Workshop 7

March 7, 2014

Topics: polynomial division, partial fractions, trigonometric substitution, integration review

Practice exercises:

1. **Polynomial Division and Partial Fractions** We use these techniques to integrate ratios of polynomials
$$\frac{f(x)}{g(x)} = \frac{f_n x^n + f_{n-1}x^{n-1} + ... + f_0}{g_m x^m + g_{m-1}x^{m-1} + ... + g_0}$$
Firstly we check if we need to use long division by asking is $\deg(f) \geq \deg(g)$, if it is we must use long division, if not we can skip straight to partial fractions.

   Integrate the following using these techniques.

   (a) $\frac{x+7}{x^2-x-6}$.

   (b) $\frac{x^3+4x^2+3}{x^2+2x+1}$.

   (c) $\frac{x-8}{x^2-x-6}$

   (d) $\frac{x}{x^2-4x-5}$

   (e) $\frac{2x+1}{x^2+2x+1}$

   (f) $\frac{1}{x^2-4}$

   (g) $\frac{x^4-x^3-x-1}{x^3-x^2}$

   (h) $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$

   Solution:

   (a) $\frac{x+7}{x^2-x-6} = \frac{2}{x-3} - \frac{1}{x+2}$ so answer is $2\ln(x-3) - \ln(x+2) + C$

   (b) Long division gives us that $\frac{x^3+4x^2+3}{x^2+2x+1} = x+2+\frac{-5x+1}{x^2+2x+1} = x+2+\frac{-5}{x+1} + \frac{6}{(x+1)^2}$ which integrates to $x^2/2 + 2x - 5\ln(x+1) - 6/(x+1)$

   (c) partial fraction decomp $= 2/(x+2) - 1/(x-3)$ which integrates to $2\ln(x+2) - \ln(x-3)$

   (d) partial fraction decomp $= \frac{5}{6(x-5)} + \frac{1}{6(x+1)}$ which integrates to $5/6\ln(x-5) + 1/6\ln(x+1)$

   (e) partial fraction decomp $= 2/(x+1) - 1/(x-1)^2$ which integrates to $2\ln(x+1) + 1/(x-1)$

   (f) partial fraction decomp $= 1/4(\frac{1}{x-1} - \frac{1}{x+2})$ which integrates to $1/4\ln|\frac{x-2}{x+2}|$

   (g) long division $= x - \frac{x+1}{x^2(x-1)}$ partial fraction decomp $= x + 2/x + 1/x^2 - 2/(x-1)$ which integrates to $x^2/2 + 2ln|x| - 1/x - 2\ln|x-1| + C$

   (h) partial fraction decomp $= 2/x - 2/(x-1) + (2x+4)/(x^2+4)$ which integrates to $2\ln(x) - 2\ln(x-1) + \ln(x^2+4) + 2\arctan(x/2)$

2. **Trigonometric substitution** There is no trick here, it just takes lot's of practice! Integrate the following using trigonometric substitution.

(a) $\frac{1}{\sqrt{9-x^2}}$.

(b) $\frac{1}{x^2+25}$.

(c) $\frac{1}{x^2-4}$

(d) $\frac{\sqrt{25x^2-4}}{x}$

(e) $\frac{1}{x^4\sqrt{9-x^2}}$

(f) $\frac{x}{\sqrt{2x^2-4x-7}}$

(g) $e^{4x}\sqrt{1+e^{2x}}$

(h) $\int_0^{1/6} \frac{x^5}{(36x^2+1)^{3/2}}$

Solution:

(a) $x = 3\sin(u)$ give $\arcsin(x/3) + C$

(b) $x = 5\tan(u)$ gives $1/5 \arctan(x/5) + C$

(c) $x = 2\sec(u)$ gives $\int(\sec^3(u) - sec(u))\, du$ Then integrating $\sec^3$ is example 8 section 6.2

(d) $x = 2/5\sec(u)$ gives $\sqrt{25x^2-4} - 2\arccos(2/5x)$

(e) $x = 3\sin(u)$, then after some simplifying we need to use $v = \cot(u)$ which gives us $\frac{(9-x^2)^{3/2}}{243x^3} - \frac{\sqrt{9-x^2}^{3/2}}{81x}$

(f) Complete the square to get the root looking like $\sqrt{2(x-1)^2-9}$ then $x = 1 + (3/\sqrt{2})\sec(u)$ gives us what we want:

$$1/\sqrt{2}\ln|\frac{\sqrt{2}(x-1)}{3} + \frac{\sqrt{2x^2-4x-7}}{3}| + \frac{\sqrt{2x^2-4x-7}}{2}$$

(g) This is a strange one! $e^x = \tan(u)$ gives us $1/5(1+e^{2x})^{5/2} - 1/3(1+e^{2x})^{3/2}$

(h) $x = 1/6\tan(u)$ and then $v = \cos(u)$ gives us $1/17496 - \frac{11\sqrt{2}}{279936}$

3. **General Integration Techniques** Just integrate, using any appropriate technique.

(a) $\int_0^{\pi} \sin^2(\theta)\, d\theta$.

(b) $\int_1^3 \frac{\ln(2x)}{x^2}\, dx$.

(c) $\int e^{-2t}\cos t\, dt$

(d) $\int \frac{3x^3-17x^2+36x-35}{x^2-4x+4}\, dx$

(e) $\int \tan^4(s)\sec^4(s)\, ds$

(f) $\int \arcsin(x-1)\, dx$

(g) $\int \sqrt{9-4x^2}\, dx$

(h) $\int \frac{2x-1}{x^2-2x+10}\, dx$

Solution:

(a) Standard use of $\sin^2(u) = 1/2(1-\cos(2u))$ gives $\pi/2$

(b) By parts $u = \ln(2x)$, $dv = 1/x^2$ gives $-\ln(6)/3 + \ln(2) - 1/3 + 1$

(c) Do by parts twice with $u = e^{-2t}$, $dv = \cos(t)$, and then the same except sin for cos, then we get a recursion which solves to give an answer of $1/5(e^{-2t}\sin(t) - 2e^{-2t}\cos(t))$

(d) After division and partial fractionation $3x - 5 + 4/(x-2) + 7/(x-2)^2$ integrates to $3/2x^2 - 5x + 4\ln|x-2| + 7/(x-2)$

(e) $u = \tan(s)$ and using some trig identities gives you $\tan^5(s)/5 + \tan^7(s)/7$

(f) This is a classic integration by parts (after the obvious substitution of $u = x - 1$) to give $(x-1)\arcsin(x-1) + \sqrt{1-(x-1)^2}$

(g) Trig substitution $x = 3/2\sin(u)$ gives the integral as $9/4\arcsin(2/3x) + 1/2x\sqrt{9-4x^2}$

(h) Complete the square on the bottom, then rewrite the top as $2(x-1)+1$, this gives an answer of $\ln(x^2-2x+10)+1/3\arctan((x-1)/3)$