end{pmatrix}$$

and their inner product is nothing other than the product of the row vector by the column vector:

$$\langle\psi|\phi\rangle = \begin{pmatrix}\alpha_1^* & \alpha_2^* & \cdots & \alpha_d^*\end{pmatrix}\begin{pmatrix}\beta_1 \\ \beta_2 \\ \vdots \\ \beta_d\end{pmatrix} = \sum_i \alpha_i^* \beta_i$$

Similarly, for a linear map M (a $d_A \times d_B$ matrix) and $|\psi\rangle \in \mathbb{C}^{d_A}$, the bra $\langle\psi|M$ is a row vector with coefficients

$$\langle\psi|M|b\rangle = \langle\psi|\mathbb{1}_A M|b\rangle = \sum_a \langle\psi|a\rangle\langle a|M|b\rangle$$

which are computed by the usual row-vector–matrix product procedure. In particular, $\langle i|M$ $(i = 1, \ldots, d_A)$ is the ith row of the matrix M.

Definition. *The* ***dual*** *(or adjoint) of the linear map* $M : \mathcal{H}_B \to \mathcal{H}_A$ *is the map* $M^* : \mathcal{H}_A \to \mathcal{H}_B$ *such that, for all* $|\phi\rangle \in \mathcal{H}_A$ *and* $|\psi\rangle \in \mathcal{H}_B$*,*

$$\langle\psi|M^*|\phi\rangle = \langle\phi|M|\psi\rangle^*$$

where the star on the right-hand side denotes complex conjugation.

Remark. This condition can also be written in the form $\langle\psi|M^*\phi\rangle = \langle\phi|M\psi\rangle^*$, that is, by conjugate symmetry, $\langle\psi|M^*\phi\rangle = \langle M\psi|\phi\rangle$.

For given orthonormal bases $|a\rangle$ of $\mathcal{H}_A$ and $|b\rangle$ of $\mathcal{H}_B$, the definition implies

$$\langle b|M^*|a\rangle = \langle a|M|b\rangle^*$$

The coefficients of the matrix of the dual M^*, of size $d_B \times d_A$, are therefore given by the formula $(M^*)_{ba} = (M_{ab})^*$, where the coefficients M_{ab} are conjugated and transposed. The matrix M^* is therefore the **conjugate transpose** of M (sometimes denoted $M^\dagger$).

Conversely, once the conjugate-transpose matrix has been defined for a given choice of bases, the corresponding linear map $M^* : \mathcal{H}_A \to \mathcal{H}_B$ indeed satisfies

$$\begin{aligned}\langle\psi|M^*|\phi\rangle = \langle\psi|\mathbb{1}_B M^* \mathbb{1}_A|\phi\rangle &= \sum_{a,b} \langle\psi|b\rangle\langle b|M^*|a\rangle\langle a|\phi\rangle \\ &= \sum_{a,b} \langle\phi|a\rangle^*\langle a|M|b\rangle^*\langle b|\psi\rangle^* = \langle\phi|\mathbb{1}_A M \mathbb{1}_B|\psi\rangle^* = \langle\phi|M|\psi\rangle^*\end{aligned}$$

for all $|\phi\rangle \in \mathcal{H}_A$ and $|\psi\rangle \in \mathcal{H}_B$. The dual M^* is therefore always well defined and uniquely characterized by this conjugate-transpose condition, for every choice of bases.

Since the initial matrix M is recovered as the conjugate transpose of its conjugate transpose M^*, we have

$$M^{**} = M$$

Example. We recover the special case of bra–ket duality: $|\psi\rangle^* = \langle\psi|$, where the conjugate transpose of a column vector with complex entries is the row vector of conjugate entries. Taking the conjugate transpose once again, we also have $\langle\psi|^* = |\psi\rangle$.

In the special case $d_A = d_B = 1$ of a matrix reduced to a scalar, M^* is simply the complex conjugate of M.

The dual $(MN)^*$ of the product MN has coefficients

$$\begin{aligned}\langle c|(MN)^*|a\rangle = \langle a|MN|c\rangle^* &= \sum_b \langle a|M|b\rangle^*\langle b|N|c\rangle^* \\ &= \sum_b \langle c|N^*|b\rangle\, \langle b|M^*|a\rangle = \langle c|N^*M^*|a\rangle\end{aligned}$$

which shows that

$$(MN)^* = N^*M^*$$

and more generally $(M_1M_2\cdots M_n)^* = M_n^*\cdots M_2^*M_1^*$: the dual of a product is the reversed product of the duals.

Example. The dual of the outer product $|\phi\rangle\langle\psi|$ is $(|\phi\rangle\langle\psi|)^* = (\langle\psi|)^*(|\phi\rangle)^* = |\psi\rangle\langle\phi|$, the outer product in the reverse order. In particular $(|\phi\rangle\langle\phi|)^* = |\phi\rangle\langle\phi|$ and, more generally, every projector $\Pi = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_n\rangle\langle\phi_n|$ is its own dual:

$$\Pi = \Pi^*$$