

Randomized Quasi-Newton Updates are Linearly Convergent Matrix Inversion Algorithms

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Abstract

We develop and analyze a broad family of stochastic/randomized algorithms for inverting a matrix. We also develop a specialized variant which maintains symmetry or positive definiteness of the iterates. All methods in the family converge globally and linearly (i.e., the error decays exponentially), with explicit rates. In special cases, we obtain stochastic block variants of several quasi-Newton updates, including bad Broyden (BB), good Broyden (GB), Powell-symmetric-Broyden (PSB), Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS). Ours are the first stochastic versions of these updates shown to converge to an inverse of a fixed matrix. Through a dual viewpoint we uncover a fundamental link between quasi-Newton updates and approximate inverse preconditioning. Further, we develop an adaptive variant of randomized block BFGS, where we modify the distribution underlying the stochasticity of the method throughout the iterative process to achieve faster convergence. By inverting several matrices from varied applications, we demonstrate that AdaRBFGS is highly competitive when compared to the well established Newton-Schulz and minimal residual methods. In particular, on large-scale problems our method outperforms the standard methods by orders of magnitude. Development of efficient methods for estimating the inverse of very large matrices is a much needed tool for preconditioning and variable metric methods in the advent of the big data era.

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Contents

1	Introduction	1
1.1	Outline	1
1.2	Notation	2
2	Contributions	2
2.1	New algorithms	2
2.2	Dual formulation	3
2.3	Quasi-Newton updates and approximate inverse preconditioning	3
2.4	Complexity: general results	3
2.5	Complexity: discrete distributions	4
2.6	Adaptive randomized BFGS	4
2.7	Previous work	4
2.8	Extensions	5
3	Randomization of Quasi-Newton Updates	5
3.1	Quasi-Newton methods	5
3.2	Quasi-Newton updates	6
3.3	Randomized quasi-Newton updates	7
4	Inverting Nonsymmetric Matrices	7
4.1	Projection viewpoint: sketch-and-project	8
4.2	Optimization viewpoint: constrain-and-approximate	8
4.3	Equivalence	9
4.4	Relation to multiple linear systems	11
5	Inverting Symmetric Matrices	11
5.1	Projection viewpoint: sketch-and-project	12
5.2	Optimization viewpoint: constrain-and-approximate	12
5.3	Equivalence	12
6	Convergence	14
6.1	Norm of the expected error	16
6.2	Expectation of the norm of the error	18
7	Discrete Random Matrices	19
7.1	Optimizing an upper bound on the convergence rate	20
7.2	Convenient sampling	22
7.3	Optimal and adaptive samplings	22
8	Randomized Quasi-Newton Updates	23
8.1	One Step Update	24
8.2	Simultaneous Randomized Kaczmarz Update	24
8.3	Randomized Bad Broyden Update	25
8.4	Randomized Powell-Symmetric-Broyden Update	25
8.5	Randomized Good Broyden Update	26
8.6	Approximate Inverse Preconditioning	26
8.7	Randomized SR1	27

8.8	Randomized DFP Update	27
8.9	Randomized BFGS Update	28
8.10	Randomized Column Update	29
9	AdaRBFGS: Adaptive Randomized BFGS	29
9.1	Motivation	29
9.2	The algorithm	30
9.3	Implementation	31
10	Numerical Experiments	32
10.1	Experiment 1: synthetic matrices	33
10.2	Experiment 2: LIBSVM matrices	33
10.3	Experiment 3: UF sparse matrices	33
11	Conclusion	34
12	Appendix: Optimizing the Convergence Rate	39
13	Appendix: Numerical Experiments with the Same Starting Matrix	39

1 Introduction

Matrix inversion is a standard tool in numerics, needed for instance, in computing a projection matrix or a Schur complement, which are common place calculations in computational methods. When only an approximate inverse is required, then iterative methods are the methods of choice, for they can terminate the iterative process when the desired accuracy is reached. This can be far more efficient than using a direct method. Calculating an approximate inverse is a much needed tool in preconditioning [32] and, if the approximate inverse is guaranteed to be positive definite, then the iterative scheme can be used to design variable metric optimization methods. Furthermore, iterative methods can make use of an initial estimate of the inverse when available.

The driving motivation of this work is the need to develop algorithms capable of computing the inverse of very large matrices, where standard techniques take an exacerating amount of time or simply fail. In particular, we develop a family of randomized/stochastic methods for inverting a matrix, with specialized variants maintaining symmetry or positive definiteness of the iterates. All methods in the family converge globally (i.e., from any starting point) and linearly (i.e., the error decays exponentially). We give an explicit expression for the convergence rate.

As special cases, we obtain stochastic block variants of several quasi-Newton updates, including bad Broyden (BB), good Broyden (GB), Powell-symmetric-Broyden (PSB), Davidon-Fletcher-Powell (DFP) and Broyden-Fletcher-Goldfarb-Shanno (BFGS). To the best of our knowledge, these are first stochastic versions of quasi-Newton updates. Moreover, this is the first time that quasi-Newton methods are shown to be iterative methods for inverting a matrix. We also offer a new interpretation of the quasi-Newton methods through a Lagrangian dual viewpoint. This new viewpoint uncovers a fundamental link between quasi-Newton updates and approximate inverse preconditioning.

We develop an adaptive variant of randomized block BFGS, in which we modify the distribution underlying the stochasticity of the method throughout the iterative process to achieve faster convergence. Through extensive numerical experiments with matrices arising from several applications, we demonstrate that AdaRBFGS is highly competitive when compared to the well established Newton-Schulz and minimal residual methods. In particular, on large-scale problems our method outperforms the standard methods by orders of magnitude.

The development of efficient methods for estimating the inverse of very large matrices is a much needed tool for preconditioning and variable metric methods in the advent of the big data era.

1.1 Outline

The rest of the paper is organized as follows. In Section 2 we summarize the main contributions of this paper. In Section 3 we describe the quasi-Newton methods, which is the main inspiration of our methods. Subsequently, Section 4 describes two algorithms, each corresponding to a variant of the inverse equation, for inverting general square matrices. We also provide insightful dual viewpoints for both methods. In Section 5 we describe a method specialized to inverting symmetric matrices. Convergence in expectation is examined in Section 6, where we consider two types of convergence: the convergence of i) the expected norm of the error, and the convergence of ii) the norm of the expected error. In Section 7 we specialize our methods to discrete distributions, and comment on how one may construct a probability distribution leading to the best complexity rate (i.e., importance sampling). We then describe a convenient probability distribution which leads to convergence rates which can be described in terms of spectral properties of the original matrix to be inverted. In Section 8 we detail several instantiations of our family of methods, and their resulting convergence rates. We show how via the choice of the parameters of the method, we

obtain *stochastic block variants* of several well known quasi Newton methods. We also describe the simultaneous randomized Kaczmarz method here. Section 9 is dedicated to the development of an adaptive variant of our randomized BFGS method, AdaRBFS, for inverting positive definite matrices. This method adaptively changes the stochasticity of the method throughout the iterative process to obtain faster practical convergence behaviour. Finally, in Section 10 we show through numerical tests that AdaRBFGS significantly outperforms state-of-the-art iterative matrix inversion methods on large-scale matrices.

1.2 Notation

Let I denote the $n \times n$ identity matrix. Let

$$\langle X, Y \rangle_{F(W^{-1})} \stackrel{\text{def}}{=} \mathbf{Tr} (X^T W^{-1} Y W^{-1}),$$

denote the weighted Frobenius inner product, where $X, Y \in \mathbb{R}^{n \times n}$ and $W \in \mathbb{R}^{n \times n}$ is a symmetric positive definite “weight” matrix. As the trace is invariant under cyclic permutations, a fact we use repeatedly throughout the article, we have

$$\|X\|_{F(W^{-1})}^2 = \mathbf{Tr} (X^T W^{-1} X W^{-1}) = \mathbf{Tr} (W^{-1/2} X^T W^{-1} X W^{-1/2}) = \left\| W^{-1/2} X W^{-1/2} \right\|_F^2, \quad (1)$$

where we have used the convention $F = F(I)$, since $\|\cdot\|_{F(I)}$ is the standard Frobenius norm. Let $\|\cdot\|_2$ denote the induced operator norm for square matrices defined via

$$\|Y\|_2 \stackrel{\text{def}}{=} \max_{\|v\|_2=1} \|Yv\|_2.$$

Finally, for positive definite $W \in \mathbb{R}^{n \times n}$, we define the weighted induced norm via

$$\|Y\|_{W^{-1}} \stackrel{\text{def}}{=} \left\| W^{-1/2} Y W^{-1/2} \right\|_2.$$

2 Contributions

In this section we describe the main contributions of this paper.

2.1 New algorithms

We develop a novel and surprisingly simple family of stochastic algorithms for inverting matrices. The problem of finding the inverse of an $n \times n$ invertible matrix A can be characterized as finding the solution to either of the *inverse equations*¹ $AX = I$ or $XA = I$. Our methods make use of randomized sketching [30, 15, 29, 31] to reduce the dimension of the inverse equations in an iterative fashion. To the best of our knowledge, these are the first stochastic algorithms for inverting a matrix with global complexity rates.

In particular, our nonsymmetric method (Algorithm 1) is based on the inverse equation $AX = I$, and performs the *sketch-and-project* iteration

$$X_{k+1} = \arg \min_{X \in \mathbb{R}^{n \times n}} \frac{1}{2} \|X - X_k\|_{F(W^{-1})}^2 \quad \text{subject to} \quad S^T A X = S^T, \quad (2)$$

¹One may use other equations uniquely defining the inverse, such as $AXA = A$, but we do not explore these in this paper.

where $S \in \mathbb{R}^{n \times q}$ is a random matrix drawn in an i.i.d. fashion from a fixed distribution \mathcal{D} , and $W \in \mathbb{R}^{n \times n}$ is the positive definite “weight” matrix. The distribution \mathcal{D} and matrix W are the parameters of the method. Note that if we choose $q \ll n$, the constraint in the projection problem (2) will be of a much smaller dimension than the original inverse equation, and hence the iteration (2) will become cheap.

In an analogous way, we design a method based on the inverse equation $XA = I$ (Algorithm 2). By adding the symmetry constraint $X = X^T$, we obtain Algorithm 3—a specialized method for inverting symmetric matrices capable of maintaining symmetric iterates.

2.2 Dual formulation

Besides the *primal formulation* described in Section 2.1—*sketch-and-project*—we also provide *dual formulations* of all three methods (Algorithms 1, 2 and 3). For instance, the dual formulation of (2) is

$$X_{k+1} = \arg_X \min_{X \in \mathbb{R}^{n \times n}, Y \in \mathbb{R}^{n \times q}} \frac{1}{2} \|X_k - A^{-1}\|_{F(W^{-1})}^2 \quad \text{subject to} \quad X = X_k + WA^T SY^T. \quad (3)$$

We call the dual formulation *constrain-and-approximate* as one seeks to perform the best approximation of the inverse (with respect to the weighted Frobenius distance) while constraining the search to a random affine space of matrices passing through X_k . While the projection (3) cannot be performed directly since A^{-1} is not known, it can be performed indirectly via the equivalent primal formulation (2).

2.3 Quasi-Newton updates and approximate inverse preconditioning

As we will discuss in Section 3, through the lens of the sketch-and-project formulation, Algorithm 3 can be seen as *randomized block extension of the quasi-Newton updates* [4, 10, 12, 34]. We distinguish here between quasi-Newton methods, which are algorithms used in optimization, and quasi-Newton updates, which are the *matrix-update* rules used in the quasi-Newton methods. Standard quasi-Newton updates work with $q = 1$ (“block” refers to the choice $q > 1$) and S chosen in a deterministic and way, depending on the sequence of iterates of the underlying optimization problem. To the best of our knowledge, this is the first time stochastic versions of quasi-Newton updates were designed and analyzed. On the other hand, through the lens of the constrain-and-approximate formulation, our methods can be seen as *new variants of the approximate inverse preconditioning (AIP) methods* [6, 32, 13, 1]. Moreover, the equivalence between these two formulations reveals deep connections between what were before seen as distinct fields: the quasi-Newton and AIP literature. Our work also provides several new insights for *deterministic* quasi-Newton updates. For instance, the *bad Broyden update* [4, 19] is a particular best rank-1 update that minimizes the distance to the inverse of A under the Frobenius norm. The *BFGS update* [4, 10, 12, 34] can be seen as a projection of A^{-1} onto a space of rank-2 symmetric matrices. To the best of our knowledge, this has not been observed before.

2.4 Complexity: general results

Our framework leads to global linear convergence (i.e., exponential decay) under very weak assumptions on \mathcal{D} . In particular, we provide an explicit convergence rate ρ for the exponential decay of the norm of the expected error of the iterates (line 2 of Table 1) and the expected norm of the error (line 3 of Table 1), where the rate is given by

$$\rho = 1 - \lambda_{\min}(W^{1/2} \mathbf{E}[Z] W^{1/2}), \quad (4)$$

$\mathbf{E} [X_{k+1} - A^{-1}] = (I - W\mathbf{E} [Z]) \mathbf{E} [X_{k+1} - A^{-1}]$	Theorem 4.1
$\ \mathbf{E} [X_{k+1} - A^{-1}]\ _{W^{-1}}^2 \leq \rho^2 \cdot \ \mathbf{E} [X_{k+1} - A^{-1}]\ _{W^{-1}}^2$	Theorem 6.1
$\mathbf{E} [\ X_{k+1} - A^{-1}\ _{F(W^{-1})}^2] \leq \rho \cdot \mathbf{E} [\ X_{k+1} - A^{-1}\ _{F(W^{-1})}^2]$	Theorem 6.2

Table 1: Our main complexity results.

where

$$Z \stackrel{\text{def}}{=} A^T S (S^T A W A^T S)^{-1} S A^T.$$

We show that the convergence rate ρ is always bounded between 0 and 1. Furthermore, we provide a lower bound on ρ that shows that the rate can potentially improve as the number of columns in S increases. This sets our method apart from current methods for inverting matrices that lack global guarantees, such as Newton-Schulz, or the self-conditioning variants of the minimal residual method.

2.5 Complexity: discrete distributions

We detail a convenient choice of probability for discrete distributions \mathcal{D} that gives easy-to-interpret convergence results depending on a scaled condition number of A . With this convenient probability distribution we obtain methods for inverting matrices with the same convergence rate as the randomized Kaczmarz method [36] and randomized coordinate descent [23] for solving linear systems. We also obtain importance sampling results by optimizing an upper bound on the convergence rate.

2.6 Adaptive randomized BFGS

We develop an additional highly efficient method—adaptive randomized BFGS (AdaRBFGS)—for calculating an approximate inverse of *positive definite matrices*. Not only does the method greatly outperform the state-of-the-art methods such as Newton-Schulz and approximate inverse preconditioning methods, but it also preserves positive definiteness, a quality not present in previous methods. Therefore, AdaRBFGS can be used to precondition positive definite systems and to design new variable-metric optimization methods. Since the inspiration behind this method comes from the desire to design an *optimal adaptive* distribution for S by examining the complexity rate ρ , this work also highlights the importance of developing algorithms with explicit convergence rates.

2.7 Previous work

A widely used iterative method for inverting matrices is the Newton-Schulz method [33] introduced in 1933, and its variants which is still subject of ongoing research [25]. The drawback of the Newton-Schulz methods is that they do not converge for any initial estimate. Instead, an initial estimate that is close to A^{-1} (in some norm) is required. In contrast, the methods we present converge globally for any initial estimate. Bingham [3] describes a method that uses the characteristic polynomial to recursively calculate the inverse, though it requires the calculating the coefficients of the polynomial when initiated, which is costly, and the method has fallen into disuse. Goldfarb [11]

uses Broyden’s method [4] for iteratively inverting matrices. Our methods include a stochastic variant of Broyden’s method.

The approximate inverse preconditioning (AIP) methods [6, 32, 13, 1] calculate an approximate inverse by minimizing in $X \in \mathbb{R}^{n \times n}$ the residual $\|XA - I\|_F$ (Frobenius norm). They accomplish this by applying a number of iterations of the steepest descent or minimal residual method. A considerable drawback of the AIP methods, is that the approximate inverses are not guaranteed to be positive definite nor symmetric, even when A is both. A solution to the lack of symmetry is to “symmetrize” the estimate between iterations, but then it is difficult to guarantee the quality of the new symmetric estimate. Another solution is to calculate directly a factored form $LL^T = X$ and minimize in L the residual $\|L^TAL - I\|_F$. But now this residual is a non-convex function, and is thus difficult to minimize. A variant of our method naturally maintains symmetry of the iterates.

2.8 Extensions

This work opens up many possible avenues for extensions. For instance, new efficient methods could be achieved by experimenting and analyzing through our framework with different sophisticated sketching matrices S , such as the Walsh-Hadamard matrix [26, 30]. Furthermore, our method produces low rank estimates of the inverse and can be adapted to calculate low rank estimates of any matrix. Our methods can be applied to non-invertible matrices A , in which case they converge to a particular pseudo-inverse.

Our results can be used to push forward work into stochastic variable metric methods. Such as the work by Leventhal and Lewis [24], where they present a randomized iterative method for estimating Hessian matrices that converge in expectation with known convergence rates for any initial estimate. Stich et al. [35] use Leventhal and Lewis’ method to design a stochastic variable metric method for black-box minimization, with explicit convergence rates, and promising numeric results. We leave these and other extensions to future work.

3 Randomization of Quasi-Newton Updates

Our methods are inspired by, and in some cases can be considered to be, randomized block variants of the quasi-Newton updates. In this section we explain how our algorithms arise naturally from the quasi-Newton setting. Readers familiar with quasi-Newton methods may jump ahead to Section 3.3.

3.1 Quasi-Newton methods

A problem of fundamental interest in optimization is the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} f(x), \tag{5}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a sufficiently smooth function. Quasi-Newton (QN) methods, first proposed by Davidon in 1959 [7], are an extremely powerful and popular class of algorithms for solving this problem, especially in the regime of moderately large n . In each iteration of a QN method, one approximates the function locally around the current iterate x_k by a quadratic of the form

$$f(x_k + s) \approx f(x_k) + (\nabla f(x_k))^T s + \frac{1}{2} s^T B_k s, \tag{6}$$

where B_k is a suitably chosen approximation of the Hessian: $B_k \approx \nabla^2 f(x_k)$. After this, a direction s_k is computed by minimizing the quadratic approximation in s , obtaining

$$s_k = -B_k^{-1} \nabla f(x_k), \tag{7}$$

if the matrix B_k is invertible. The next iterate is then set to

$$x_{k+1} = x_k + h_k, \quad h_k = \alpha_k s_k,$$

for a suitable choice of stepsize α_k , often chosen by a line-search procedure (i.e., by approximately minimizing $f(x_k + \alpha s_k)$ in α).

Gradient descent arises as a special case of this process by choosing B_k to be constant throughout the iterations. A popular choice is $B_k = LI$, where I is the identity matrix and $L \in \mathbb{R}_+$ is the Lipschitz constant of the gradient of f . In such a case, the quadratic approximation (6) is a global upper bound on $f(x_k + s)$, which means that $f(x_k + s_k)$ is guaranteed to be at least as good (i.e., smaller or equal) as $f(x_k)$, leading to guaranteed descent. Newton's method also arises as a special case: by choosing $B_k = \nabla^2 f(x_k)$. These two algorithms are extreme cases on the opposite end of a spectrum. Gradient descent benefits from a trivial update rule for B_k and from cheap iterations due to the fact that no linear systems need to be solved. However, curvature information is largely ignored, which slows down the practical convergence of the method. Newton's method utilizes the full curvature information contained in the Hessian, but requires the computation of the Hessian in each step, which is expensive for large n . QN methods aim to find a sweet spot on the continuum between these two extremes. In particular, the QN methods choose B_{k+1} to be a matrix for which the *secant equation* is satisfied:

$$B_{k+1}(x_{k+1} - x_k) = \nabla f(x_{k+1}) - \nabla f(x_k). \quad (8)$$

The basic reasoning behind this requirement is the following: if f is a convex quadratic then the Hessian matrix satisfies the secant equation for all pairs of vectors x_{k+1} and x_k . If f is not a quadratic, the reasoning is as follows. Using the fundamental theorem of calculus, we have that

$$\left(\int_0^1 \nabla^2 f(x_k + th_k) dt \right) (x_{k+1} - x_k) = \nabla f(x_{k+1}) - \nabla f(x_k).$$

By selecting B_{k+1} that satisfies the secant equation, we are enforcing that B_{k+1} mimics the action of the integrated Hessian along the line segment joining x_k and x_{k+1} . Unless $n = 1$, the secant equation (8) does not have a unique solution in B_{k+1} . All QN methods differ only in which particular solution is used. The formulas transforming B_k to B_{k+1} are called *QN updates*.

Since these matrices are used to compute the direction s_k via (7), it is often more reasonable to instead maintain a sequence of inverses $X_k = B_k^{-1}$. By multiplying both sides of (8) by X_{k+1} , we arrive at the *secant equation for the inverse*:

$$X_{k+1}(\nabla f(x_{k+1}) - \nabla f(x_k)) = x_{k+1} - x_k. \quad (9)$$

The most popular classes of QN updates choose X_{k+1} as the closest matrix to X_k , in a suitable norm (usually a weighted Frobenius norm with various weight matrices), subject to the secant equation, often with an explicit symmetry constraint:

$$X_{k+1} = \arg \min_X \{ \|X - X_k\| : X y_k = h_k, X = X^T \}, \quad (10)$$

where $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$,

3.2 Quasi-Newton updates

Consider now problem (5) with the quadratic objective

$$f(x) = \frac{1}{2} x^T A x - b^T x + c, \quad (11)$$

where A is an $n \times n$ symmetric positive definite matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Granted, this is not a typical problem for which QN methods would be used by a practitioner. Indeed, the Hessian of f does not change, and hence one *does not have to* track it. The problem can simply be solved by setting the gradient to zero, which leads to the system $Ax = b$, the solution being $x_* = A^{-1}b$. As solving a linear system is much simpler than computing the inverse A^{-1} , approximately tracking the (inverse) Hessian of f along the path of the iterates $\{x_k\}$ —the basic strategy of all QN methods—seems like too much effort for what is ultimately a much simpler problem.

However, and this is one of the main insights of this work, instead of viewing QN methods as optimization algorithms, we can alternatively interpret them as iterative algorithms producing a sequence of matrices, $\{B_k\}$ or $\{X_k\}$, hopefully converging to some matrix of interest. In particular, one would hope that if a QN method is applied to the quadratic problem (11), with any symmetric positive definite initial guess X_0 , then the sequence $\{X_k\}$ converges to A^{-1} .

For f given by (11), the QN updates of the minimum distance variety given by (10) take the form

$$X_{k+1} = \arg \min_X \{ \|X - X_k\| : XAh_k = h_k, X = X^T \}. \quad (12)$$

3.3 Randomized quasi-Newton updates

While the motivation for our work comes from optimization, having arrived at the update (12), we can dispense of some of the implicit assumptions and propose and analyze a wider class of methods. In particular, in this paper we analyze a large class of *randomized algorithms* of the type (12), where the vector h_k is replaced by a random matrix S and A is *any* invertible², and not necessarily symmetric or positive definite matrix. This constitutes a randomized block extension of the QN updates.

4 Inverting Nonsymmetric Matrices

In this paper we are concerned with the development and complexity analysis of a family of stochastic algorithms for computing the inverse of a nonsingular matrix $A \in \mathbb{R}^{n \times n}$. The starting point in the development of our methods is the simple observation that the inverse A^{-1} is the (unique) solution of a linear matrix equation, which we shall refer to as *inverse equation*:

$$AX = I. \quad (13)$$

Alternatively, one can use the inverse equation $XA = I$ instead. Since (13) is difficult to solve directly, our approach is to iteratively solve a small randomly relaxed version of (13). That is, we choose a random matrix $S \in \mathbb{R}^{n \times q}$, with $q \ll n$, and instead solve the following *sketched inverse equation*:

$$S^T AX = S^T. \quad (14)$$

If we base the method on the second inverse equation, the sketched inverse equation $XAS = S$ should be used instead. Note that A^{-1} satisfies (14). If $q \ll n$, the sketched inverse equation is of a much smaller dimension than the original inverse equation, and hence easier to solve. However, the equation will no longer have a unique solution and in order to design an algorithm, we need a way of picking a particular solution. Our algorithm defines X_{k+1} to be the solution that is closest

²In fact, one can apply the method to an arbitrary real matrix A , in which case the iterates $\{X_k\}$ converge to the Moore-Penrose pseudoinverse of A . However, this development is outside the scope of this paper, and is left for future work.

to the current iterate X_k in a weighted Frobenius norm. This is repeated in an iterative fashion, each time drawing S independently from a fixed distribution \mathcal{D} .

The distribution \mathcal{D} and the matrix W can be seen as parameters of our method. The flexibility of being able to adjust \mathcal{D} and W is important: by varying these parameters we obtain various specific instantiations of the generic method, with varying properties and convergence rates. This gives the practitioner the flexibility to adjust the method to the structure of A , to the computing environment and so on. As we shall see in Section 8, for various choices of these parameters we recover stochastic block variants of several well known quasi-Newton updates.

4.1 Projection viewpoint: sketch-and-project

The next iterate X_{k+1} is the nearest point to X_k that satisfies a *sketched* version of the inverse equation:

$$\boxed{X_{k+1} = \arg \min_X \frac{1}{2} \|X - X_k\|_{F(W^{-1})}^2 \quad \text{subject to} \quad S^T A X = S^T} \quad (15)$$

In the special case when $S = I$, the only such matrix is the inverse itself, and (15) is not helpful. However, if S is “simple”, (15) will be easy to compute and the hope is that through a sequence of such steps, where the matrices S are sampled in an i.i.d. fashion from some distribution, X_k will converge to A^{-1} .

Alternatively, we can sketch the equation $XA = I$ and project onto $XAS = S$:

$$\boxed{X_{k+1} = \arg \min_X \frac{1}{2} \|X - X_k\|_{F(W^{-1})}^2 \quad \text{subject to} \quad XAS = S} \quad (16)$$

While the method (15) sketches the rows of A , the method (16) sketches the columns of A . Thus we refer to (15) as the row variant and to (16) as the column variant. The two variants (15) and (16) both converge to the inverse of A , as will be established in Section 6.

If A is singular, then the iterates of (16) converge to the left inverse, while the iterates of (15) converge to the right inverse, an observation we leave to future work.

4.2 Optimization viewpoint: constrain-and-approximate

The row sketch-and-project method can be cast in an apparently different yet equivalent viewpoint:

$$\boxed{X_{k+1} = \arg_X \min_{X,Y} \frac{1}{2} \|X - A^{-1}\|_{F(W^{-1})}^2 \quad \text{subject to} \quad X = X_k + W A^T S Y^T} \quad (17)$$

In this viewpoint, at each iteration (17), we select a random affine space that passes through X_k . After that, we select the point in this space that is as close as possible to the inverse. This random search space is special in that, independently of the input pair (W, S) we can efficiently compute the projection of A^{-1} onto this space, without knowing A^{-1} explicitly.

The column variant (16) also has an equivalent constrain-and-approximate formulation:

$$\boxed{X_{k+1} = \arg_X \min_{X,Y} \frac{1}{2} \|X - A^{-1}\|_{F(W^{-1})}^2 \quad \text{subject to} \quad X = X_k + Y S^T A^T W} \quad (18)$$

These two variants (17) and (18) can be viewed as new variants of the approximate inverse preconditioner (AIP) methods [1, 13, 22, 21]. The AIP methods are a class of methods for computing approximate inverses of A by minimizing $\|XA - I\|_F$ via iterative optimization algorithms.

In particular, the AIP methods use variants of the steepest descent or a minimal residual method to minimize $\|XA - I\|_F$. The idea behind the AIP methods is to minimize the distance of X from A^{-1} in some sense. Our variants do just that, but under a weight Frobenius norm. Furthermore, our methods project onto a randomly generated affine space instead of employing steepest descent of a minimal residual method.

4.3 Equivalence

We now prove that (15) and (16) are equivalent to (17) and (18), respectively, and give their explicit solution.

Theorem 4.1. *The viewpoints (15) and (17) are equivalent to (16) and (18), respectively. Furthermore, if S has full column rank, then the explicit solution to (15) is*

$$\boxed{X_{k+1} = X_k + WA^T S(S^T AWA^T S)^{-1} S^T (I - AX_k)} \quad (19)$$

and the explicit solution to (16) is

$$\boxed{X_{k+1} = X_k + (I - X_k A^T) S(S^T A^T W A S)^{-1} S^T A^T W} \quad (20)$$

Proof. We will prove all the claims for the row variant, that is, we prove that (15) are (17) equivalent and that their solution is given by (19). The remaining claims, that (16) are (18) are equivalent and that their solution is given by (20), follow with analogous arguments.

It suffices to consider the case when $W = I$, as we can perform a change of variables to recover the solution for any W . Indeed, in view of (1), with the change of variables

$$\hat{X} \stackrel{\text{def}}{=} W^{-1/2} X W^{-1/2}, \quad \hat{X}_k \stackrel{\text{def}}{=} W^{-1/2} X_k W^{-1/2}, \quad \hat{A} \stackrel{\text{def}}{=} W^{1/2} A W^{1/2} \quad \text{and} \quad \hat{S} \stackrel{\text{def}}{=} W^{-1/2} S, \quad (21)$$

(15) becomes

$$\min_{\hat{X} \in \mathbb{R}^{n \times n}} \frac{1}{2} \left\| \hat{X} - \hat{X}_k \right\|_F^2 \quad \text{subject to} \quad \hat{S}^T \hat{A} \hat{X} = \hat{S}^T. \quad (22)$$

If we moreover let $\hat{Y} = W^{-1/2} Y$, then (17) becomes

$$\min_{\hat{X} \in \mathbb{R}^{n \times n}, \hat{Y} \in \mathbb{R}^{n \times q}} \frac{1}{2} \left\| \hat{X} - \hat{A}^{-1} \right\|_F^2 \quad \text{subject to} \quad \hat{X} = \hat{X}_k + \hat{A}^T \hat{S} \hat{Y}^T. \quad (23)$$

By substituting the constraint in (23) into the objective function, then differentiating to find the stationary point, we obtain that

$$\hat{X} = \hat{X}_k + \hat{A}^T \hat{S} (\hat{S}^T \hat{A} \hat{A}^T \hat{S})^{-1} \hat{S}^T (I - \hat{A} \hat{X}_k), \quad (24)$$

is the solution to (23). After changing the variables back using (21), the update (24) becomes (37).

Now we prove the equivalence of (22) and (23) using Lagrangian duality. The sketch-and-project viewpoint (22) has a convex quadratic objective function with linear constraints, thus strong duality holds. Introducing Lagrangian multiplier $\hat{Y} \in \mathbb{R}^{n \times q}$, the Lagrangian dual of (22) is given by

$$L(\hat{X}, \hat{Y}) = \frac{1}{2} \left\| \hat{X} - \hat{X}_k \right\|_F^2 - \left\langle \hat{Y}^T, \hat{S}^T \hat{A} (\hat{X} - \hat{A}^{-1}) \right\rangle_F. \quad (25)$$

Algorithm 1 Stochastic Iterative Matrix Inversion (SIMI) – nonsymmetric row variant

- 1: **input:** invertible matrix $A \in \mathbb{R}^{n \times n}$
 - 2: **parameters:** \mathcal{D} = distribution over random matrices; positive definite matrix $W \in \mathbb{R}^{n \times n}$
 - 3: **initialize:** arbitrary square matrix $X_0 \in \mathbb{R}^{n \times n}$
 - 4: **for** $k = 0, 1, 2, \dots$ **do**
 - 5: Sample an independent copy $S \sim \mathcal{D}$
 - 6: Compute $\Lambda = S(S^T A W A^T S)^{-1} S^T$
 - 7: $X_{k+1} = X_k + W A^T \Lambda (I - A X_k)$ ▷ This is equivalent to (15) and (17)
 - 8: **output:** last iterate X_k
-

Clearly

$$(22) = \min_{X \in \mathbb{R}^{n \times n}} \max_{\hat{Y} \in \mathbb{R}^{n \times q}} L(\hat{X}, \hat{Y}).$$

We will now prove that

$$(23) = \max_{\hat{Y} \in \mathbb{R}^{n \times q}} \min_{X \in \mathbb{R}^{n \times n}} L(\hat{X}, \hat{Y}),$$

thus proving that (22) and (23) are equivalent by strong duality. Differentiating the Lagrangian in \hat{X} and setting to zero gives

$$\hat{X} = \hat{X}_k + \hat{A}^T \hat{S} \hat{Y}^T. \quad (26)$$

Substituting back into (25) gives

$$\begin{aligned} L(\hat{X}, \hat{Y}) &= \frac{1}{2} \left\| \hat{A}^T \hat{S} \hat{Y}^T \right\|_F^2 - \left\langle \hat{A}^T \hat{S} \hat{Y}^T, \hat{X}_k + \hat{A}^T \hat{S} \hat{Y}^T - \hat{A}^{-1} \right\rangle_F \\ &= -\frac{1}{2} \left\| \hat{A}^T \hat{S} \hat{Y}^T \right\|_F^2 - \left\langle \hat{A}^T \hat{S} \hat{Y}^T, \hat{X} - \hat{A}^{-1} \right\rangle_F. \end{aligned}$$

Adding $\pm \frac{1}{2} \left\| \hat{X}_k - \hat{A}^{-1} \right\|_F^2$ to the above gives

$$L(\hat{X}, \hat{Y}) = -\frac{1}{2} \left\| \hat{A}^T \hat{S} \hat{Y}^T + \hat{X}_k - \hat{A}^{-1} \right\|_F^2 + \frac{1}{2} \left\| \hat{X}_k - \hat{A}^{-1} \right\|_F^2.$$

Finally, substituting (26) into the above, minimizing in \hat{X} then maximizing in \hat{Y} , and dispensing of the term $\frac{1}{2} \left\| \hat{X}_k - \hat{A}^{-1} \right\|_F^2$ as it does not depend on \hat{Y} nor \hat{X} , we have that the dual problem is

$$\max_{\hat{Y}} \min_{\hat{X}} L(\hat{X}, \hat{Y}) = \min_{\hat{X}, \hat{Y}} \frac{1}{2} \left\| \hat{X} - \hat{A}^{-1} \right\|_F^2 \quad \text{subject to} \quad \hat{X} = \hat{X}_k + \hat{A}^T \hat{S} \hat{Y}^T.$$

It now remains to change variables using (21) and set $Y = W^{1/2} \hat{Y}$ to obtain (17). □

Based on Theorem 4.1, we can summarize the methods described in this section as Algorithm 1 and Algorithm 2.

The explicit formulas (19) and (20) for (15) and (16) allow us to efficiently implement these methods, and facilitate convergence analysis. In particular, we can now see that the convergence analysis of (20) will follow trivially from analyzing (19). This is because (19) and (20) differ only in terms of a transposition. That is, transposing (20) gives

$$X_{k+1}^T = X_k^T + W A S (S^T A^T W A S)^{-1} S^T (I - A^T X_k^T),$$

Algorithm 2 Stochastic Iterative Matrix Inversion (SIMI) – nonsymmetric column variant

- 1: **input:** invertible matrix $A \in \mathbb{R}^{n \times n}$
 - 2: **parameters:** \mathcal{D} = distribution over random matrices; positive definite matrix $W \in \mathbb{R}^{n \times n}$
 - 3: **initialize:** arbitrary square matrix $X_0 \in \mathbb{R}^{n \times n}$
 - 4: **for** $k = 0, 1, 2, \dots$ **do**
 - 5: Sample an independent copy $S \sim \mathcal{D}$
 - 6: Compute $\Lambda = S(S^T A^T W A S)^{-1} S^T$
 - 7: $X_{k+1} = X_k + (I - X_k A^T) \Lambda A^T W$ ▷ This is equivalent to (16) and (18)
 - 8: **output:** last iterate X_k
-

which is the solution to the row variant of the sketch-and-project viewpoint but where the equation $A^T X^T = I$ is sketched instead of $AX = I$. Thus it suffices to study the convergence of (19), then the convergence of (20) follows by simply swapping the role of A for A^T . We collect this observation in the following remark.

Remark 4.1. *The expression for the rate of Algorithm 2 is the same as the expression for the rate of Algorithm 1, but with every occurrence of A swapped for A^T .*

4.4 Relation to multiple linear systems

Any iterative method for solving linear systems can be applied to the n linear systems that define the inverse through $AX = I$ to obtain an approximate inverse. Though not all methods for solving linear systems can be applied to solve these n linear systems simultaneously, that is calculating each column of X simultaneously, which is necessary for an efficient matrix inversion method.

The recently proposed methods in [15] for solving linear systems can be easily and efficiently generalized to inverting a matrix, and the resulting method is equivalent to our row variant method (15) and (17). To show this, we perform the change of variables $\hat{X}_k = X_k W^{-1/2}$, $\hat{A} = W^{1/2} A$ and $\hat{S} = W^{-1/2} S$ then (15) becomes

$$\hat{X}_{k+1} \stackrel{\text{def}}{=} X_{k+1} W^{-1/2} = \arg \min_{\hat{X} \in \mathbb{R}^{n \times n}} \frac{1}{2} \left\| W^{-1/2} (\hat{X} - \hat{X}_k) \right\|_F^2 \quad \text{subject to} \quad \hat{S}^T \hat{A} \hat{X} = \hat{S}^T.$$

The above is a separable problem and each column of \hat{X}_{k+1} can be calculated separately. Let \hat{x}_{k+1}^i be the i th column of \hat{X}_{k+1} which can be calculated through

$$\hat{x}_{k+1}^i = \arg \min_{\hat{x} \in \mathbb{R}^n} \frac{1}{2} \left\| W^{-1/2} (\hat{x} - \hat{x}_k^i) \right\|_2^2 \quad \text{subject to} \quad \hat{S}^T \hat{A} \hat{x} = \hat{S}^T e_i.$$

The above was proposed as a method for solving linear systems in [15] applied to the system $\hat{A} \hat{x} = e_i$. Thus the convergence results established in [15] carry over to our row variant (15) and (17). In particular, the theory in [15] proves that the expected norm difference of each column of $W^{-1/2} X_k$ converges to $W^{-1/2} A^{-1}$ with rate ρ as defined in (4). This equivalence breaks down when we impose additional matrix properties through constraints, such as symmetry.

5 Inverting Symmetric Matrices

When A is symmetric, it may be useful to maintain symmetry in the iterates, in which case the nonsymmetric methods—Algorithms 1 and 2—have an issue, as they do not guarantee that the iterates are symmetric. However, we can modify (15) by adding a symmetry constraint. The resulting *symmetric* method naturally maintains symmetry in the iterates.

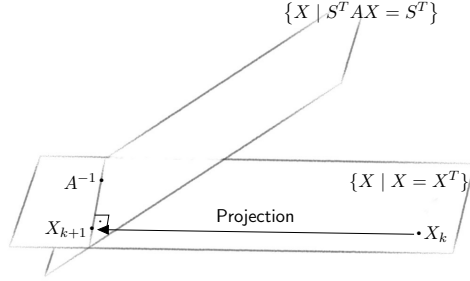


Figure 1: The new estimate X_{k+1} is obtained by projecting X_k onto the affine space formed by intersecting $\{X \mid X = X^T\}$ and $\{X \mid S^T A X = S^T\}$.

5.1 Projection viewpoint: sketch-and-project

The new iterate X_{k+1} is the result of projecting X_k onto the space of matrices that satisfy a sketched inverse equation and that are also symmetric, that is

$$X_{k+1} = \arg \min_X \frac{1}{2} \|X - X_k\|_{F(W^{-1})}^2 \quad \text{subject to} \quad S^T A X = S^T, \quad X = X^T \quad (27)$$

See Figure 1 for an illustration of the symmetric update (27).

This viewpoint can be seen as a randomized block version of the quasi-Newton methods [12, 18], as detailed in Section 3. The flexibility in using a weighted norm is important for choosing a norm that better reflects the geometry of the problem. For instance, when A is symmetric positive definite, it turns out that $W^{-1} = A$ results in a good method. This added freedom of choosing an appropriate weighting matrix has proven very useful in the quasi-Newton literature, in particular, the highly successful BFGS method [4, 10, 12, 34] selects W^{-1} as an estimate of the Hessian matrix.

5.2 Optimization viewpoint: constrain-and-approximate

The viewpoint (27) also has an interesting dual viewpoint:

$$X_{k+1} = \arg \min_{X, Y} \frac{1}{2} \|X - A^{-1}\|_{F(W^{-1})}^2 \quad \text{subject to} \quad X = X_k + \frac{1}{2}(Y S^T A W + W A^T S Y^T) \quad (28)$$

The minimum is taken over matrices $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{n \times q}$. The next iterate X_{k+1} is the best approximation to A^{-1} restricted to a random affine space of symmetric matrices. Furthermore, (28) is a symmetric equivalent of (17); that is, the constraint in (28) is the result of projecting the constraint in (17) onto the space of symmetric matrices.

When A is symmetric positive definite and we choose $W^{-1} = A$ in (17) and (18), then

$$\|X - A^{-1}\|_{F(A)}^2 = \mathbf{Tr}((X - A^{-1})A(X - A^{-1})A) = \|XA - I\|_F^2.$$

This is exactly the objective function used in most approximate inverse preconditioners (AIP) [1, 13, 22, 21].

5.3 Equivalence

We now prove that the two viewpoints (27) and (28) are equivalent, and show their explicit solution.

Theorem 5.1. *If A and X_k are symmetric, then the viewpoints (27) and (28) are equivalent. That is, they define the same X_{k+1} . Furthermore, if S has full column rank, then the explicit solution to (27) and (28) is*

$$\boxed{X_{k+1} = X_k - (X_k A S - S) \Lambda S^T A W + W A S \Lambda (S^T A X_k - S^T) (A S \Lambda S^T A W - I)} \quad (29)$$

where $\Lambda \stackrel{\text{def}}{=} (S^T A W A S)^{-1}$.

Proof. It was recently shown in [14, Section 2] and [20, Section 4]³ that (29) is the solution to (27). We now prove the equivalence of (27) and (28) using Lagrangian duality. It suffices to prove the claim for $W = I$ as we did in the proof of Theorem 4.1, since using the change of variables (21) applied to (27) we have that (27) is equivalent to

$$\min_{\hat{X} \in \mathbb{R}^{n \times n}} \frac{1}{2} \left\| \hat{X} - \hat{X}_k \right\|_F^2 \quad \text{subject to} \quad \hat{S}^T \hat{A} \hat{X} = \hat{S}^T, \quad \hat{X} = \hat{X}^T. \quad (30)$$

Since (27) has a convex quadratic objective with linear constraints, strong duality holds. Thus we will derive a dual formulation for (30) then use the change of coordinates (21) to recover the solution to (27). Let $\Gamma \in \mathbb{R}^{n \times q}$ and $\Lambda \in \mathbb{R}^{n \times n}$ and consider the Lagrangian of (30) which is

$$L(\hat{X}, \Gamma, \Lambda) = \frac{1}{2} \left\| \hat{X} - \hat{X}_k \right\|_F^2 - \left\langle \Gamma^T, \hat{S}^T \hat{A} (\hat{X} - \hat{A}^{-1}) \right\rangle_F - \left\langle \Lambda, \hat{X} - \hat{X}^T \right\rangle_F. \quad (31)$$

Differentiating in \hat{X} and setting to zero gives

$$\hat{X} = \hat{X}_k + \hat{A}^T \hat{S} \Gamma^T + \Lambda - \Lambda^T. \quad (32)$$

Applying the symmetry constraint $X = X^T$ gives

$$\Lambda - \Lambda^T = \frac{1}{2} \left(\Gamma \hat{S}^T \hat{A} - \hat{A}^T \hat{S} \Gamma^T \right).$$

Substituting the above into (32) gives

$$\hat{X} = \hat{X}_k + \frac{1}{2} \left(\Gamma \hat{S}^T \hat{A} + \hat{A}^T \hat{S} \Gamma^T \right). \quad (33)$$

Now let $\Theta = \frac{1}{2} (\Gamma \hat{S}^T \hat{A} + \hat{A}^T \hat{S} \Gamma^T)$ and note that, since the matrix $\Theta + \hat{X}_k - \hat{A}^{-1}$ is symmetric, we get

$$\left\langle \hat{A}^T \hat{S} \Gamma^T, \Theta + \hat{X}_k - \hat{A}^{-1} \right\rangle_F = \left\langle \Theta, \Theta + \hat{X}_k - \hat{A}^{-1} \right\rangle_F. \quad (34)$$

Substituting (33) into (31) gives

$$\begin{aligned} L(\hat{X}, \Gamma, \Lambda) &= \frac{1}{2} \|\Theta\|_F^2 - \left\langle \hat{A}^T \hat{S} \Gamma^T, \Theta + \hat{X}_k - \hat{A}^{-1} \right\rangle_F \stackrel{(34)}{=} \frac{1}{2} \|\Theta\|_F^2 - \left\langle \Theta, \Theta + \hat{X}_k - \hat{A}^{-1} \right\rangle_F \\ &= -\frac{1}{2} \|\Theta\|_F^2 - \left\langle \Theta, \hat{X}_k - \hat{A}^{-1} \right\rangle_F. \end{aligned} \quad (35)$$

Adding $\pm \frac{1}{2} \left\| \hat{X}_k - \hat{A}^{-1} \right\|_F^2$ to (35) gives

$$L(\hat{X}, \Gamma, \Lambda) = -\frac{1}{2} \left\| \Theta + \hat{X}_k - \hat{A}^{-1} \right\|_F^2 + \frac{1}{2} \left\| \hat{X}_k - \hat{A}^{-1} \right\|_F^2.$$

³To re-interpret methods for solving linear systems through Bayesian inference, Hennig constructs estimates of the inverse system matrix using the sampled action of a matrix taken during a linear solve [20].

Finally, using (33) and maximizing over Γ then minimizing over X gives the dual problem

$$\min_{\hat{X}, \Gamma} \frac{1}{2} \left\| \hat{X} - \hat{A}^{-1} \right\|_F^2 \quad \text{subject to} \quad \hat{X} = \hat{X}_k + \frac{1}{2} (\Gamma \hat{S}^T \hat{A} + \hat{A}^T \hat{S} \Gamma^T).$$

It now remains to change variables according to (21) and set $Y = W^{1/2} \Gamma$. □

Algorithm 3 Stochastic Iterative Matrix Inversion (SIMI) – symmetric variant

- 1: **input:** symmetric invertible matrix $A \in \mathbb{R}^{n \times n}$
 - 2: **parameters:** \mathcal{D} = distribution over random matrices; symmetric positive definite $W \in \mathbb{R}^{n \times n}$
 - 3: **initialize:** symmetric matrix $X_0 \in \mathbb{R}^{n \times n}$
 - 4: **for** $k = 0, 1, 2, \dots$ **do**
 - 5: Sample an independent copy $S \sim \mathcal{D}$
 - 6: Compute $\Lambda = S(S^T A W A S)^{-1} S^T$
 - 7: Compute $\Theta = \Lambda A W$
 - 8: Compute $M_k = X_k A - I$
 - 9: $X_{k+1} = X_k - M_k \Theta - (M_k \Theta)^T + \Theta^T (A X_k A - A) \Theta$ ▷ This is equivalent to (27) & (28)
 - 10: **output:** last iterate X_k
-

6 Convergence

We now analyze the convergence of the *error*, $X_k - A^{-1}$, for iterates of Algorithms 1, 2 and 3. For the sake of economy of space, we only analyze Algorithms 1 and 3. Convergence of Algorithm 2 follows from convergence of Algorithm 1 by observing Remark 4.1.

The first analysis we present in Section 6.1 is concerned with the convergence of

$$\left\| \mathbf{E} [X_k - A^{-1}] \right\|^2,$$

that is, the *norm of the expected error*. We then analyze the convergence of

$$\mathbf{E} \left[\left\| X_k - A^{-1} \right\|^2 \right],$$

the *expected norm of the error*. The latter is a stronger type of convergence, as explained in the following proposition.

Proposition 6.1. *Let $X \in \mathbb{R}^{n \times n}$ be a random matrix, $\|\cdot\|$ a matrix norm induced by an inner product, and fix $A^{-1} \in \mathbb{R}^{n \times n}$. Then*

$$\left\| \mathbf{E} [X - A^{-1}] \right\|^2 = \mathbf{E} \left[\left\| X - A^{-1} \right\|^2 \right] - \mathbf{E} \left[\left\| X - \mathbf{E}[X] \right\|^2 \right].$$

Proof. Note that $\mathbf{E} \left[\left\| X - \mathbf{E}[X] \right\|^2 \right] = \mathbf{E} \left[\left\| X \right\|^2 \right] - \left\| \mathbf{E}[X] \right\|^2$. Adding and subtracting $\left\| A^{-1} \right\|^2 - 2 \langle \mathbf{E}[X], A^{-1} \rangle$ from the right hand side, then grouping the appropriate terms, yields the desired result. □

This shows that if $\mathbf{E} \left[\|X_k - A^{-1}\|^2 \right]$ converges to zero, then $\|\mathbf{E} [X_k - A^{-1}]\|^2$ converges to zero. But the converse is not necessarily true. Rather, the variance $\mathbf{E} \left[\|X_k - \mathbf{E} [X_k]\|^2 \right]$ must converge to zero for the converse to be true⁴.

The convergence of Algorithms 1 and 3 can be entirely characterized by studying the following random matrix

$$Z \stackrel{\text{def}}{=} A^T S (S^T A W A^T S)^{-1} S^T A. \quad (36)$$

With this definition, the update step of Algorithm 1 can be re-written as a simple fixed point formula

$$X_{k+1} - A^{-1} = (I - WZ) (X_k - A^{-1}). \quad (37)$$

We can also simplify the iterates of Algorithm 3 to

$$X_{k+1} - A^{-1} = (I - WZ) (X_k - A^{-1}) (I - ZW). \quad (38)$$

The only stochastic component in both methods is contained in the matrix Z , and ultimately, the convergence of the iterates will depend on $\mathbf{E} [Z]$, the expected value of this matrix. Thus we start with two lemmas concerning the Z and $\mathbf{E} [Z]$ matrices.

Lemma 6.1. *If Z is defined as in (36), then*

1. *the eigenvalues of $W^{1/2} Z W^{1/2}$ are either 0 or 1,*
2. *the matrix $W^{1/2} Z W^{1/2}$ projects onto the q -dimensional subspace $\mathbf{Range} (W^{1/2} A^T S)$.*

Proof. Using (36), simply verify that $(W^{1/2} Z W^{1/2})^2 = W^{1/2} Z W^{1/2}$ proves that it is a projection matrix, and thus has eigenvalues 0 or 1. Furthermore, the matrix $W^{1/2} Z W^{1/2}$ projects onto $\mathbf{Range} (W^{1/2} A^T S)$, which follows by verifying

$$W^{1/2} Z W^{1/2} (W^{1/2} A^T S) = W^{1/2} A^T S \quad \text{and} \quad W^{1/2} Z W^{1/2} y = 0, \quad \forall y \in \mathbf{Null} (W^{1/2} A^T S).$$

Finally $\dim (\mathbf{Range} (W^{1/2} A^T S)) = \mathbf{Rank} (W^{1/2} A^T S) = \mathbf{Rank} (S) = q$. □

Lemma 6.2. *Let Z be defined as in (36). The spectrum of $W^{1/2} \mathbf{E} [Z] W^{1/2}$ is contained in $[0, 1]$.*

Proof. Let $\hat{Z} = W^{1/2} Z W^{1/2}$, thus $W^{1/2} \mathbf{E} [Z] W^{1/2} = \mathbf{E} [\hat{Z}]$. Since the mapping $A \mapsto \lambda_{\max}(A)$ is convex, by Jensen's inequality we get $\lambda_{\max}(\mathbf{E} [\hat{Z}]) \leq \mathbf{E} [\lambda_{\max}(\hat{Z})]$. Applying Lemma 6.1, we conclude that $\lambda_{\max}(\mathbf{E} [\hat{Z}]) \leq 1$. The inequality $\lambda_{\min}(\mathbf{E} [\hat{Z}]) \geq 0$ can be shown analogously using convexity of the mapping $A \mapsto -\lambda_{\min}(A)$. □

⁴The convergence of $\|\mathbf{E} [X_k - A^{-1}]\|^2$ is also known in the probability literature as L^2 -norm convergence. It also follows trivially from the Markov's inequality that convergence in L^2 -norm implies convergence in probability.

6.1 Norm of the expected error

We start by proving that the norm of the expected error of the iterates of Algorithm 1 and Algorithm 3 converges to zero. The following theorem is remarkable in that we do not need to make any assumptions on the distribution S , except that S has full column rank. Rather, the theorem pinpoints that convergence depends solely on the spectrum of $I - W^{-1/2}\mathbf{E}[Z]W^{-1/2}$.

Theorem 6.1. *Let S be a random matrix which has full column rank with probability 1 (so that Z is well defined). Then the iterates X_{k+1} of Algorithm 1 satisfy*

$$\mathbf{E}[X_{k+1} - A^{-1}] = (I - W\mathbf{E}[Z])\mathbf{E}[X_k - A^{-1}]. \quad (39)$$

Let $X_0 \in \mathbb{R}^{n \times n}$. If X_k is calculated in either one of these two ways

1. Applying k iterations of Algorithm 1,
2. Applying k iterations of Algorithm 3 (assuming A and X_0 are symmetric),

then X_k converges to the inverse exponentially fast, according to

$$\|\mathbf{E}[X_k - A^{-1}]\|_{W^{-1}} \leq \rho^k \|X_0 - A^{-1}\|_{W^{-1}}, \quad (40)$$

where

$$\rho \stackrel{\text{def}}{=} 1 - \lambda_{\min}(W^{1/2}\mathbf{E}[Z]W^{1/2}). \quad (41)$$

Moreover, we have the following lower and upper bounds on the convergence rate:

$$0 \leq 1 - \frac{\mathbf{E}[q]}{n} \leq \rho \leq 1. \quad (42)$$

Proof. For all k , define $R_k \stackrel{\text{def}}{=} W^{-1/2}(X_k - A^{-1})W^{-1/2}$. Left and right multiplying (37) by $W^{-1/2}$ gives

$$R_{k+1} = (I - W^{1/2}ZW^{1/2})R_k. \quad (43)$$

Taking expectation with respect to S in (43) gives

$$\mathbf{E}[R_{k+1} | R_k] = (I - W^{1/2}\mathbf{E}[Z]W^{1/2})R_k. \quad (44)$$

Taking full expectation in (43) and using the tower rule gives

$$\begin{aligned} \mathbf{E}[R_{k+1}] &= \mathbf{E}[\mathbf{E}[R_{k+1} | R_k]] \\ &\stackrel{(44)}{=} \mathbf{E}[(I - W^{1/2}\mathbf{E}[Z]W^{1/2})R_k] \\ &= (I - W^{1/2}\mathbf{E}[Z]W^{1/2})\mathbf{E}[R_k]. \end{aligned} \quad (45)$$

Applying the norm in (45) gives

$$\begin{aligned} \|\mathbf{E}[X_{k+1} - A^{-1}]\|_{W^{-1}} &= \|\mathbf{E}[R_{k+1}]\|_2 \leq \left\| I - W^{1/2}\mathbf{E}[Z]W^{1/2} \right\|_2 \|\mathbf{E}[R_k]\|_2 \\ &= \left\| I - W^{1/2}\mathbf{E}[Z]W^{1/2} \right\|_2 \|\mathbf{E}[X_k - A^{-1}]\|_{W^{-1}}. \end{aligned} \quad (46)$$

Furthermore

$$\begin{aligned} \left\| I - W^{1/2}\mathbf{E}[Z]W^{1/2} \right\|_2 &= \lambda_{\max}\left(I - W^{1/2}\mathbf{E}[Z]W^{1/2}\right) \\ &= 1 - \lambda_{\min}(W^{1/2}\mathbf{E}[Z]W^{1/2}) \stackrel{(41)}{=} \rho, \end{aligned} \quad (47)$$

where we used to symmetry of $(I - W^{1/2}\mathbf{E}[Z]W^{1/2})$ when passing from the operator norm to the spectral radius. Note that the symmetry of $\mathbf{E}[Z]$ derives from the symmetry of Z . It now remains to unroll the recurrence in (46) to get (40).

Now we analyse the iterates of Algorithm 3. Left and right multiplying (38) by $W^{-1/2}$ we have

$$R_{k+1} = P(R_k) \stackrel{\text{def}}{=} \left(I - W^{1/2}ZW^{1/2} \right) R_k \left(I - W^{1/2}ZW^{1/2} \right). \quad (48)$$

Defining $\bar{P}(R) \stackrel{\text{def}}{=} \mathbf{E}[P(R)]$, taking expectation in (48), conditioned on R_k , gives

$$\mathbf{E}[R_{k+1} | R_k] = \bar{P}(R_k).$$

As \bar{P} is a linear operator, taking expectation again yields

$$\mathbf{E}[R_{k+1}] = \mathbf{E}[\bar{P}(R_k)] = \bar{P}(\mathbf{E}[R_k]). \quad (49)$$

Let $\|\bar{P}\|_2 \stackrel{\text{def}}{=} \max_{\|R\|_2=1} \|\bar{P}(R)\|_2$ be the operator induced norm. Applying norm in (49) gives

$$\|\mathbf{E}[X_{k+1} - A^{-1}]\|_{W^{-1}} = \|\mathbf{E}[R_{k+1}]\|_2 \quad (50)$$

$$\begin{aligned} &\leq \|\bar{P}\|_2 \|\mathbf{E}[R_k]\|_2 \\ &= \|\bar{P}\|_2 \|\mathbf{E}[X_k - A^{-1}]\|_{W^{-1}}. \end{aligned} \quad (51)$$

Clearly, P is a *positive linear map*, that is, it is linear and maps positive semi-definite matrices to positive semi-definite matrices. Thus, by Jensen's inequality, the map \bar{P} is also a positive linear map. As every positive linear map attains its norm at the identity matrix (see Corollary 2.3.8 in [2]), we have that

$$\begin{aligned} \|\bar{P}\|_2 &= \|\bar{P}(I)\|_2 \\ &\stackrel{(48)}{=} \left\| \mathbf{E} \left[\left(I - W^{1/2}ZW^{1/2} \right) I \left(I - W^{1/2}ZW^{1/2} \right) \right] \right\|_2 \\ &\stackrel{(\text{Lemma 6.1})}{=} \left\| \mathbf{E} \left[I - W^{1/2}ZW^{1/2} \right] \right\|_2 \\ &\stackrel{(47)}{=} \rho. \end{aligned}$$

Inserting the above equivalence in (51) and unrolling the recurrence gives (40).

Finally to prove (42), as proven in Lemma 6.2, the spectrum of $W^{1/2}\mathbf{E}[Z]W^{1/2}$ is contained in $[0, 1]$ consequently $0 \leq \rho \leq 1$. Furthermore, as the trace of a matrix is equal to the sum of its eigenvalues, we have

$$\begin{aligned} \mathbf{E}[q] &\stackrel{(\text{Lemma 6.1})}{=} \mathbf{E} \left[\text{Tr} \left(W^{1/2}ZW^{1/2} \right) \right] \\ &= \text{Tr} \left(\mathbf{E} \left[W^{1/2}ZW^{1/2} \right] \right) \\ &\geq n \lambda_{\min} \left(\mathbf{E} \left[W^{1/2}ZW^{1/2} \right] \right), \end{aligned} \quad (52)$$

where we used that $W^{1/2}ZW^{1/2}$ projects onto a q -dimensional subspace (Lemma 6.1), and thus $\text{Tr}(W^{1/2}ZW^{1/2}) = q$. Rearranging (52) gives (42). \square

If $\rho = 1$, this theorem does not guarantee convergence. But when $\mathbf{E}[Z]$ is positive definite, as it will transpire in all practical variants of our method, some of which we describe in Section 8, the rate ρ will be strictly less than one, and the norm of the expected error will converge to zero.

6.2 Expectation of the norm of the error

Now we consider the convergence of the expected norm of the error. This form of convergence is preferred, as it also proves that the variance of the iterates converges to zero (see Proposition 6.1).

Theorem 6.2. *Let S be a random matrix that has full column rank with probability 1 and such that $\mathbf{E}[Z]$ is positive definite, where Z is defined in (36). Let $X_0 \in \mathbb{R}^{n \times n}$. If X_k is calculated in either one of these two ways*

1. *Applying k iterations of Algorithm 1,*
2. *Applying k iterations of Algorithm 3 (assuming both A and X_0 are symmetric matrices),*

then X_k converges to the inverse according to

$$\mathbf{E} \left[\|X_k - A^{-1}\|_{F(W^{-1})}^2 \right] \leq \rho^k \|X_0 - A^{-1}\|_{F(W^{-1})}^2. \quad (53)$$

Proof. First consider Algorithm 1, where X_{k+1} is calculated by iteratively applying (37). If we let

$$R_k \stackrel{\text{def}}{=} X_k - A^{-1}, \quad \hat{R}_k \stackrel{\text{def}}{=} W^{-1/2} R_k W^{-1/2}, \quad \hat{Z} \stackrel{\text{def}}{=} W^{1/2} Z W^{1/2}, \quad (54)$$

then from (37) we have

$$\hat{R}_{k+1} = (I - \hat{Z}) \hat{R}_k. \quad (55)$$

From this we obtain

$$\begin{aligned} \|R_{k+1}\|_{F(W^{-1})}^2 &\stackrel{(1)}{=} \|\hat{R}_{k+1}\|_F^2 \\ &\stackrel{(55)}{=} \|(I - \hat{Z}) \hat{R}_k\|_F^2 \\ &= \mathbf{Tr} \left((I - \hat{Z}) (I - \hat{Z}) \hat{R}_k \hat{R}_k^T \right) \\ &\stackrel{(\text{Lemma 6.1})}{=} \mathbf{Tr} \left((I - \hat{Z}) \hat{R}_k \hat{R}_k^T \right) \\ &= \|\hat{R}_k\|_F^2 - \mathbf{Tr} \left(\hat{Z} \hat{R}_k \hat{R}_k^T \right). \end{aligned} \quad (56)$$

Taking expectations, conditioned on \hat{R}_k , we get

$$\mathbf{E} \left[\|\hat{R}_{k+1}\|_F^2 \mid \hat{R}_k \right] = \|\hat{R}_k\|_F^2 - \mathbf{Tr} \left(\mathbf{E} [\hat{Z}] \hat{R}_k \hat{R}_k^T \right).$$

Using that $\mathbf{Tr} \left(\mathbf{E} [\hat{Z}] \hat{R}_k \hat{R}_k^T \right) \geq \lambda_{\min} \left(\mathbf{E} [\hat{Z}] \right) \mathbf{Tr} \left(\hat{R}_k \hat{R}_k^T \right)$, which relies on the symmetry of $\mathbf{E} [\hat{Z}]$, we have that

$$\mathbf{E} \left[\|\hat{R}_{k+1}\|_F^2 \mid \hat{R}_k \right] \leq \left(1 - \lambda_{\min} \left(\mathbf{E} [\hat{Z}] \right) \right) \|\hat{R}_k\|_F^2 = \rho \cdot \|\hat{R}_k\|_F^2.$$

In order to arrive at (53), it now remains to take full expectation, unroll the recurrence and use the substitution (54)

Now we assume that A and X_0 are symmetric and $\{X_k\}$ are the iterates computed by Algorithm 3. Left and right multiplying (38) by $W^{-1/2}$ we have

$$\hat{R}_{k+1} = (I - \hat{Z}) \hat{R}_k (I - \hat{Z}). \quad (58)$$

Taking norm we have

$$\begin{aligned}
\left\| \hat{R}_{k+1} \right\|_F^2 &\stackrel{\text{(Lemma 6.1)}}{=} \mathbf{Tr} \left(\hat{R}_k \left(I - \hat{Z} \right) \hat{R}_k \left(I - \hat{Z} \right) \right) \\
&= \mathbf{Tr} \left(\hat{R}_k \hat{R}_k \left(I - \hat{Z} \right) \right) - \mathbf{Tr} \left(\hat{R}_k \hat{Z} \hat{R}_k \left(I - \hat{Z} \right) \right) \\
&\leq \mathbf{Tr} \left(\hat{R}_k \hat{R}_k \left(I - \hat{Z} \right) \right),
\end{aligned} \tag{59}$$

where in the last inequality we used the fact that $I - \hat{Z}$ is symmetric, whence

$$\mathbf{Tr} \left(\hat{R}_k \hat{Z} \hat{R}_k \left(I - \hat{Z} \right) \right) = \mathbf{Tr} \left(\hat{Z}^{1/2} \hat{R}_k \left(I - \hat{Z} \right) \hat{R}_k \hat{Z}^{1/2} \right) \geq 0.$$

The remainder of the proof follows similar steps as those we used in the first part of the proof from (57) onwards. □

Theorem 6.2 establishes that for all three methods, the expected norm of the error converges exponentially fast to zero. Moreover, the convergence rate ρ is the same that appeared in Theorem 6.1, where we established the convergence of the norm of the expected error.

Both of the convergence results in Theorems 6.1 and 6.2 can be recast as iteration complexity bounds. For instance, using standard arguments, from Theorem 6.1 we observe that for a given $0 < \epsilon < 1$ we have that

$$k \geq \left(\frac{1}{2} \right) \frac{1}{1 - \rho} \log \left(\frac{1}{\epsilon} \right) \quad \Rightarrow \quad \left\| \mathbf{E} [X_k - A^{-1}] \right\|_{W^{-1}}^2 \leq \epsilon \left\| X_0 - A^{-1} \right\|_{W^{-1}}^2. \tag{60}$$

On the other hand, from Theorem 6.2 we have

$$k \geq \frac{1}{1 - \rho} \log \left(\frac{1}{\epsilon} \right) \quad \Rightarrow \quad \mathbf{E} \left[\left\| X_k - A^{-1} \right\|_{F(W^{-1})}^2 \right] \leq \epsilon \left\| X_0 - A^{-1} \right\|_{F(W^{-1})}^2. \tag{61}$$

To push the expected norm of the error below the ϵ tolerance (61), we require double the amount of iterates, as compared to bringing the norm of expected error below the same tolerance (60). This is because in Theorem 6.2 we determined that ρ is the rate at which the expectation of the *squared* norm error converges, while in Theorem 6.1 we determined that ρ is the rate at which the norm, without the square, of the expected error converges. Though it takes double the number of iterations to decrease the expectation of the norm error, as proven in Proposition 6.1, the former is a stronger form of convergence. Thus, Theorem 6.1 does not give a stronger result than Theorem 6.2, but rather, these theorems give qualitatively different results and ultimately enrich our understanding of the iterative process.

7 Discrete Random Matrices

We now consider the case of a discrete random matrix S . We show that when S is a *complete discrete sampling*, then $\mathbf{E} [Z]$ is positive definite, and thus from Theorems 6.1 and 6.2, Algorithms 1, 2 and 4 converge.

Definition 7.1 (Complete Discrete Sampling). *The random matrix S has a finite discrete distribution with r outcomes. In particular, $S = S_i \in \mathbb{R}^{n \times q_i}$ with probability $p_i > 0$ for $i = 1, \dots, r$, where S_i is of full column rank. We say that S is a complete discrete sampling when $\mathbf{S} \stackrel{\text{def}}{=}} [S_1, \dots, S_r] \in \mathbb{R}^{n \times n}$ has full row rank.*

As an example of a complete discrete sampling, let $S = e_i$ (the i th unit coordinate vector in \mathbb{R}^n) with probability $p_i = 1/n$, for $i = 1, \dots, n$. Then \mathbf{S} , as defined in Definition 7.1, is equal to the identity matrix: $\mathbf{S} = I$. Consequently, S is a complete discrete sampling. In fact, from any basis of \mathbb{R}^n we could construct a complete discrete sampling in an analogous way.

Next we establish that for discrete distribution of S , that S having a complete discrete distribution is a necessary and sufficient condition for $\mathbf{E}[Z]$ to be positive definite. We also determine simple formula for $\mathbf{E}[Z]$. This will allow us to determine an optimized distribution for S in Section 7.1.

Proposition 7.1. *Let S be a discrete random matrix with r outcomes. The matrix $\mathbf{E}[Z]$ is positive definite if and only if S is a complete discrete sampling. Furthermore*

$$\mathbf{E}[Z] = A^T \mathbf{S} D^2 \mathbf{S}^T A, \quad (62)$$

where

$$D \stackrel{\text{def}}{=} \text{Diag}\left(\sqrt{p_1}(S_1^T A W A^T S_1)^{-1/2}, \dots, \sqrt{p_r}(S_r^T A W A^T S_r)^{-1/2}\right). \quad (63)$$

Proof. Taking the expectation of Z as defined in (36) gives

$$\begin{aligned} \mathbf{E}[Z] &= \sum_{i=1}^r A^T S_i (S_i^T A W A^T S_i)^{-1} S_i^T A p_i \\ &= A^T \left(\sum_{i=1}^r S_i \sqrt{p_i} (S_i^T A W A^T S_i)^{-1/2} (S_i^T A W A^T S_i)^{-1/2} \sqrt{p_i} S_i^T \right) A \\ &= (A^T \mathbf{S} D) (D \mathbf{S}^T A), \end{aligned}$$

and $\mathbf{E}[Z]$ is clearly positive semi-definite. Note that, since we assume throughout that S has full column rank with probability 1, the matrix D is well defined and nonsingular. Let $v \in \mathbf{Null}(\mathbf{E}[Z])$, thus

$$0 = v^T A^T \mathbf{S} D^2 \mathbf{S}^T A v = \|D \mathbf{S}^T A v\|_2^2,$$

which shows that $\mathbf{S}^T A v = 0$ and thus $A v \in \mathbf{Null}(\mathbf{S}^T)$. Assume that S is a complete discrete sampling then \mathbf{S}^T has full column rank. Thus $A v = 0$ together with the non-singularity of A gives $v = 0$. Conversely, assume that \mathbf{S}^T does not have full column rank, then there exists a nonzero $w \in \mathbf{Null}(\mathbf{S}^T)$ and consequently $0 \neq A^{-1} w \in \mathbf{Null}(\mathbf{E}[Z])$, which shows that $\mathbf{E}[Z]$ would not be positive definite. \square

With a closed form expression for $\mathbf{E}[Z]$ we can optimize ρ over the possible distributions of S to yield a better convergence rate.

7.1 Optimizing an upper bound on the convergence rate

So far we have proven two different types of convergence for Algorithms 1, 2 and 3 in Theorems 6.1 and 6.2. Furthermore, both forms of convergence depend on the same convergence rate ρ for which we have a closed form expression (41).

The availability of a closed form expression for the convergence rate opens up the possibility of designing particular distributions for S optimizing the rate. In [15] it was shown that (in the context of solving linear systems) for a complete discrete sampling, computing the optimal probability distribution, assuming the the matrices $\{S_i\}_{i=1}^r$ are fixed, leads to a semi-definite program (SDP). In some cases, the gain in performance from the optimal probabilities is much larger than the loss incurred by having to solve the SDP. However, this is not always the case.

Here we propose an alternative: to optimize the following upper bound on the convergence rate:

$$\rho = 1 - \lambda_{\min}(W^{1/2}\mathbf{E}[Z]W^{1/2}) \leq 1 - \frac{1}{\mathbf{Tr}(W^{-1/2}(\mathbf{E}[Z])^{-1}W^{-1/2})} \stackrel{\text{def}}{=} \gamma.$$

To emphasize the dependence of γ and Z on the probability distribution $p = (p_1, \dots, p_r) \in \mathbb{R}^r$, let us denote

$$\gamma(p) \stackrel{\text{def}}{=} 1 - \frac{1}{\mathbf{Tr}(W^{-1/2}(\mathbf{E}[Z_p])^{-1}W^{-1/2})}, \quad (64)$$

where we have added a subscript to Z to indicate that it is a function of p . We now minimize $\gamma(p)$ over the probability simplex:

$$\Delta_r \stackrel{\text{def}}{=} \left\{ p = (p_1, \dots, p_r) \in \mathbb{R}^r : \sum_{i=1}^r p_i = 1, p_i \geq 0 \right\}.$$

Theorem 7.1. *Let S be a complete discrete sampling and let $\bar{S}_i \in \mathbb{R}^{n \times q_i}$, for $i = 1, 2, \dots, r$, be such that $\mathbf{S}^{-T} = [\bar{S}_1, \dots, \bar{S}_r]$. Then*

$$\min_{p \in \Delta_r} \gamma(p) = 1 - \frac{1}{\left(\sum_{i=1}^r \left\| W^{1/2} A^T S_i \bar{S}_i^T A^{-T} W^{-1/2} \right\|_F \right)^2}. \quad (65)$$

Proof. In view of (64), minimizing γ in p is equivalent to minimizing $\mathbf{Tr}(W^{-1/2}(\mathbf{E}[Z_p])^{-1}W^{-1/2})$ in p . Further, we have

$$\begin{aligned} \mathbf{Tr}\left(W^{-1/2}(\mathbf{E}[Z_p])^{-1}W^{-1/2}\right) &\stackrel{(62)}{=} \mathbf{Tr}\left(W^{-1/2}(A^T \mathbf{S} D^2 \mathbf{S}^T A)^{-1}W^{-1/2}\right) \\ &= \mathbf{Tr}\left(W^{-1/2}A^{-1}\mathbf{S}^{-T}D^{-2}\mathbf{S}^{-1}A^{-T}W^{-1/2}\right) \\ &\stackrel{(63)}{=} \sum_{i=1}^r \frac{1}{p_i} \mathbf{Tr}\left(W^{-1/2}A^{-1}\bar{S}_i(S_i^T A W A^T S_i)\bar{S}_i^T A^{-T}W^{-1/2}\right) \\ &= \sum_{i=1}^r \frac{1}{p_i} \left\| W^{1/2}A^{-1}\bar{S}_i S_i^T A W^{-1/2} \right\|_F^2. \end{aligned} \quad (66)$$

$$(67)$$

Applying Lemma 12.1 in the Appendix, the optimal probabilities are given by

$$p_i = \frac{\left\| W^{1/2}A^{-1}\bar{S}_i S_i^T A W^{-1/2} \right\|_F}{\sum_{j=1}^r \left\| W^{1/2}A^{-1}\bar{S}_j S_j^T A W^{-1/2} \right\|_F}, \quad i = 1, 2, \dots, r \quad (68)$$

Plugging this into (67) gives the result (65). \square

Observe that in general, the optimal probabilities (68) cannot be calculated, since the formula involves the inverse of A , which is not known. However, if A is symmetric positive definite, we can choose $W = A^2$, which eliminates this issue. If A is not symmetric positive definite, or if we do not wish to choose $W = A^2$, we can approach the formula (68) as a recipe for a heuristic choice of the probabilities: we can use the iterates $\{X_k\}$ as a proxy for A^{-1} . With this setup, the resulting method is not guaranteed to converge by the theory developed in this paper. However, in practice one would expect it to work well. We have not done extensive experiments to test this, and leave this to future research. To illustrate, let us consider a concrete simple example. Choose $W = I$

and $S_i = e_i$ (the unit coordinate vector in \mathbb{R}^n). We have $\mathbf{S} = [e_1, \dots, e_n] = I$, whence $\bar{S}_i = e_i$ for $i = 1, \dots, r$. Plugging into (68), we obtain

$$p_i = \frac{\|X_k e_i e_i^T A\|_F}{\sum_{j=1}^r \|X_k e_j e_j^T A\|_F} = \frac{\|X_k e_i\|_2 \|e_i^T A\|_2}{\sum_{j=1}^r \|X_k e_j\|_2 \|e_j^T A\|_2}.$$

7.2 Convenient sampling

We now ask the following question: given matrices S_1, \dots, S_r defining a complete discrete sampling, assign probabilities p_i to S_i so that the convergence rate ρ becomes *easy to interpret*. The following result was first stated in [15] in the context of solving linear systems, and gives a convenient choice of probabilities resulting in the rate ρ which depends on a (scaled) condition number of the original data matrix A .

Proposition 7.2. *Let S be a complete discrete sampling where $S = S_i$ with probability*

$$p_i = \frac{\|W^{1/2} A^T S_i\|_F^2}{\|W^{1/2} A^T \mathbf{S}\|_F^2}. \quad (69)$$

Then the convergence rate takes the form

$$\rho = 1 - \frac{1}{\kappa_{2,F}^2(W^{1/2} A^T \mathbf{S})}, \quad (70)$$

where

$$\kappa_{2,F}(W^{1/2} A^T \mathbf{S}) \stackrel{\text{def}}{=} \left\| (W^{1/2} A^T \mathbf{S})^{-1} \right\|_2 \left\| W^{1/2} A^T \mathbf{S} \right\|_F = \sqrt{\frac{\text{Tr}(\mathbf{S}^T A W A^T \mathbf{S})}{\lambda_{\min}(\mathbf{S}^T A W A^T \mathbf{S})}} \geq \sqrt{n}. \quad (71)$$

Proof. For the proof of (70), see Theorem 5.1 in [15]. The bound in (71) follows trivially. \square

Following from Remark 4.1, we can determine a convergence rate for Algorithm 2 based on the Theorem 7.2.

Remark 7.1. *Let S be a complete discrete sampling where $S = S_i$ with probability*

$$p_i = \frac{\|W^{1/2} A S_i\|_F^2}{\|W^{1/2} A \mathbf{S}\|_F^2}. \quad (72)$$

Then Algorithm 2 converges at the rate $\rho_2 = 1 - 1/\kappa_{2,F}^2(W^{1/2} A \mathbf{S})$.

7.3 Optimal and adaptive samplings

Having decided on the probabilities p_1, \dots, p_r associated with the matrices S_1, \dots, S_r in Proposition 7.2, we can now ask the following question. How should we choose the matrices $\{S_i\}$ if we want ρ to be as small as possible? Since the rate improves as the *condition number* $\kappa_{2,F}^2(W^{1/2} A^T \mathbf{S})$ decreases, we should aim for matrices that minimize the condition number. Notice that the lower bound in (71) is reached for $\mathbf{S} = (W^{1/2} A^T)^{-1} = A^{-T} W^{-1/2}$. While we do not know A^{-1} , we can use our best current approximation of it, X_k , in its place. This leads to a method which *adapts* the probability distribution governing S throughout the iterative process. This observation inspires a very efficient modification of Algorithm 3, which we call AdaRBFBS (Adaptive Randomized BFGS), and describe in Section 9.

Notice that, luckily and surprisingly, our twin goals of computing the inverse and optimizing the convergence rate via the above adaptive trick are compatible. Indeed, we wish to find A^{-1} , whose knowledge gives us the optimal rate. This should be contrasted with the SDP approach mentioned earlier: i) the SDP could potentially be harder than the inversion problem, and ii) having found the optimal probabilities $\{p_i\}$, we are still not guaranteed the optimal rate. Indeed, optimality is relative to the choice of the matrices S_1, \dots, S_r , which can be suboptimal.

Remark 7.2 (Adaptive sampling). *The convergence rate (70) suggests how one can select a sampling distribution for S that would result in faster practical convergence. We now detail several practical choices for W and indicate how to sample S . These suggestions require that the distribution of S depends on the iterate X_k , and thus no longer fit into our framework. Nonetheless, we collect these suggestions here in the hope that others will wish to extend these ideas further, and as a demonstration of the utility of developing convergence rates.*

1. *If $W = I$, then Algorithm 1 converges at the rate $\rho = 1 - 1/\kappa_{2,F}^2(A^T \mathbf{S})$, and hence S should be chosen so that \mathbf{S} is a preconditioner of A^T . For example $\mathbf{S} = X_k^T$, that is, S should be a sampling of the rows of X_k .*
2. *If $W = I$, then Algorithm 2 converges at the rate $\rho = 1 - 1/\kappa_{2,F}^2(\mathbf{A}\mathbf{S})$, and hence S should be chosen so that \mathbf{S} is a preconditioner of A . For example $\mathbf{S} = X_k$; that is, S should be a sampling of the columns of X_k .*
3. *If A is symmetric positive definite, we can choose $W = A^{-1}$, in which case Algorithm 3 converges at the rate $\rho = 1 - 1/\kappa_{2,F}^2(A^{1/2}\mathbf{S})$. This rate suggests that S should be chosen so that \mathbf{S} is an approximation of $A^{-1/2}$. In Section 9 we develop these ideas further, and design the AdaRBFSG algorithm.*
4. *If $W = (A^T A)^{-1}$, then Algorithm 1 can be efficiently implemented with $S = AV$, where V is a complete discrete sampling. Furthermore $\rho = 1 - 1/\kappa_{2,F}^2(A\mathbf{V})$, where $\mathbf{V} \stackrel{\text{def}}{=} [V_1, \dots, V_r]$. This rate suggests that V should be chosen so that \mathbf{V} is a preconditioner of A . For example $\mathbf{V} = X_k$; that is, V should be a sampling of the rows of X_k .*
5. *If $W = (AA^T)^{-1}$, then Algorithm 2 can be efficiently implemented with $S = A^T V$, where V is a complete discrete sampling. From Remark 7.1, the convergence rate of the resulting method is given by $1 - 1/\kappa_{2,F}^2(A^T \mathbf{V})$. This rate suggests that V should be chosen so that \mathbf{V} is a preconditioner of A^T . For example, $\mathbf{V} = X_k^T$; that is, V should be a sampling of the columns of X_k .*
6. *If A is symmetric positive definite, we can choose $W = A^2$, in which case Algorithm 3 can be efficiently implemented with $S = AV$. Furthermore $\rho = 1 - 1/\kappa_{2,F}^2(A\mathbf{V})$. This rate suggests that V should be chosen so that \mathbf{V} is a preconditioner of A . For example $\mathbf{V} = X_k$, that is, V should be a sampling of the rows or the columns of X_k .*

8 Randomized Quasi-Newton Updates

Algorithms 1, 2 and 3 are in fact families of algorithms indexed by the two parameters: i) positive definite matrix W and ii) distribution \mathcal{D} (from which we pick random matrices S). This allows us to design a myriad of specific methods by varying these parameters. Here we highlight some of these possibilities, focusing on complete discrete distributions for S so that convergence of the

iterates is guaranteed through Theorems 6.1 and 6.2. We also compute the convergence rate ρ for these special methods for the convenient probability distribution given by (69) (Proposition 7.2) so that the rate ρ depends on a condition number which is easy to interpret. We will also make some connections to existing quasi-Newton and Approximate Inverse Preconditioning methods. Table 2 provides a guide through this section.

A	W	S	Inverse Equation	Randomized Update	Section
any	any	invertible	any	One Step	8.1
any	I	e_i	$AX = I$	Simultaneous Kaczmarz (SK)	8.2
any	I	vector	$XA = I$	Bad Broyden (BB)	8.3
sym.	I	vector	$AX = I, X = X^T$	Powell-Symmetric-Broyden (PSB)	8.4
any	I	vector	$XA^{-1} = I$	Good Broyden (GB)	8.5
sym.	$A^{-1} - X_k$	vector	$AX = I$ or $XA = I$	Symmetric Rank 1 (SR1)	8.7
s.p.d.	A	vector	$XA^{-1} = I, X = X^T$	Davidon-Fletcher-Powell (DFP)	8.8
s.p.d.	A^{-1}	vector	$AX = I, X = X^T$	Broyden-Fletcher-Goldfarb-Shanno (BFGS)	8.9
any	$(A^T A)^{-1}$	vector	$AX = I$	Column	8.10

Table 2: Specific randomized updates for inverting matrices discussed in this section, obtained as special cases of our algorithms. First column: “sym” means “symmetric” and “s.p.d.” means “symmetric positive definite”. Block versions of all these updates are obtained by choosing S as a matrix with more than one column (i.e., not as a vector).

8.1 One Step Update

We have the freedom to select S as almost any random matrix that has full column rank. This includes choosing S to be a constant and invertible matrix, such as the identity matrix I , in which case X_1 must be equal to the inverse. Indeed, the sketch-and-project formulations of all our algorithms reveal that. For Algorithm 1, for example, the sketched system is $S^T A X = S^T$, which is equivalent to $AX = I$, which has as its unique solution $X = A^{-1}$. Hence, $X_1 = A^{-1}$, and we have convergence in one iteration/step. Through inspection of the complexity rate, we see that $W^{1/2} \mathbf{E}[Z] W^{1/2} = I$ and $\rho = \lambda_{\min}(W^{1/2} \mathbf{E}[Z] W^{1/2}) = 1$, thus this one step convergence is predicted the theory, Theorems 6.1 and 6.2.

8.2 Simultaneous Randomized Kaczmarz Update

Perhaps the most natural choice for the weighting matrix W is the identity $W = I$. With this choice, Algorithm 1 is equivalent to applying the randomized Kaczmarz update simultaneously to the n linear systems encoded in $AX = I$. To see this, note that the sketch-and-project viewpoint (15) of Algorithm 1 is

$$X_{k+1} = \arg \min_X \frac{1}{2} \|X - X_k\|_F^2 \quad \text{subject to} \quad S^T A X = S^T, \quad (73)$$

which, by (19), results in the explicit update

$$X_{k+1} = X_k + A^T S (S^T A A^T S)^{-1} S^T (I - A X_k). \quad (74)$$

If S is a random coordinate vector, then (73) is equivalent to projecting the j th column of X_k onto the solution space of $A_{i:j} x = \delta_{ij}$, which is exactly an iteration of the randomized Kaczmarz

update applied to solving $Ax = e_j$. In particular, if $S = e_i$ with probability $p_i = \|A_{i:}\|_2^2 / \|A\|_F^2$ then according to Proposition 7.2, the rate of convergence of update (74) is given by

$$\mathbf{E} \left[\|X_k - A^{-1}\|_F^2 \right] = \left(1 - \frac{1}{\kappa_{2,F}^2(A)} \right)^k \|X_0 - A^{-1}\|_F^2$$

where we used that $\kappa_{2,F}(A) = \kappa_{2,F}(A^T)$. This is exactly the rate of convergence given by Strohmer and Vershynin in [36] for the randomized Kaczmarz method.

8.3 Randomized Bad Broyden Update

The update (74) can also be viewed as an adjoint form of the bad Broyden update [4, 19]. To see this, if we use Algorithm 2 with $W = I$, then the iterative process is

$$X_{k+1} = X_k + (I - X_k A) S (S^T A^T A S)^{-1} S^T A^T. \quad (75)$$

This update (75) is a randomized block form of the *bad Broyden update* [4, 19]. In the quasi-Newton setting, S is not random, but rather the previous step direction $S = \delta \in \mathbb{R}^n$. Furthermore, if we rename $\gamma \stackrel{\text{def}}{=} AS \in \mathbb{R}^n$, then (75) becomes

$$X_{k+1} = X_k + \frac{\delta - X_k \gamma}{\|\gamma\|_2^2} \gamma^T, \quad (76)$$

which is the standard way of writing the bad Broyden update [19]. The update (74) is an adjoint form of the bad Broyden in the sense that, if we transpose (74), then set $S = \delta$ and denote $\gamma = A^T S$, we obtain the bad Broyden, but applied to X_k^T instead.

From the constrain-and-approximate viewpoint (18) we give a new interpretation to the bad Broyden update, namely, the update (76) can be written as

$$X_{k+1} = \arg_X \min_{X \in \mathbb{R}^{n \times n}, y \in \mathbb{R}^n} \frac{1}{2} \|X - A^{-1}\|_F^2 \quad \text{subject to} \quad X = X_k + y \gamma^T.$$

Thus, *the bad Broyden update is the best rank-one update approximating the inverse.*

We can determine the rate at which our randomized variant of the BB update (75) converges by using Remark 7.1. In particular, if $S = S_i$ with probability $p_i = \|AS_i\|_F^2 / \|AS\|_F^2$, then (81) converges with the rate

$$\mathbf{E} \left[\|X_k - A^{-1}\|_F^2 \right] = \left(1 - \frac{1}{\kappa_{2,F}^2(AS)} \right)^k \|X_0 - A^{-1}\|_F^2.$$

8.4 Randomized Powell-Symmetric-Broyden Update

If A is symmetric and we use Algorithm 3 with $W = I$, the iterates are given by

$$\begin{aligned} X_{k+1} = & X_k + AS^T (S^T A^2 S)^{-1} SA (X_k AS - S) ((S^T A^2 S)^{-1} S^T A - I) \\ & - (X_k AS - S) (S^T A^2 S)^{-1} S^T A, \end{aligned} \quad (77)$$

which is a randomized block form of the Powell-Symmetric-Broyden update [14]. If $S = S_i$ with probability $p_i = \|AS_i\|_F^2 / \|AS\|_F^2$, then according to Proposition 7.2, the iterates (77) and (74) converge according to

$$\mathbf{E} \left[\|X_k - A^{-1}\|_F^2 \right] \leq \left(1 - \frac{1}{\kappa_{2,F}^2(A^T S)} \right)^k \|X_0 - A^{-1}\|_F^2.$$

8.5 Randomized Good Broyden Update

Next we present a method that shares certain properties with Gaussian elimination and can be viewed as a randomized block variant of the good Broyden update [4, 19]. This method requires the following adaptation of Algorithm 2: instead of sketching the inverse equation, consider the update (78) that performs a column sketching of the equation $XA^{-1} = I$ by right multiplying with Ae_i , where e_i is the i th coordinate vector. Projecting an iterate X_k onto this sketched equation gives

$$X_{k+1} = \arg \min_{X \in \mathbb{R}^{n \times n}} \frac{1}{2} \|X - X_k\|_F^2 \quad \text{subject to} \quad Xe_i = Ae_i. \quad (78)$$

The iterates defined by the above are given by

$$X_{k+1} = X_k + (A - X_k)e_i e_i^T. \quad (79)$$

Given that we are sketching and projecting onto the solution space of $XA^{-1} = I$, the iterates of this method converge to A . Therefore the inverse iterates X_k^{-1} converge to A^{-1} . We can efficiently compute the inverse iterates by using the Woodbury formula [37] which gives

$$X_{k+1}^{-1} = X_k^{-1} - \frac{(X_k^{-1}A - I)e_i e_i^T X_k^{-1}}{e_i^T X_k^{-1} A e_i}. \quad (80)$$

This update (80) behaves like Gaussian elimination in the sense that, if i is selected in a cyclic fashion, that is $i = k$ on the k th iteration, then from (79) it is clear that

$$X_{k+1}e_i = Ae_i, \quad \text{thus} \quad X_{k+1}^{-1}Ae_i = e_i, \quad \text{for } i = 1 \dots k.$$

That is, on the k th iteration, the first k columns of the matrix $X_{k+1}^{-1}A$ are equal to the first k columns of the identity matrix. Consequently, $X_n = A$ and $X_n^{-1} = A^{-1}$. If instead, we select i uniformly at random, then we can adapt Proposition 7.2 by swapping each occurrence of A^T for A^{-1} and observing that $S_i = Ae_i$ thus $\mathbf{S} = A$. Consequently the iterates (79) converge to A at a rate of

$$\rho = 1 - \kappa_{2,F}^2(A^{-1}A) = 1 - \frac{1}{n},$$

and thus the lower bound (42) is achieved and X_k converges to A according to

$$\mathbf{E} \left[\|X_k - A\|_F^2 \right] \leq \left(1 - \frac{1}{n} \right)^k \|X_0 - A\|_F^2.$$

Despite this favourable convergence rate, this does not say anything about how fast X_k^{-1} converges to A^{-1} . Therefore (80) is not an efficient method for calculating an approximate inverse. If we replace e_i by a *step* direction $\delta_k \in \mathbb{R}^d$, then the update (80) is known as the *good Broyden* update [4, 19].

8.6 Approximate Inverse Preconditioning

When A is symmetric positive definite, we can choose $W = A^{-1}$, and Algorithm 1 is given by

$$X_{k+1} = X_k + S(S^T A S)^{-1} S^T (I - A X_k). \quad (81)$$

The constrain-and-approximate viewpoint (17) of this update is

$$X_{k+1} = \arg_X \min_{X,Y} \frac{1}{2} \left\| A^{1/2} X A^{1/2} - I \right\|_F^2 \quad \text{subject to} \quad X = X_k + S Y^T.$$

This viewpoint reveals that the update (81) is akin to the Approximate Inverse Preconditioning (AIP) methods [1, 13, 22, 21].

We can determine the rate at which (81) converges using Remark 7.1. In particular, if $S = S_i$ with probability $p_i = \mathbf{Tr}(S_i^T A S_i) / \mathbf{Tr}(S^T A S)$, then (81) converges with rate

$$\rho = 1 - \frac{1}{\kappa_{2,F}^2(A^{1/2} S)} = 1 - \frac{\lambda_{\min}(S^T A S)}{\mathbf{Tr}(S^T A S)}, \quad (82)$$

and according to

$$\mathbf{E} \left[\left\| A^{1/2} X_k A^{1/2} - I \right\|_F^2 \right] \leq \left(1 - \frac{\lambda_{\min}(S^T A S)}{\mathbf{Tr}(S^T A S)} \right)^k \left\| A^{1/2} X_0 A^{1/2} - I \right\|_F^2.$$

8.7 Randomized SR1

The Symmetric Rank-1 (SR1) update [8, 27] does not explicitly fit into our framework, and nor does it fit into the traditional quasi-Newton framework, since it requires a W that is not positive definite. Despite this, we present the update since it is still commonly used.

When A is symmetric and $W = A^{-1} - X_k$ then from (19) or (20) we get

$$X_{k+1} = X_k + (I - A X_k)^T S (S^T (A - A X_k A) S)^{-1} S^T (I - A X_k). \quad (83)$$

This choice for W presents problems, namely, the update (83) is not always well defined because it requires inverting $S^T (A - A X_k A) S$ which is not necessarily invertible. To fix this, we should select the sketching matrix S so that $S^T (A - A X_k A) S$ is invertible. But this in turn means that S will depend on X_k and most likely cannot be sampled in an i.i.d fashion. Alternatively, we can alter the definition of the update, and use the pseudo inverse of $S^T (A - A X_k A) S$ in place of the inverse.

Since W is not positive definite, our convergence theory says nothing about this update.

8.8 Randomized DFP Update

If A is symmetric positive definite then we choose $W = A$. Furthermore, if we adapt the sketch-and-project formulation (15) to sketch the equation $X A^{-1} = I$ by right multiplying by AS , and additionally impose symmetry on the iterates, we arrive at the following update.

$$X_{k+1} = \arg \min_{X \in \mathbb{R}^{n \times n}} \frac{1}{2} \|X - X_k\|_{F(A)}^2 \quad \text{subject to} \quad X S = A S, \quad X = X^T. \quad (84)$$

The solution to the above is given by

$$X_{k+1} = A S (S^T A S)^{-1} S^T A + (I - A S (S^T A S)^{-1} S^T) X_k (I - S (S^T A S)^{-1} S^T A). \quad (85)$$

Using the Woodbury formula [37], we find that

$$X_{k+1}^{-1} = X_k^{-1} + A S (S^T A S)^{-1} S^T A - X_k^{-1} S (S^T X_k^{-1} S)^{-1} S^T X_k^{-1}. \quad (86)$$

The update (86) is a randomized variant of the Davidon-Fletcher-Powell (DFP) update [7, 10]. We can adapt Proposition 7.2 to determine the rate at which X_k converges to A by swapping each

occurrence of A^T for A^{-1} . Indeed, for example, let $S_i = Ae_i$ with probability $p_i = \lambda_{\min}(A)/\mathbf{Tr}(A)$, then the iterates (79) converge to A at a rate of

$$\mathbf{E} \left[\|X_k - A\|_{F(A^{-1})}^2 \right] \leq \left(1 - \frac{\lambda_{\min}(A)}{\mathbf{Tr}(A)} \right)^k \|X_0 - A\|_{F(A^{-1})}^2. \quad (87)$$

Thus X_k converges to A at a favourable rate. But this does not indicate at what rate does X_k^{-1} converge to A^{-1} . This is in contrast to the randomized BFGS, which produces iterates that converge to A^{-1} at this same favourable rate, as we show in the next section. This sheds new light on why BFGS update performs better than the DFP update.

8.9 Randomized BFGS Update

If A is symmetric and positive definite, we can choose $W = A^{-1}$ and apply Algorithm 3 to maintain symmetry of the iterates. The iterates are given by

$$X_{k+1} = S(S^T AS)^{-1}S^T + (I - S(S^T AS)^{-1}S^T A) X_k (I - AS(S^T AS)^{-1}S^T). \quad (88)$$

This is a block variant, see [14], of the BFGS update [4, 10, 12, 34]. The constrain-and-approximate viewpoint gives a new interpretation to the Block BFGS update. That is, from (27), the iterates (88) can be equivalently defined by

$$X_{k+1} = \arg_X \min_{X,Y} \frac{1}{2} \|XA - I\|_F^2 \quad \text{subject to} \quad X = X_k + SY^T + YS^T.$$

Thus the block BFGS update, and the standard BFGS update, can be seen as a method for calculating an approximate inverse subject to a particular symmetric affine space passing through X_k . This is a completely new way of interpreting the BFGS update.

If $p_i = \mathbf{Tr}(S_i^T AS_i) / \mathbf{Tr}(SAS^T)$, then according to Proposition 7.2, the updates (88) and (81) converge according to

$$\mathbf{E} \left[\|X_k A - I\|_F^2 \right] \leq \left(1 - \frac{1}{\kappa_{2,F}^2(A^{1/2}S)} \right)^k \|X_0 A - I\|_F^2. \quad (89)$$

A remarkable property of the update (88) is that it preserves positive definiteness of A . Indeed, assume that X_k is positive definite and let $v \in \mathbb{R}^n$ and $P \stackrel{\text{def}}{=} S(S^T AS)^{-1}S^T$. Left and right multiplying (88) by v^T and v , respectively, gives

$$v^T X_{k+1} v = v^T P v + v^T (I - PA) X_k (I - AP) v \geq 0.$$

Thus $v^T X_{k+1} v = 0$ implies that $Pv = 0$ and $(I - AP)v = 0$, which when combined gives $v = 0$. This proves that X_{k+1} is positive definite. Thus the update (88) is particularly well suited for calculating the inverse of a positive definite matrices.

In the next section, we detail a update designed to improve the convergence rate in (89). The result is a method that is able to invert large scale positive definite matrices orders of magnitude faster than the state-of-the-art.

8.10 Randomized Column Update

We now describe an update that has no connection to any previous updates, yet the convergence rate we determine (92) is favourable, and comparable to all the other updates we develop.

For this update, we need to perform a linear transformation of the sampling matrices. For this, let V be a complete discrete sampling where $V = [V_1, \dots, V_r]$ with probability $p_i > 0$, for $i = 1, \dots, r$. Let $\mathbf{V} = [V_1, \dots, V_r]$. Let the sampling matrices be defined as $S_i = AV_i \in \mathbb{R}^{n \times q_i}$ for $i = 1, \dots, r$. As A is nonsingular, and $\mathbf{S} = A\mathbf{V}$, then S is a complete discrete sampling. With these choices and $W^{-1} = A^T A$, the sketch-and-project viewpoint (15) is given by

$$X_{k+1} = \arg \min_X \frac{1}{2} \|X - X_k\|_{F(A^T A)}^2 \quad \text{subject to} \quad V_i^T A^T A X = V_i^T A^T.$$

The solution to the above are the iterates of Algorithm 1, which is given by

$$X_{k+1} = X_k + V_i (V_i^T A^T A V_i)^{-1} V_i^T (A^T - A^T A X_k). \quad (90)$$

From the constrain-and-approximate viewpoint (17), this can be written as

$$X_{k+1} = \arg \min_{X \in \mathbb{R}^{n \times n}} \frac{1}{2} \|A(XA^T - I)\|_F^2 \quad \text{subject to} \quad X = X_k + V_i Y^T, \quad Y \in \mathbb{R}^{n \times q}.$$

With these same parameter choices for S and W , the iterates of Algorithm 3 are given by

$$\begin{aligned} X_{k+1} = & X_k + V_i (V_i^T A^2 V_i)^{-1} V_i^T (A X_k - I) (A^2 V_i (V_i^T A^2 V_i)^{-1} V_i^T - I) \\ & - (X_k A - I) A V_i (V_i^T A^2 V_i)^{-1} V_i^T. \end{aligned} \quad (91)$$

If we choose $p_i = \|(AA^T)^{-1/2} AA^T V_i\|_F^2 / \|(AA^T)^{-1/2} AA^T \mathbf{V}\|_F^2 = \|A^T V_i\|_F^2 / \|A^T \mathbf{V}\|_F^2$, then according to Proposition 7.2, the iterates (90) and (91) converge exponentially in expectation to the inverse according to

$$\mathbf{E} \left[\|A(X_k A^T - I)\|_F^2 \right] \leq \left(1 - \frac{1}{\kappa_{2,F}^2(A\mathbf{V})} \right)^k \|A(X_0 A^T - I)\|_F^2. \quad (92)$$

There also exists an analogous ‘‘row’’ variant of (90), which arises by using Algorithm 2, but we do not explore it here.

9 AdaRBFGS: Adaptive Randomized BFGS

All the updates we have developed thus far use a sketching matrix S that is sampled in an i.i.d. fashion from a fixed distribution \mathcal{D} at each iteration. In this section we assume that A is symmetric positive definite, and propose AdaRBFGS: a variant of the RBFGS update, discussed in Section 8.9, which *adaptively* changes the distribution \mathcal{D} throughout the iterative process. Due to this change, Theorems 6.1 and 6.2 and Proposition 7.2 are no longer applicable. Superior numerical efficiency of this update is verified through extensive numerical experiments in Section 10.

9.1 Motivation

We now motivate the design of this new update by examining the convergence rate (89) of the RBFGS iterates (88). Recall that in RBFGS we choose $W = A^{-1}$ and $S = S_i$ with probability

$$p_i = \mathbf{Tr}(S_i^T A S_i) / \mathbf{Tr}(\mathbf{S} A \mathbf{S}^T), \quad i = 1, 2, \dots, r, \quad (93)$$

where S is a complete discrete sampling and $\mathbf{S} = [S_1, \dots, S_r]$. The convergence rate is

$$\rho = 1 - \frac{1}{\kappa_{2,F}^2(A^{1/2}\mathbf{S})} \stackrel{(71)}{=} 1 - \frac{\lambda_{\min}(\mathbf{S}^T \mathbf{A} \mathbf{S})}{\mathbf{Tr}(\mathbf{S}^T \mathbf{A} \mathbf{S})}.$$

Consider now the question of choosing the matrix \mathbf{S} in such a way that ρ is as small as possible. Note that the optimal choice is any \mathbf{S} such that

$$\mathbf{S}^T \mathbf{A} \mathbf{S} = I.$$

Indeed, then $\rho = 1 - 1/n$, and the lower bound (71) is attained. For instance, the choice $\mathbf{S} = A^{-1/2}$ would be optimal. This means that in each iteration we would choose S to be a random column (or random column submatrix) of $A^{-1/2}$. Clearly, this is not a feasible choice, as we do not know the inverse of A . In fact, it is A^{-1} which we are trying to find! However, this leads to the following interesting observation: *the goals of finding the inverse of A and of designing an optimal distribution \mathcal{D} are in synchrony.*

9.2 The algorithm

While we do not know $A^{-1/2}$, we can use the information of the iterates $\{X_k\}$ themselves to construct a good *adaptive* sampling. Indeed, the iterates contain information about the inverse and hence we can use them to design a better sampling S . In order to do so, it will be useful to maintain a factored form of the iterates,

$$X_k = L_k L_k^T, \tag{94}$$

where $L_k \in \mathbb{R}^{n \times n}$ is invertible. With this in place, let us choose S to be a random column submatrix of L_k . In particular, let C_1, C_2, \dots, C_r be nonempty subsets of $[n] = \{1, 2, \dots, n\}$ forming a partition of $[n]$, and at iteration k choose

$$S = L_k I_{:C_i} \stackrel{\text{def}}{=} S_i, \tag{95}$$

with probability p_i given by (93) for $i = 1, 2, \dots, r$. For simplicity, assume that $C_1 = \{1, \dots, c_1\}$, $C_2 = \{c_1 + 1, \dots, c_2\}$ and so on, so that, by the definition of \mathbf{S} , we have

$$\mathbf{S} = [S_1, \dots, S_r] = L_k. \tag{96}$$

Note that now both \mathbf{S} and p_i depend on k . The method described above satisfies the following recurrence.

Theorem 9.1. *Consider one step of the AdaRBFPS method described above. Then*

$$\mathbf{E} \left[\|X_{k+1} - A^{-1}\|_{F(A)}^2 \mid X_k \right] \leq \left(1 - \frac{\lambda_{\min}(AX_k)}{\mathbf{Tr}(AX_k)} \right) \|X_k - A^{-1}\|_{F(A)}^2. \tag{97}$$

Proof. Using the same arguments as those in the proof of Theorem 6.2, we obtain

$$\mathbf{E} \left[\|X_{k+1} - A^{-1}\|_{F(A)}^2 \mid X_k \right] \leq \left(1 - \lambda_{\min} \left(A^{-1/2} \mathbf{E} [Z \mid X_k] A^{-1/2} \right) \right) \|X_k - A^{-1}\|_{F(A)}^2, \tag{98}$$

where

$$Z \stackrel{(99)}{=} AS_i (S_i^T AS_i)^{-1} S_i^T A. \tag{99}$$

So, we only need to show that

$$\lambda_{\min} \left(A^{-1/2} \mathbf{E} [Z | X_k] A^{-1/2} \right) \geq \frac{\lambda_{\min}(AX_k)}{\mathbf{Tr}(AX_k)}.$$

Since S is a complete discrete sampling, Proposition 7.1 applied to our setting says that

$$\mathbf{E} [Z | X_k] = \mathbf{A} \mathbf{S} D^2 \mathbf{S}^T \mathbf{A}, \quad (100)$$

where

$$D \stackrel{\text{def}}{=} \text{Diag} \left(\sqrt{p_1} (S_1^T \mathbf{A} S_1)^{-1/2}, \dots, \sqrt{p_r} (S_r^T \mathbf{A} S_r)^{-1/2} \right). \quad (101)$$

We now have

$$\begin{aligned} \lambda_{\min} \left(A^{-1/2} \mathbf{E} [Z | X_k] A^{-1/2} \right) &\stackrel{(100)+(96)}{\geq} \lambda_{\min} \left(A^{1/2} L_k L_k^T A^{1/2} \right) \lambda_{\min}(D^2) \\ &\stackrel{(94)}{=} \frac{\lambda_{\min}(AX_k)}{\lambda_{\max}(D^{-2})} \\ &\stackrel{(101)}{=} \frac{\lambda_{\min}(AX_k)}{\max_i \lambda_{\max}(S_i^T \mathbf{A} S_i) / p_i} \\ &\geq \frac{\lambda_{\min}(AX_k)}{\max_i \mathbf{Tr}(S_i^T \mathbf{A} S_i) / p_i} \\ &\stackrel{(93)+(96)}{=} \frac{\lambda_{\min}(AX_k)}{\mathbf{Tr}(AX_k)}, \end{aligned}$$

where in the second equality we have used the fact that the largest eigenvalue of a block diagonal matrix is equal to the maximum of the largest eigenvalues of the blocks. \square

If X_k converges to A^{-1} , then necessarily the one-step rate of AdaRBFGS proved in Theorem 9.1 asymptotically reaches the lower bound

$$\rho_k \stackrel{\text{def}}{=} 1 - \frac{\lambda_{\min}(AX_k)}{\mathbf{Tr}(AX_k)} \rightarrow 1 - \frac{1}{n}.$$

In other words, as long as this method works, the convergence rate gradually improves, and becomes asymptotically optimal and independent of the condition number. We leave a deeper analysis of this and other adaptive variants of the methods developed in this paper to future work.

9.3 Implementation

To implement the AdaRBFGS update, we need to maintain the iterates X_k in the factored form (94). Fortunately, a factored form of the update (88) was introduced in [17], which we shall now describe. Assuming that X_k is symmetric positive definite such that $X_k = L_k L_k^T$, we shall describe how to obtain a corresponding factorization of X_{k+1} . Letting $G_k = (S^T L_k^{-T} L_k^{-1} S)^{1/2}$ and $R_k = (S^T \mathbf{A} S)^{-1/2}$, it can be verified through direct inspection [17] that $X_{k+1} = L_{k+1} L_{k+1}^T$, where

$$L_{k+1} = L_k + S R_k \left(G_k^{-1} S^T L_k^{-T} - R_k^T S^T \mathbf{A} L_k \right). \quad (102)$$

If we instead of (95) consider the more general update $S = L_k \tilde{S}$, where \tilde{S} is chosen in an i.i.d. fashion from some fixed distribution \tilde{D} , then

$$L_{k+1} = L_k + L_k \tilde{S} R_k \left((\tilde{S}^T \tilde{S})^{-1/2} \tilde{S}^T - R_k^T \tilde{S}^T L_k^T \mathbf{A} L_k \right). \quad (103)$$

Algorithm 4 Adaptive Randomized BFGS (AdaRBFGS)

- 1: **input:** symmetric positive definite matrix A
 - 2: **parameter:** $\tilde{\mathcal{D}}$ = distribution over random matrices with n rows
 - 3: **initialize:** pick invertible $L_0 \in \mathbb{R}^{n \times n}$
 - 4: **for** $k = 0, 1, 2, \dots$ **do**
 - 5: Sample an independent copy $\tilde{S} \sim \tilde{\mathcal{D}}$
 - 6: Compute $S = L_k \tilde{S}$ $\triangleright S$ is sampled adaptively, as it depends on k
 - 7: Compute $R_k = (\tilde{S}^T A \tilde{S})^{-1/2}$
 - 8: $L_{k+1} = L_k + S R_k \left((\tilde{S}^T \tilde{S})^{-1/2} \tilde{S}^T - R_k^T S^T A L_k \right)$ \triangleright Update the factor
 - 9: **output:** $X_k = L_k L_k^T$
-

The above can now be implemented efficiently, see Algorithm 4.

In Section 10 we test two variants based on (103). The first is the *AdaRBFGS_gauss* update, in which the entries of \tilde{S} are standard Gaussian. The second is *AdaRBFGS_cols*, where $\tilde{S} = I_{:C_i}$, as described above, and $|C_i| = q$ for all i for some q .

10 Numerical Experiments

Given the demand for approximate inverses of positive definite matrices in preconditioning and in variable metric methods in optimization, and the authors own interests in the aforementioned applications, we restrict our test to inverting positive definite matrices.

We test four iterative methods for inverting matrices. This rules out the all-or-nothing direct methods such as Gaussian elimination of LU based methods.

For our tests we use two variants of Algorithm 4: *AdaRBFGS_gauss*, where $\tilde{S} \in \mathbb{R}^{n \times q}$ is a normal Gaussian matrix, and *AdaRBFGS_cols*, where \tilde{S} consists of a collection of q distinct coordinate vectors in \mathbb{R}^n , selected uniformly at random. At each iteration the AdaRBFGS methods compute the inverse of a small matrix $S^T A S$ of dimension $q \times q$. To invert this matrix we use MATLAB's inbuilt `inv` function, which uses *LU* decomposition or Gaussian elimination, depending on the input. Either way, `inv` costs $O(q^3)$. We selected $q = \sqrt{n}$ so that the cost of inverting $O(q^3)$ and forming $O(n^2 q)$ the matrix $S^T A S$ are of the same order.

We compare our method to two well established and competitive methods, the *Newton-Schulz method* [33] and the global self-conditioned Minimal Residual (*MR*) method [6]. The Newton-Schulz method arises from applying the Newton-Raphson method to solve the inverse equation $I - AX = 0$, which gives

$$X_{k+1} = 2X_k - X_k A X_k. \quad (104)$$

The MR method was designed to calculate approximate inverses, and it does so by minimizing the norm of the residual along the preconditioned residual direction, that is

$$\|I - AX_{k+1}\|_F^2 = \min_{\alpha \in \mathbb{R}} \left\{ \|I - AX\|_F^2 \quad \text{subject to} \quad X = X_k + \alpha X_k (I - AX_k) \right\}, \quad (105)$$

see [32, chapter 10.5] for a didactic introduction to MR methods. The resulting iterates of the MR are given by

$$X_{k+1} = X_k + \frac{\mathbf{Tr}(R_k^T A X_k R_k)}{\mathbf{Tr}((A X_k R_k)^T A X_k R_k)} X_k R_k, \quad (106)$$

where $R_k = I - AX_k$.

We perform two sets of tests. On the first set, we choose a different starting matrix for each method which is optimized, in some sense, for that method. We then compare the empirical convergence of each method, including the time take to calculate X_0 . In particular, the Newton-Schulz is only guaranteed to converge for an initial matrix X_0 such that $\rho(I - X_0A) < 1$. Indeed, the Newton-Schulz method did not converge in most of our experiments when X_0 was not carefully chosen according to this criteria. To remedy this, we choose $X_0 = 0.99 \cdot A^T / \sigma^2(A)$ for the Newton-Schulz method, so that $\rho(I - X_0A) < 1$ is satisfied. To compute $\sigma(A)$ we used the inbuilt MATLAB function `normest` which is coded in C++. While for MR we followed the suggestion in [32] and used the projected identity for the initial matrix $X_0 = (\mathbf{Tr}(A) / \mathbf{Tr}(AA^T)) \cdot I$. For our AdaRBFGS methods we simply used $X_0 = I$, as this worked well in practice.

In the second set of tests, which we relegate to the Appendix, we compare the empirical convergence of the methods starting from the same matrix, namely the identity matrix $X_0 = I$.

We run each method until the relative error $\|I - AX_k\|_F / \|I - AX_0\|_F$ is below 10^{-2} . All experiments were performed and run in MATLAB R2014b. To appraise the performance of each method we plot the relative error against time taken and against the number of floating point operations (*flops*).

10.1 Experiment 1: synthetic matrices

First we compare the four methods on synthetic matrices generated using the matrix function `rand`. To appraise the difference in performance of the methods as the dimension of the problem grows, we tested for $n = 1000, 2000$ and 5000 . As the dimension grows, only the two variants of the AdaRBFGS method are able to reach the 10^{-2} desired tolerance in a reasonable amount time and number of flops (see Figure 2).

10.2 Experiment 2: LIBSVM matrices

Next we invert the Hessian matrix $\nabla^2 f(x)$ of four ridge-regression problems of the form

$$\min_{x \in \mathbb{R}^n} f(x) \stackrel{\text{def}}{=} \frac{1}{2} \|Ax - b\|_2^2 + \frac{\lambda}{2} \|x\|_2^2, \quad \nabla^2 f(x) = A^T A + \lambda I, \quad (107)$$

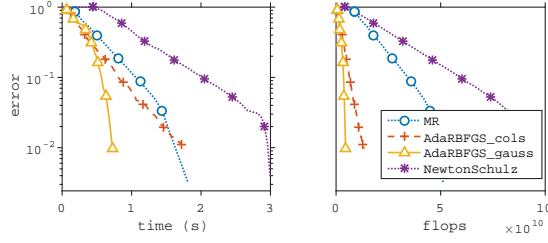
using data from LIBSVM [5], see Figure 3. We use $\lambda = 1$ as the regularization parameter. On the two problems of smaller dimension, `aloi` and `protein`, the four methods have a similar performance, and encounter the inverse in less than one second. On the two larger problems, `gisette-scale` and `real-sim`, the two variants of AdaRBFGS significantly outperform the MR and the Newton-Schulz method.

10.3 Experiment 3: UF sparse matrices

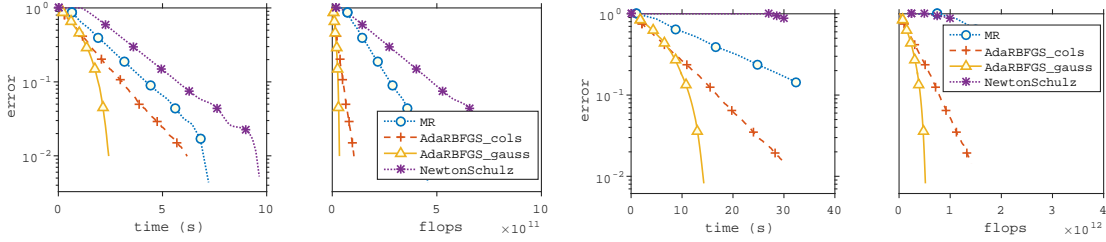
For our final batch of tests, we invert several sparse matrices from the Florida sparse matrix collection [9]. We have selected six problems from six different applications, so that the set of matrices display a varied sparsity pattern and structure, see Figure 4.

On the matrix `Bates/Chem97ZtZ` of moderate size, the four methods perform well, with the Newton-Schulz method converging first in time and AdaRBFGS_cols first in flops. While on the matrices of larger dimension, the two variants of AdaRBFGS converge much faster, often orders of magnitude before the MR and Newton-Schulz method reach the required precision.

The significant difference between the performance of the methods on large scale problems can be, in part, explained by their iteration cost. The iterates of the Newton-Schulz and MR method



(a) rand with $n = 10^4$



(b) rand with $n = 2 \cdot 10^4$

(c) rand with $n = 5 \cdot 10^4$

Figure 2: Synthetic MATLAB generated problems. Uniform random matrix $A = \bar{A}^T \bar{A}$ where $\bar{A} = \text{rand}(n)$.

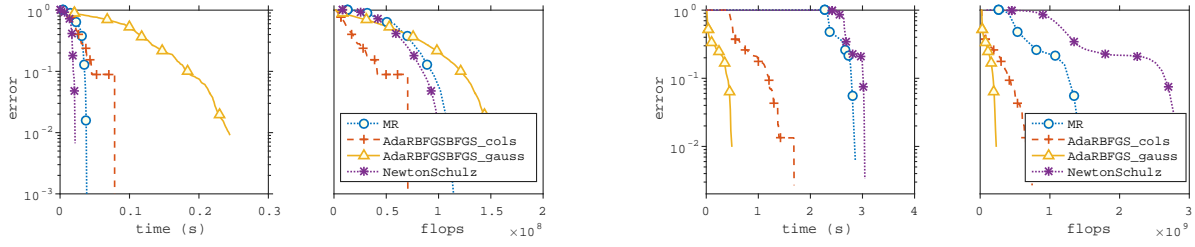
compute $n \times n$ matrix-matrix products. While the cost of an iteration of the AdaRBFGS methods is dominated by the cost of a $n \times n$ matrix by $n \times q$ matrix product. As a result, and because we set $q = \sqrt{n}$, this is difference of n^3 to $n^{2+1/2}$ in iteration cost, which clearly shows on the larger dimensional instances. On the other hand, both the Newton-Schulz and MR method are quadratically locally convergent, thus when the iterates are close to the solution, these methods enjoy a notable speed-up.

11 Conclusion

We develop a family of stochastic methods for iteratively inverting matrices, with a specialized variant for asymmetric, symmetric and positive definite matrices. The methods have two dual viewpoints, a sketch-and-project viewpoint which is an extension of the least-change formulation of the quasi-Newton methods, and a constrain-and-approximate viewpoint which is related to the approximate inverse preconditioning (API) methods. The equivalence between these two viewpoints reveals a new connection between the quasi-Newton and the API methods, which were previously considered to be unrelated.

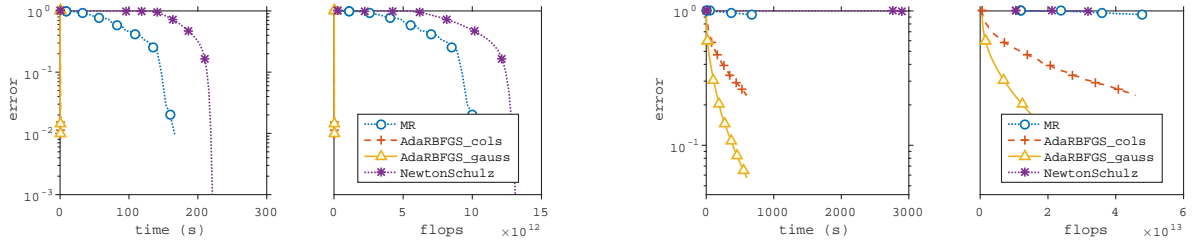
Under mild conditions, we prove convergence rates through two different perspectives, the convergence of the expected norm of the error, and the norm of the expected error. Our convergence theorems are general enough to accommodate discrete samplings and continuous samplings, though we only explore discrete sampling here in more detail.

For discrete samplings, we determine a probability distribution for which the convergence rates are equal to a scaled condition number, and thus are easily interpretable. Furthermore, for discrete sampling, we determining a practical optimized sampling distribution, that is obtained by mini-



(a) aloi

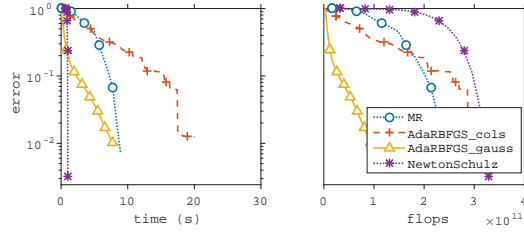
(b) protein



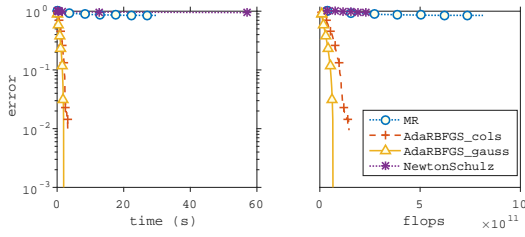
(c) gisette_scale

(d) real_sim

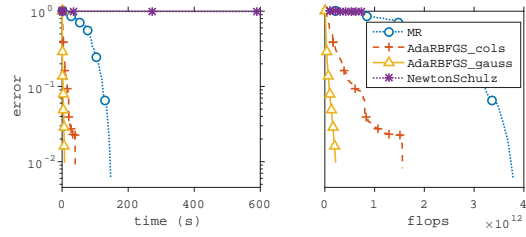
Figure 3: The performance of Newton-Schulz, MR, AdaRBFSGS_gauss and AdaRBFSGS_cols methods on the Hessian matrix of four LIBSVM test problems: (a) **aloi**: $(m; n) = (108, 000; 128)$ (b) **protein**: $(m; n) = (17, 766; 357)$ (c) **gisette_scale**: $(m; n) = (6000; 5000)$ (d) **real_sim**: $(m; n) = (72, 309; 20, 958)$.



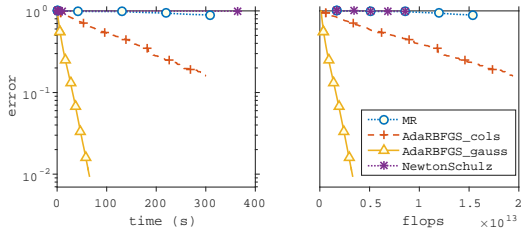
(a) Bates/Chem97ztZ



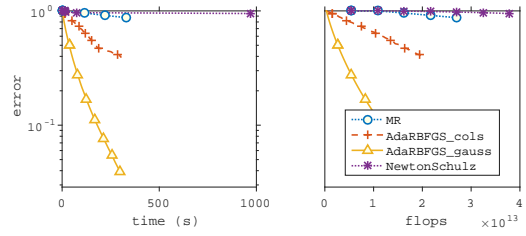
(b) FIDAP/ex9



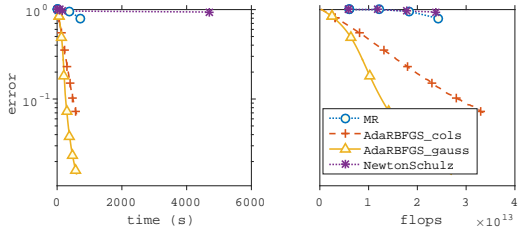
(c) Nasa/nasa4704



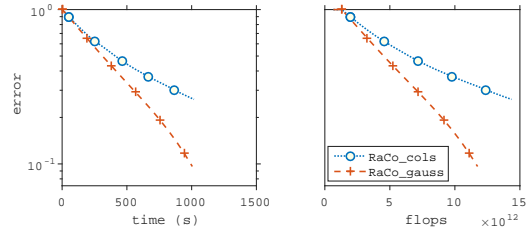
(d) HB/bcsstk18



(e) Pothen/bodyy4



(f) ND/nd6k



(g) GHS_psdef/wathen100

Figure 4: The performance of Newton-Schulz, MR, AdaRBFGS_gauss and AdaRBFGS_cols on (a) Bates-Chem97ztZ: $n = 2541$, (b) FIDAP/ex9: $n = 3,363$, (c) Nasa/nasa4704: $n = 4,704$, (d) HB/bcsstk18: $n = 11,948$, (e) Pothen/bodyy4: $n = 17,546$ (f) ND/nd6k: $n = 18,000$ (g) GHS_psdef/wathen100: $n = 30,401$.

mizing an upper bound on the convergence rate. We develop new randomized block variants of the quasi-Newton updates, including the BFGS update, complete with convergence rates, and provide new insights into these methods using our dual viewpoint.

For positive definite matrices, we develop an Adaptive Randomized BFGS methods (AdaRBFGS), which in large-scale numerical experiments, prove to be orders of magnitude faster (in time and flops) than the self-conditioned minimal residual method and the Newton-Schulz method. In particular, only the AdaRBFGS methods are able to approximately invert the $20,958 \times 20,958$ ridge regression matrix based on the `real-sim` data set in reasonable time and flops.

This work opens up many possible venues for future work, including, developing methods that use continuous random sampling, implementing a limited memory approach akin to the LBFGS [28] method, which could maintain an operator that serves as an approximation to the inverse. As recently shown in [16], an analogous method applied to linear systems converges with virtually no assumptions on the system matrix. This can be extended to calculating the pseudo inverse matrix, something we leave for future work.

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12 Appendix: Optimizing the Convergence Rate

Lemma 12.1. *Let a_1, \dots, a_r be positive real numbers. Then*

$$\left[\frac{\sqrt{a_1}}{\sum_{i=1}^r \sqrt{a_i}}, \dots, \frac{\sqrt{a_r}}{\sum_{i=1}^r \sqrt{a_i}} \right] = \arg \min_{p \in \Delta_r} \sum_{i=1}^r \frac{a_i}{p_i}.$$

Proof. Incorporating the constraint $\sum_{i=1}^r p_i = 1$ into the Lagrangian we have

$$\min_{p \geq 0} \sum_{i=1}^r \frac{a_i}{p_i} + \mu \sum_{i=1}^r (p_i - 1),$$

where $\mu \in \mathbb{R}$. Differentiating in p_i and setting to zero, then isolating p_i gives

$$p_i = \sqrt{\frac{a_i}{\mu}}, \quad \text{for } i = 1, \dots, r. \tag{108}$$

Summing over i gives

$$1 = \sum_{i=1}^r \sqrt{\frac{a_i}{\mu}} \Rightarrow \mu = \left(\sum_{i=1}^r \sqrt{a_i} \right)^2.$$

Inserting this back into (108) gives $p_i = \sqrt{a_i} / \sum_{i=1}^r \sqrt{a_i}$. □

13 Appendix: Numerical Experiments with the Same Starting Matrix

We now investigate the empirical convergence of the methods MR, AdaRBFGS_cols and AdaRGFBS_gauss when initiated with the same starting matrix $X_0 = I$, see Figures 5 and 6. We did not include the Newton-Schultz method in these figures because it diverged on all experiments when initiated from $X_0 = I$. Again we observe that, as the dimension grows, only the two variants of the AdaRBFGS are capable of inverting the matrix to the desired 10^{-2} precision in a reasonable amount of time. Furthermore, the AdaRBFGS_gauss variant had the overall best performance.

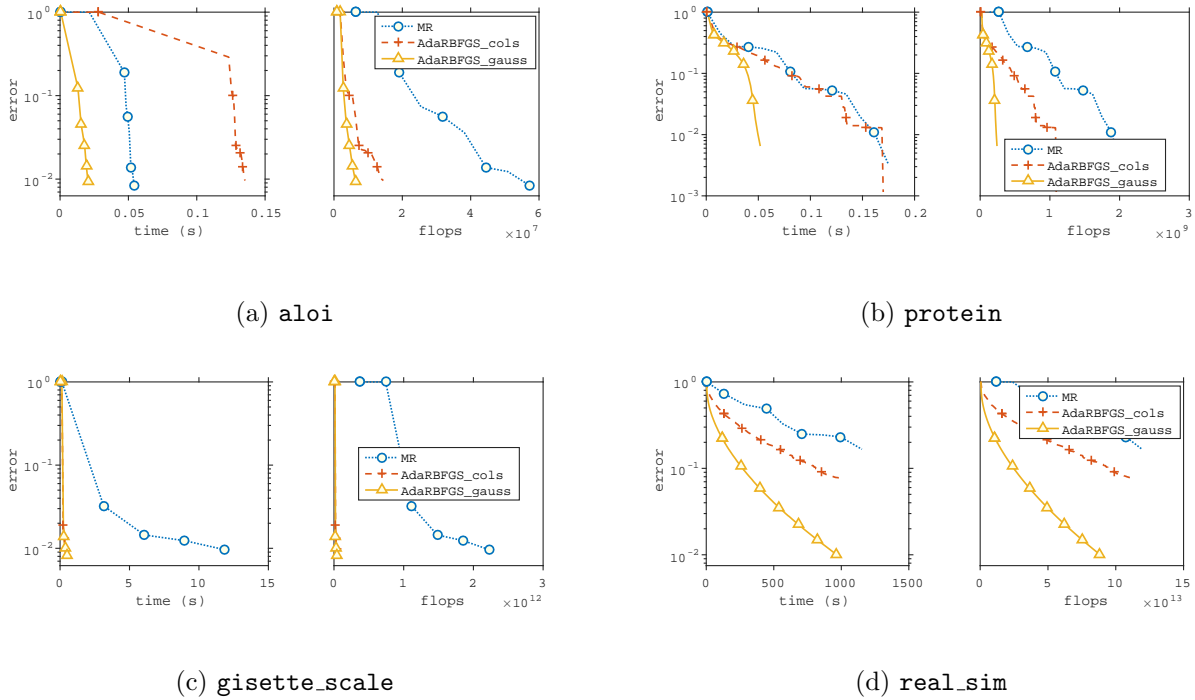
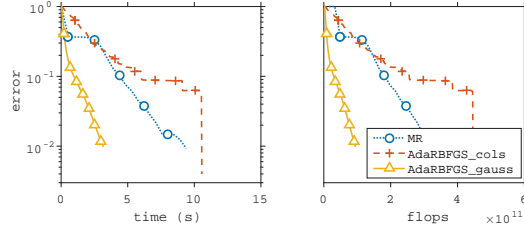
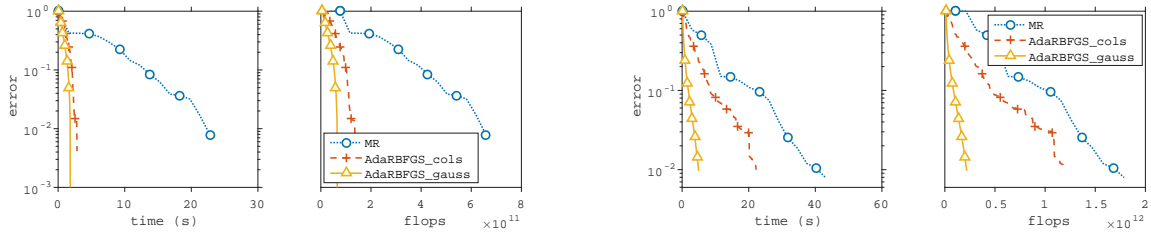


Figure 5: The performance of Newton-Schulz, MR, AdaRBFSG_gauss and AdaRBFSG_cols methods on the Hessian matrix of four LIBSVM test problems: (a) **aloi**: $(m; n) = (108, 000; 128)$ (b) **protein**: $(m; n) = (17, 766; 357)$ (c) **gisette_scale**: $(m; n) = (6000; 5000)$ (d) **real_sim**: $(m; n) = (72, 309; 20, 958)$. The starting matrix $X_0 = I$ was used for all methods.

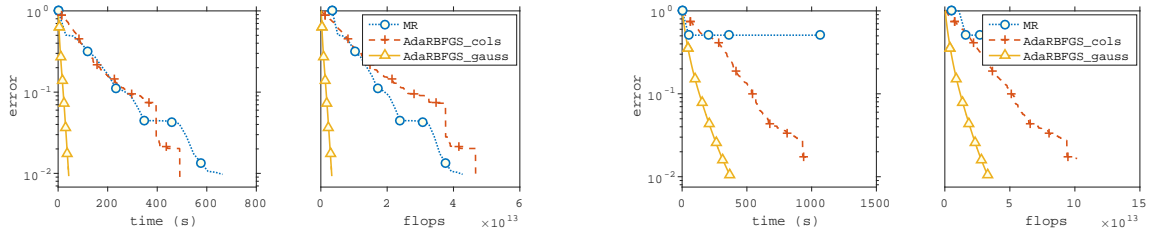


(a) Bates/Chem97ztZ



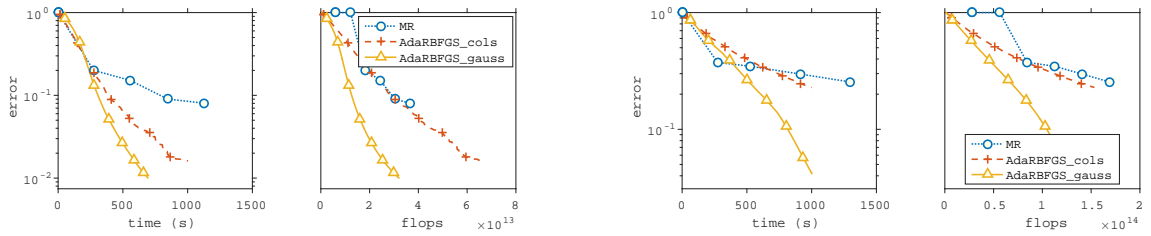
(b) FIDAP/ex9

(c) Nasa/nasa4704



(d) HB/bcsstk18

(e) Pothen/bodyy4



(f) ND/nd6k

(g) GHS_psdef/wathen100

Figure 6: The performance of Newton-Schulz, MR, AdaRBFGS_gauss and AdaRBFGS_cols on (a) Bates-Chem97ztZ: $n = 2541$, (b) FIDAP/ex9: $n = 3,363$, (c) Nasa/nasa4704: $n = 4,704$, (d) HB/bcsstk18: $n = 11,948$, (e) Pothen/bodyy4: $n = 17,546$ (f) ND/nd6k: $n = 18,000$ (g) GHS_psdef/wathen100: $n = 30,401$. The starting matrix $X_0 = I$ was used for all methods.