Workshop 10

March 31, 2014

Topics: Sequences and Series

1. **Explicit Sequences** Determine if the following sequences diverge or converge as $n \to \infty$. If they converge, give the limit (with proof!). If they diverge, prove that they diverge.
   (a) $a_n = \frac{3n^2-1}{10n+5n^2}$
   (b) $(-1)^n$
   (c) $\frac{(-1)^n}{n}$
   (d) $\frac{n^n}{n!}$
   (e) $\frac{2^n}{n!}$
   (f) $\frac{n+47}{\sqrt{n^2+3n}}$
   (g) $\sqrt{n+47} - \sqrt{n}$

2. **Recurrsive Sequences.** Use the "Bounded Monotic Theorem" to prove convergence. You will need to use mathematical induction to do this.
   (a) $a_1 = 1$ and $a_{n+1} = 3 - 1/a_n$ for $n \geq 1$. Prove that $1 \leq a_n < 3$ for all n and that the sequence is increasing. Find $\lim_{n\to\infty} a_n$.
   (b) $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$. Prove that $a_n \leq 3$, that it is increasing and converges. Find $\lim_{n\to\infty} a_n$.

3. **Series.** Using basic properties, sum of geometric series and comparison test, see if the series converges of diverges.

$$\textbf{Geometric Series:} \quad \sum_{n=0}^{\infty} \frac{1}{r^n} = \frac{1}{1-1/r}.$$

   (a) $\sum_{n=1}^{\infty} 1/2$
   Calculate the following Geometric Series:
   (b) $\sum_{n=0}^{\infty} e^{1-2n}$. Calculate the limit sum.
   (c) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{2n+1}}$. Calculate the limit sum.
   Use comparison
   (d) $\sum_{n=1}^{\infty} \frac{2+n}{n^3}$
   (e) $\sum_{n=1}^{\infty} \frac{2}{n^2+1}$
   (f) $\sum_{n=1}^{\infty} \frac{1+n}{n+n^{3/2}}$
   (g) $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$.
   (h) $\sum_{n=1}^{\infty} 1/\ln(2+2n)$

4. **Series.** Using basic properties, alternating series, comparison, ratio tests, see if it converges or diverges

- **Alternating Test:** If $\sum_{n=1}^{\infty}(-1)^n a_n$ is such that $\lim_{n\to\infty} a_n = 0$ and $0 \leq a_{n+1} \leq a_n$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.
- **Ratio test:** If $\sum_{n=1}^{\infty} a_n$ is such that
$$\lim_{\infty} \frac{a_{n+1}}{a_n} = L$$
The ratio test states that:
if $L < 1$ then the series converges absolutely;
if $L > 1$ then the series does not converge;
if $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.
- **Root test:** If $\sum_{n=1}^{\infty} a_n$ is such that
$$\lim_{\infty} \sqrt[n]{a_n} = C$$
The ratio test states that:
if $C < 1$ then the series converges absolutely;
if $C > 1$ then the series does not converge;
if $C = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

(a) **Apply Alternating test:** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2n)}$

(b) **Apply Alternating test or Comparison?** $\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/10}}$

(d) **Apply the Ratio Test:**
$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{2n+1}}$. (Was the Ratio test necessary here?)

(e) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(f) $\sum_{n=1}^{\infty} \frac{n^4}{(2n)!}$

(g) $\sum_{n=1}^{\infty} \frac{3^{n^3}}{n!}$

(h) **Try Root Test:**
$\sum \frac{(-3n)^n}{(2n\sqrt{n}+2)^n}$

(i) $\sum_{n=1}^{\infty}(1+1/n)^n$. **TIP:** $\lim_{n\to\infty}(1+1/n)^n = e$.

(j) $\sum_{n=1}^{\infty}(1+1/n)^{n^2}$