Substitution and Integration by-parts

Support Workshop A

04-03-2014

In this session we look at two of the main tools for solving integrals. The substitution rule, which is almost analogous to the chain-rule and integration by-parts, which is almost analogous to the product rule in differentiation.

Substitution If $u(x)$ is differentiable and $f(x)$ continuous then the **indefinite** integral

$$\boxed{\int f(u(x))\frac{du(x)}{dx}dx = \int f(u)du}$$

while the **definite** integral

$$\boxed{\int_a^b f(u(x))\frac{du(x)}{dx}dx = \int_{u(a)}^{u(b)} f(u)du}$$

Example:

$$\int e^{x^2}2xdx = \int e^{u(x)}\frac{du(x)}{dx}dx,$$

where $u(x) = x^2$. Thus

$$\int e^{x^2}2xdx = \int e^u du = e^u = e^{x^2}.$$

Integration-by-parts: Given two differentiable functions $u(x)$ and $v(x)$,

$$\boxed{\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx.} \tag{1}$$

Example Let $u(x) = x$ and $v(x) = e^x$ thus

$$\int xe^x dx = xe^x - \int e^x dx = e^x(x-1).$$

Sometimes we need to repeat, for instance with $u(x) = x^2$ and $v(x) = e^x$

$$\int x^2e^x dx = x^2e^x - 2\int xe^x dx = x^2e^x + 2e^x(1-x).$$

Repeated use of integration by parts kills-off polynomials. Repeated use of integration by parts is also used with *cyclic* functions such as $\cos(x)$ and $\sin(x)$. As an example where $u(x) = \cos(x)$ and $v(x) = e^x$ we have

$$\int \cos(x)e^x dx = \cos(x)e^x + \underbrace{\int \sin(x)e^x dx}_{I}. \tag{2}$$

To solve I, apply by-parts

$$\int \sin(x)e^x dx = \sin(x)e^x - \int \cos(x)e^x dx.$$

Using this in (2) we have

$$\int \cos(x)e^x dx = (\cos(x) + \sin(x))e^x - \int \cos(x)e^x dx.$$

Rearranging

$$\int \cos(x)e^x dx = \frac{1}{2}(\cos(x) + \sin(x))e^x + C.$$

1. **Substitution.** Solve the following **indefinite** integrals using substitution
   (a) $\int \sin(10x)dx$
   (b) $\displaystyle\int \frac{1}{1-x}dx$
   (c) $\displaystyle\int \frac{x}{1-x}dx$
   (d) $\displaystyle\int \frac{x^3}{\sqrt{x^2+3}}dx$
   (e) $\int \sin(x)\cos(x)dx$
   (f) $\int -\cos^2(x)\sin(x)dx$
   (g) $\int \sin^3(x)dx$. First try and rearrange until a $\cos(x)$ and a $\sin(x)$ appear. Hint: $\cos^2(x)+\sin^2(x) = 1$.
   (h) $\displaystyle\int \frac{\ln(x)}{x}dx$
   (i) $\int e^{\sin(x)}\cos(x)dx$

Solution:

(a) $u(x) = 10x$ thus $\displaystyle\int \sin(10x)dx = \int \sin(u)\frac{du}{10} = -\frac{\cos(u)}{10} + C = -\frac{\cos(10x)}{10} + C.$

(b) $u(x) = 1-x$ thus $\displaystyle\int \frac{1}{1-x}dx = -\int \frac{1}{u}du = -\ln(|u|) + C = -\ln(|1-x|) + C.$

(c) $u(x) = 1-x$ thus $\displaystyle\int \frac{x}{1-x}dx = -\int \frac{1-u}{u}du = \int 1 - \frac{1}{u}du = u - \ln(|u|) + C = (1-x) - \ln(|1-x|) + C.$

(d) $u(x) = x^2 + 3$ thus

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^2+3}}xdx &= \int \frac{u-3}{\sqrt{u}}\frac{du}{2}\\ &= \int u^{1/2} - 3u^{-1/2}\frac{du}{2} = 2/6u^{3/2} - 3u^{1/2} + C\\ &= 2/6(x^2+3)^{3/2} - 3(x^2+3)^{1/2} + C\end{aligned}$$

(e) $u(x) = \sin(x)$ thus $\int \sin(x)\cos(x)dx = \int udu = u^2/2 + C = \sin(x)^2/2 + C.$

(f) $u(x) = \cos(x)$ thus $\int -\cos^2(x)\sin(x)dx = \int u^2du = u^3/3 + C = \cos^3(x)/3 + C.$

(g) First, note that $\sin^3(x) = \sin(x)(1-\cos^2(x))$. Now substitute $u = \cos(x)$, we have $\int \sin^3(x)dx = \int \sin(x)(1-\cos^2(x))dx = \int -(1-u^2)du = -u + u^3/3 + C = -\cos(x) + \cos^3(x)/3 + C.$

(h) $u(x) = \ln(x)$ thus $\displaystyle\int \frac{\ln(x)}{x}dx = \int u(x)u'(x)dx = \int udu = u^2/2 + C = \ln(x)^2/2 + C.$

(i) $u(x) = \sin(x)$ thus $\int e^{\sin(x)}\cos(x)dx = \int e^{u(x)}u'(x)dx = \int e^u u = e^u + C = e^{\sin(x)} + C.$

2. **Substitution.** Solve the following **definite** integrals using substitution
   (a) $\int_0^1 e^{10x} dx$
   (b) $\displaystyle\int_1^5 x^2\sqrt{x-1}dx$
   (c) $\int_0^a x\sqrt{a^2-x^2}dx$
   (d) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}dx$. Did you really *have to* calculate this?
   (e) **EXTRA, it's a trick!** Try and "guess" the solution based on the last question $\displaystyle\int_{-\pi}^{\pi} \frac{\cos(e^{|x|})}{\tan(x)^2}x^{311}dx$

Solution:

(a) $u(x) = 10x$ thus $\displaystyle\int_0^1 e^{10x}dx = \frac{1}{10}\int_0^{10} e^u u' du = e^u|_0^{10} = e^{10} - 1.$

(b) $u(x) = x - 1$ thus $\displaystyle\int_1^5 x^2\sqrt{x-1}dx = \int_0^4 (1+u)^2 u^{1/2} du = \left(2/3u^{3/2} + 4/5u^{5/2} + 2/7u^{7/2}\right)\Big|_0^4$
Wolfram alpha Failed to solve this!!

(c) $u(x) = a^2 - x^2$ thus $\int_0^a x\sqrt{a^2-x^2}dx = \int_{a^2}^0 \sqrt{u}\frac{-1}{2}du = -\frac{1}{3}u^{3/2}|_{a^2}^0 = a^3/3.$

(d) $u(x) = \cos(x)$ thus $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)}dx = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{1}{u} - du = -\ln(|u|)|_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} = 0!$ Think about this, is $\tan(x)$ and odd or even function?

(e) The function $\frac{\cos(e^{|x|})}{\tan(x)^2}$ is even, while x^{311} is odd, thus multiplied together form an odd functions. Integral of odd functions are odd. Use the fundamental theorem of calculus to show that the integral in zero (or ask a tutor).

3. **Integration by parts.** Solve the following integrals using integration by parts
   (a) $\int x\sin(x)dx =$
   (b) $\int e^{2x}e^{5x}dx =$
   (c) $\int \cos(x)\sin(x)dx =$
   (d) $\int x^2\ln(x)dx =$
   (e) Try $u(x) = \ln(x)$ and $v(x) = 1$ in $\int \ln(x)dx =$
   (f) Try $u(x) = (\ln(x))^2$ and $v(x) = 1$ in $\int (\ln(x))^2 dx =$
   (g) $\int x^n \log_{10}(x)dx =$, for any $n \in \mathbb{N}$?

Solution:

(a) $\int x\sin(x)dx = \frac{1}{2}(\cos(x) + \sin(x))e^x + C.$

(b) $\int e^{2x}e^{5x}dx = \int e^{7x}dx = \frac{1}{7}e^{7x} + C$. If you used by-parts for this, I tricked you!

(c) $\int \cos(x)\sin(x)dx = -\cos(x)^2 - \int \cos(x)\sin(x)dx$, thus rearranging $\int \cos(x)\sin(x)dx = -\frac{1}{2}\cos(x)^2 + C.$

(d) $\displaystyle\int x^2\ln(x)dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3x^{-1}dx = 19x^3(3\ln(x) - 1) + C.$

(e) Try $u(x) = \ln(x)$ and $v(x) = 1$ in $\int \ln(x)dx = \ln(x)x - \int \frac{x}{x}dx = x(\ln(x) - 1) + C$.

(f) Try $u(x) = (\ln(x))^2$ and $v(x) = 1$ in $\int (\ln(x))^2 dx = (\ln(x))^2 x - 2\int \ln(x)\frac{x}{x}dx = (\ln(x))^2 x - 2x(\ln(x) - 1) + C$

(g) $\displaystyle \int x^n \log_{10}(x)dx = \frac{1}{n}x^{n+1}\log_{10}(x) - \int \frac{1}{n+1}x^{n+1}\frac{1}{x\ln(10)} = \frac{1}{(n+1)^2}x^{n+1}\left((n+1)\log_{10}(x) - \frac{1}{\ln(10)}\right)$