OR MSc Maths Revision Course

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13 September 2013

General Information

Today

- JCMB 1501, 09:30-12:30
  - Mathematics revision class: A revision class on fundamental Mathematics skills

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Weeks 1 and 2

- JCMB 1501
  - Revision course tutorials (MRev)
    - Wednesday 23/9 15:00-17:00
    - Thursday 24/9 16:00-18:00
    - Wednesday 30/9 15:00-17:00
    - Thursday 1/10 16:00-18:00
  - Mathematics assessment test **II**

Functions

Functions: Logarithms

The logarithm $\log_b(x)$ for a base b and a number x is the answer to: What power should we raise b so that the result is x? That is

$$x = b^y \quad \Leftrightarrow \quad y = \log_b(x).$$

The logarithm $\log_b(x)$ is the **inverse** of taking b to the power of x. This means that $y = \log_b(b^y)$ and $y = b^{(\log_b y)}$

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- **Multiplication and division** identities include

$$\begin{array}{rcl|rcl} \log_b(xy) & = & \log_b x + \log_b y & b^{x+y} & = & b^x b^y \\ \log_b(x/y) & = & \log_b x - \log_b y & b^{x-y} & = & b^x / b^y \\ \log_b x^n & = & n \log_b x & (b^x)^n & = & b^{xn} \end{array}$$

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Exercise

Simplify $\ln(y) = 4\ln(2) - \frac{1}{2}\ln(25)$, .

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Exercise

Simplify $\ln(y) = 4\ln(2) - \frac{1}{2}\ln(25)$, $\quad y = 16/5$.

Functions: Linear, Quadratic and Roots

- **Linear functions** are in the form $f(x) = mx + c$, where m and c are constants. Example: $f(x) = 3x - 2$.
- **Quadratic functions** take the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, b and c are constants. Ex: $f(x) = 5x^2 - 2x + 1$.
  The **roots** are the solutions to $f(x) = 0$.
  Three techniques are commonly used:
  - Factorize into two brackets by inspection, e.g., $x^2 - 2\alpha x + \alpha^2 = (x - \alpha)(x - \alpha)$.
  - Find the roots using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - Complete the square, so $f(x) = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a^2}$

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Exercises

- Find the roots of $x^2 + 7x - 8 = 0$.
- Find the solution set for $x^2 - 3x - 4 \geq 0$.

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- Find the roots of $x^2 + 7x - 8 = 0$. $(x - 1)(x + 8) = 0$.
- Find the solution set for $x^2 - 3x - 4 \geq 0$.
  $(x - 4)(x + 1) \geq 0, \quad x \in (-\infty, -1] \cup [4, \infty)$

Calculus

Calculus: Derivatives

Derivative

A function $f(x)$ is said to have derivative at x, written $f'(x)$, if the limit of $\dfrac{f(x+h)-f(x)}{h}$ exists as $h \to 0$. $f'(x)$ is defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Derivatives of common functions

$f(x)$	a	x	x^n	e^x	a^x	$\ln x$	$\sin x$	$\cos x$
$f'(x)$	0	1	nx^{n-1}	e^x	$a^x \ln a$	$1/x$	$\cos x$	$-\sin x$

Calculus: Derivative rules

- *Constant Factor Rule:* If $f(x) = cu(x)$, where c constant, then
$$f'(x) = cu'(x)$$
- *Sum rule:* If $f(x) = u(x) + v(x)$, then
$$f'(x) = u'(x) + v'(x)$$
- *Product rule:* If $f(x) = u(x)v(x)$, then
$$f'(x) = u'(x)v(x) + u(x)v'(x)$$
- *Quotient rule:* If $f(x) = \frac{u(x)}{v(x)}$, then
$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$
- *Chain rule:* If $f(x) = g(u(x))$, then
$$f'(x) = g'(u(x)) \cdot u'(x)$$

Exercises

Find $f'(x)$:

- $f(x) = \cos(x)\ln(\cos x)$.
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- $f(x) = 1 + \dfrac{2^x}{x^2}$. $f'(x) = \dfrac{2^x}{x^2}(\ln(2) - 2)$

Application

Given a curve defined by $y = f(x)$, the tangent line of the curve at point $(a, f(a))$ is

$$y - f(a) = f'(a)(x - a)$$

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Calculus: Higher Order Derivatives

Higher order derivatives

Since $f'(x)$ is a function, we can find its derivative: the second derivative of $f(x)$, written as $f''(x)$ Similarly, third, fourth, higher derivatives are written $f'''(x)$, $f^{(4)}(x)$, $\ldots$, $f^{(n)}(x)$

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Taylor Series

An approximation of a function, local to a point a is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$

- First $p+1$ terms are the Taylor polynomial of degree p
- If $a = 0$ the expansion is called a *Maclaurin series*

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Determine the degree 3 polynomial for $f(x) = e^{2x}$ about $x = 0$.

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Exercise

Determine the degree 3 polynomial for $f(x) = e^{2x}$ about $x = 0$.
$f(x) \approx 1 + 2x + 2x^2$

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- **Sufficient condition:** Suppose a is a critical point
  If $f''(a) < 0$ then a is a maximizer
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Exercise

Find the critical points of $f(x) = 4x^3 - 21x^2 + 18x + 6$. Discuss their nature $f'(x) = 6(2x^2 - 7x + 3) = 0 \Rightarrow x = \frac{1}{2}, 3.$
$f''(x) = 6(4x - 7)$, thus $f''(1/2) = -30 < 0$ and $f''(3) = 30 > 0$

Calculus: Partial Derivatives

First partial derivatives

Let $z = f(x, y)$ be a function of x and y. Partial derivatives of z wrt x and y, respectively, are defined as

$$
\begin{aligned}
f_x(x, y) &= \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
f_y(x, y) &= \frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}
\end{aligned}
$$

Second partial derivatives

Second-order partial derivatives can also be written in either notation:

$$
\begin{aligned}
f_{xx}(x,y) &= \frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \\
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Exercise

Find $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ given that $f(x,y) = x^3 + 2xy - y^2$.

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Second-order partial derivatives can also be written in either notation:

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Exercise

Find $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ given that $f(x,y) = x^3 + 2xy - y^2$.

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & 2 \\ 2 & -2 \end{bmatrix}$$

Hessian matrix

The *Hessian H* is a square matrix containing all the second partial derivatives. Note that for smooth functions, $f_{xy} = f_{yx}$, so the matrix is symmetric. In this example with two variables,

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The Hessian can be used to test the nature of critical points

Calculus: Integrals

Integrals

- Consider two functions $f(x)$ and $F(x)$
- If $F'(x) = f(x)$ then $F(x)$ is the **indefinite integral** of $f(x)$
- This is written as

$$F(x) = \int f(x)\, dx$$

Calculus: Integrals

Integration examples and rules

$$
\begin{aligned}
\int (1)\ dx &= x + C \\
\int x^n\ dx &= \frac{x^{n+1}}{n+1} + C \qquad (n \neq -1) \\
\int x^{-1} &= \ln|x| + C \\
\int e^x\ dx &= e^x + C \\
\int a^x\ dx &= \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1) \\
\int af(x)\ dx &= a \int f(x)\ dx \\
\int [f(x) + g(x)]\ dx &= \int f(x)\ dx + \int g(x)\ dx \\
\int [f(x)]^n f'(x)\ dx &= \frac{[f(x)]^{n+1}}{n+1} + C \qquad (n \neq -1) \\
\int f(x)^{-1} f'(x)\ dx &= \ln f(x) + C \\
\int e^{f(x)} f'(x)\ dx &= e^{f(x)} + C \\
\int a^{f(x)} f'(x)\ dx &= \frac{a^{f(x)}}{\ln a} + C \qquad (a > 0, a \neq 1)
\end{aligned}
$$

Calculus: Integrals

Integration by parts

For two functions $u(x)$ and $v(x)$

$$\int u(x)v'(x)\ dx = u(x)v(x) - \int v(x)u'(x)\ dx$$

Integration by Substitution

Substitution If $u(x)$ is differentiable and $f(x)$ continuous then

$$\boxed{\int f(u(x))\frac{du(x)}{dx}dx = \int f(u)du}$$

Fundamental Theorem of Calculus

If $f(x)$ is continuous for all x satisfying $a \leq x \leq b$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where $F(x)$ is any indefinite integral of $f(x)$
$F(b) - F(a)$ is often written as $[F(x)]_a^b$

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Exercises

Find the following integrals:

- $F(x) = \int (2x+5)^3 dx.$
- $F(x) = \int_1^2 \frac{\ln x}{x^2} dx.$

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- $F(x) = \int (2x+5)^3 dx$. Subst $u = 2x+5$, $F(x) = (2x+5)^4/8$
- $F(x) = \int_1^2 \frac{\ln x}{x^2} dx$. By parts, $u(x) = \ln x$ and $v'(x) = x^{-2}$. $F(x) = -\log(x)x^{-1} - x^{-1}$.

Linear Algebra

Linear Algebra: Matrices

Matrices

- A **matrix** is any rectangular array of numbers
- The ij**th element** of A, written as a_{ij}, is the number in the ith row and jth column of A
- Two matrices A and B are **equal** if $a_{ij} = b_{ij}$ for all i and j

Linear Algebra: Vectors

Vectors

- **Column vector** is a matrix with only one column
- **Row vector** is a matrix with only one row
- **Vector** is a column vector or row vector
- **Dimension** of a vector is the number of elements in it
- **Zero vector** is vector with all elements equal 0

Scalar Product

If $\mathbf{u}$ is row vector and $\mathbf{v}$ is column vector with the same dimension n, then the **scalar product** of $\mathbf{u}$ and $\mathbf{v}$, written $\mathbf{uv}$, is the number

$$u_1 v_1 + \cdots + u_n v_n$$

For two column vectors $\mathbf{u}$ and $\mathbf{v}$, the scalar product is $\mathbf{u}^T\mathbf{v}$

Linear Algebra: Vector norms

Norm

- The **norm** of a vector is a quantity that describes in some way length or size of the vector.
- The p-norm $\|\mathbf{x}\|_p$ for $p = 1, 2, \ldots$ is defined as

$$\|\mathbf{x}\|_p = (\sum_i |x_i|^p)^{1/p}$$

Commonly used versions of the p-norm are

$$p = 1 \qquad \|\mathbf{x}\|_1 = \sum_i |x_i|$$

$$p = 2 \qquad \|\mathbf{x}\|_2 = \sqrt{\left(\sum_i |x_i|^2\right)}$$

$$p = \infty \qquad \|\mathbf{x}\|_\infty = \max_i |x_i|.$$

Norm properties

Some useful properties, which are true for all norms, are

- $\|\mathbf{x}\| > 0$ when $\mathbf{x} \neq \mathbf{0}$ and $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$
- $\|k\mathbf{x}\| = |k|\|\mathbf{x}\|$ for any scalar k
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

Linear Algebra: Matrices

Special Matrices

- For an $m \times n$ matrix A, the $n \times m$ matrix A^T which has elements $[A^T]_{ij} = a_{ji}$ is called the **transpose** of A
- A **square matrix** has equal numbers of rows and columns
- A matrix A for which $A = A^T$ is **symmetric**
- A **diagonal matrix** is one where only the elements a_{ii} are non-zero
- The **identity matrix** is diagonal with $a_{ii} = 1$
- An **orthogonal matrix** has the property $AA^T = I$

Positive definiteness

A real symmetric matrix is **positive definite** iff

- $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$
- All the leading submatrices of A have positive determinants
- All the eigenvalues of A satisfy $\lambda_i > 0$
- There exists a non-singular matrix W such that $A = W^T W$

If one of these conditions is true, it implies that the others are also true

Linear Algebra: Critical points

Critical point test for functions of two variables $f(x, y)$

For critical point (x, y) [where $f_x = 0, f_y = 0$] and Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

$\det H_1 = f_{xx}$	$\det H = f_{xx}f_{yy} - f_{xy}^2$	Conclusion
> 0	> 0	H is positive definite, so a (local) minimizer
< 0	> 0	H is negative definite, so a (local) maximizer
$\neq 0$	< 0	H is indefinite, so a saddle point

The test is inconclusive if either of the determinants is 0

Linear Algebra: Critical points

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$f(x, y) = 2x^2 - 4xy + y^4 + 2$

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$\det(H_1) = f_{xx} = 4$. $\det(H) = 48y^2 - 16 > 0$ for $y = 1$ or -1.
Computing $f(1, 1) = 1 = f(-1, -1)$ two global minima!

Linear Algebra: Matrix operations

Matrix operations

- **Scalar multiple of a matrix:** If A is a matrix, c is a number, then matrix cA is obtained by multiplying each element of A by c
- **Addition of two matrices:** If A and B are two matrices with same order (that is, $m \times n$), then matrix $C = A + B$ is obtained by defining $c_{ij} = a_{ij} + b_{ij}$
- **Matrix multiplication:** The matrix product of two matrices A and B, written AB, is defined if and only if

  $$\text{number of columns in } A = \text{number of rows in } B$$

  Suppose A is $m \times r$ matrix and B is $r \times n$ matrix, then the **matrix product** $C = AB$ is a $m \times n$ matrix whose ijth element c_{ij} is the scalar product of ith row of A and j column of B

Linear Algebra: Matrix operations

Inversion and Transposition

- **Matrix inversion**: Some $n \times n$ square matrices are invertible and it is possible to find A^{-1} such that

$$AA^{-1} = I$$

- **Inverse of matrix product:** Let A and B both be $n \times n$ square invertible matrices. Then $(AB)^{-1} = B^{-1}A^{-1}$
- **Transpose matrices:** from the definition of a transpose matrix above, it is not too hard to see these properties

$$\begin{aligned} (A^T)^T &= A \\ (A+B)^T &= A^T + B^T \\ (cA)^T &= cA^T \\ (AB)^T &= B^T A^T \\ (A^T)^{-1} &= (A^{-1})^T \end{aligned}$$

Exercises

- Given the matrices

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

  evaluate AB, BA^T

- Considering C and D as matrices of dimension $n \times n$, simplify $C(CD)^{-1}D(C^{-1}D)^{-1}$

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$$C(CD)^{-1}D(C^{-1}D)^{-1} \quad = CD^{-1}C^{-1}D(C^{-1}D)^{-1}$$

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Linear Algebra: Matrix operations

Determinant

The **determinant** of a 2×2 matrix can be calculated

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

For larger matrices, the determinant may be found by calculating the determinants of minor matrices M_{ij} recursively

$$\det(A) = \sum_{i=1}^{k} a_{ij}(-1)^{i+j} M_{ij}$$

Linear Algebra: Matrix operations

Exercises

Given

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

calculate $det(A)$ and $det(B)$

Methods for inverting matrices

- The formula for a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- Augment the matrix A to $[A \quad I]$ and use Gauss method (see below)
- Cramer's rule (in theory)

Linear Algebra: Matrix operations

Elementary row operations (`ero`)

If matrix A' is obtained by a set of elementary row operations from matrix A, then A and A' are equivalent

- **Type 1** `ero`**:** A' is obtained by multiplying any row of A by a nonzero number
- **Type 2** `ero`**:** A' is obtained by first multiplying any row of A (say, row i) by a nonzero number, then adding it to another row of A (say, row j), that is,

$$\text{row } j \text{ of } A' = c(\text{row } i \text{ of } A) + \text{row } j \text{ of } A$$

- **Type 3** `ero`**:** interchange any two rows of A

Exercises

Given

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

determine A^{-1} and B^{-1} and verify your results

Exercises

Given

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix},$$

determine A^{-1} and B^{-1} and verify your results

$$A^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1/3 & -2/3 & 0 \\ 1/3 & 1/3 & 0 \\ 1 & -2 & -1 \end{bmatrix},$$

Linear Algebra: Solving systems

Systems of linear equations

- Suppose A is $m \times n$ matrix and $\mathbf{b}$ is a vector of dimension m
- A **system of linear equations** is

$$A\mathbf{x} = \mathbf{b}$$

  with unknown column vector $\mathbf{x}$ of dimension n
- A column vector $\mathbf{x}$ is **solution** of a system of linear equations if it satisfies $A\mathbf{x} = \mathbf{b}$
- **Find a solution by Gauss method:**
  For the system $A\mathbf{x} = \mathbf{b}$, construct the augmented matrix $[A \quad \mathbf{b}]$ and use type 1, type 2 and type 3 eros
- **Special cases:**
  No solution or infinite number of solutions

Linear Algebra: Solving systems

Exercise

Solve by Gaussian elimination:

$$\begin{aligned} x - y + z &= 1 \\ 2x - y - 3z &= 0 \\ -x + y + 2z &= 2 \end{aligned}$$

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$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & -3 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

$(x, y, z) = (3, 3, 1).$

Linear Algebra: Differentiation

Differentiating linear algebra expressions

The symbol ∇ denotes the vector derivative or gradient operator

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

Here is a brief summary of how to apply this operator to expressions involving vectors and matrices. τ is a scalar, $\mathbf{c}, \mathbf{x} \in R^n$ are vectors, while A, Q are matrices, all of suitable dimensions

$f(\mathbf{x})$	$\mathbf{c}^T\mathbf{x}$	$\tau\mathbf{c}^T\mathbf{x}$	$A^T\mathbf{x}$	$\frac{1}{2}\mathbf{x}^T Q\mathbf{x}$
$\nabla f(\mathbf{x})$	$\mathbf{c}$	$\tau\mathbf{c}$	A	$\frac{1}{2}(Q\mathbf{x} + Q^T\mathbf{x})$

Further exercises

- Express $\{x : |x+3| < 2\}$ as intervals
- Solve by Gaussian elimination

$$\begin{aligned} 4x - y &= 3 \\ -2x + 5y &= 21 \end{aligned}$$

- Assume $\mathbf{c}, \mathbf{s}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are vectors, while $A, Q \in \mathbb{R}^{n \times n}$ are matrices, all of suitable dimensions.
  Find $\nabla_{\mathbf{x}} L(\mathbf{x})$ for the following expressions
  - $L(\mathbf{x}) = \mathbf{c}^T\mathbf{x} - \mathbf{x}^T\mathbf{s}$
  - $L(\mathbf{x}) = \mathbf{y}^T(A\mathbf{x} - \mathbf{b})$
  - $L(\mathbf{x}) = \mathbf{c}^T\mathbf{x} + \frac{1}{2}\mathbf{x}^T Q\mathbf{x}$

Further exercises

- Express $\{x : |x+3| < 2\}$ as intervals $\quad -5 < x < -1.$
- Solve by Gaussian elimination

$$\begin{aligned} 4x - y &= 3 \\ -2x + 5y &= 21 \end{aligned}$$

  $(x, y) = (2, 5).$

- Assume $\mathbf{c}, \mathbf{s}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are vectors, while $A, Q \in \mathbb{R}^{n\times n}$ are matrices, all of suitable dimensions.
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  - $\nabla_x L(\mathbf{x}) = \mathbf{c} - \mathbf{s}$
  - $\nabla_x L(\mathbf{x}) = A^T\mathbf{y}$
  - $\nabla_x L(\mathbf{x}) = \mathbf{c} + \frac{1}{2}(Q + Q^T)\mathbf{x}.$