Quotient Rule for Derivatives and Limits

Support Workshop A

28-1-2014

In this session we will start with a review of the Quotient rule for derivatives. Then we move on to solving some finite limits, check the existence of limits, limits of quocients of functions and finally applications of the squeeze theorem for solving limits.

The quotient rule: Given two differentiable functions $f(x)$ and $g(x)$,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}. \tag{1}$$

Practice exercises:

1. **Quotient rule.** Find the derivatives of the following functions using the Quotient rule (1)

   (a) $F(x) = x/(1 + x)$.

   (b) $F(x) = x/\ln(x)$.

   (c) $F(x) = e^x/x^2$.

   (d) $F(x) = \tan(x)$.

   (e) $F(t) = t\sin(t)/\log_7(t)$.

   (f) See if you can use the chain-rule and product rule to repeat the demonstration of the Quotient Rule shown at the beginning of the tutorial.

2. **Limit Laws.** Solve the following limits using the *Limit Laws* of p.35 of the book.

   (a) $\lim_{x \to 2}(x^2 + 5)$

   (b) $\lim_{x \to 0} 3$

   (c) $\lim_{x \to 0}(\tan(x) - \sin(x))$

   (d) $\lim_{x \to 0} x\cos(x)$

   (e) $\lim_{x \to 1} \frac{x^2 - 5}{x + 3}$

   (f) $\lim_{x \to \pi}(x - \pi)\sin(x/10)\log(\pi x)$

3. **Existence of limits.** For the following questions, if the limit exists, calculate it. If it doesn't exist, justify why not. Remember: A limit $\lim_{x \to a} f(x)$ is said to exist if the left and right limits are the same $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$ **and** they are not **infinite**.

   (a) $\lim_{x \to \infty}(x - 3)$

   (b) $\lim_{x \to \infty}(x - x)$. **Remember you CANNOT use the *Limit Laws* when the *individual* limits do not exist.**

   (c) $\lim_{x \to \infty} 10$
   Let

   $$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 < x \le 2 \\ 1 - x & \text{if } 2 < x. \end{cases}$$

(d) $\lim_{x \to 1} f(x)$

(e) $\lim_{x \to 2} f(x)$

(f) $\lim_{x \to \infty} e^x$

(g) $\lim_{x \to 0} \lfloor x \rfloor + x$

(h) $\lim_{x \to 1} \dfrac{x}{|x|}$

4. **Limits of Quotient of Functions.**

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right). \tag{3}$$

We know from the *Limit Laws* that when the individual limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, we can divide one by the other to get (3). But what do we do when the quotient of their limits doesn't exist? Such as "$\frac{0}{0}$"? What to do when the individual limits do not exist? Let us figure this out through examples

(a) $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2}$.

(b) $\lim_{t \to 0} \dfrac{(-5 + t)^2 - 25}{t}$.

(c) $\lim_{x \to 2} \dfrac{x^3 - x^2 - 2x}{x - 2}$.

(d) $\lim_{x \to \infty} \dfrac{5x}{3x}$.

(e) $\lim_{x \to \infty} \dfrac{5x + 1}{3x - 1}$.

(f) $\lim_{x \to \infty} \dfrac{5x^2 - 2x + 1}{3x^2 - 3}$.

(g) $\lim_{x \to 0} \dfrac{\sqrt{9 + 2h} - 3}{h}$.

5. **The Squeeze Theorem.** If $\ell(x) \le f(x) \le u(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} \ell(x) = L = \lim_{x \to a} u(x),$$

then

$$\lim_{x \to a} \ell(x) \le \lim_{x \to a} f(x) \le \lim_{x \to a} u(x),$$

and the limit of $f(x)$ is *squeezed*

$$L \le \lim_{x \to a} f(x) \le L,$$

thus $\lim_{x \to a} f(x) = L$. Now try and apply this theorem to following problems, indicating the lower function $\ell(x)$ and the upper function $u(x)$.

(a) $\lim_{x \to 0} x \sin \frac{1}{2x}$.

(b) $\lim_{x \to \infty} 1/x \sin(3\pi x)$.

(c) $\lim_{x \to 0} x^{1/3} \left(3 - \sin(3\pi/x) \right)$.

(d) $\lim_{x \to 8} (x^{1/3} - 2)(x - \lfloor x \rfloor)$.

(e) $\lim_{x \to 0} \sin(x) \frac{|x|}{x}$.

(f) $\lim_{x \to \infty} e^{-x} \cos \frac{3}{\pi x}$.